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Flat locally homogeneous affine connections with torsion on 2-dimensional manifolds

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Abstract. The aim of this paper is to review and to obtain results about flat locally homogeneous affine connections with torsion on 2-dimensional manifolds using the new information provides by the recently obtained classification [1].

Keywords: two-dimensional manifolds with flat affine connection, locally homogeneous connections, Lie algebras of vector fields, Killing vector fields

MSC 2000 classification: primary 53B05, secondary 53C30

Dedicated to Prof. Oldřich Kowalski for the important advances made in differential geometry.

1 Introduction and main result

The field of affine differential geometry is well-established and still in quick development (see e.g. [8]). Also, many basic facts about affine transformation groups and affine Killing vector fields are known from the literature (see [4, vol.I] and [3]).

On the other hand, homogeneity is one of the fundamental notions in geometry although its meaning must be always specified for the concrete situations. In this paper, we consider the homogeneity of manifolds equipped with affine connections. This homogeneity means that, for every two points of a manifold, there is an affine diffeomorphism which sends one point into another. We shall treat a local version of the homogeneity, that is, we admit that the affine diffeomorphisms are given only locally, i. e., from a neighborhood onto a neighborhood.

Let (M^n, ∇) be an affine manifold. We say that the affine connection ∇ is *flat* if and only if the curvature tensor $\mathcal{R}$ vanishes on M. Moreover, the following

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result it is well-known. (See [2, p. 49]).

1 Theorem. *Let (M^n, ∇) be an affine manifold. Then, the curvature tensor and the torsion tensor vanish on M if and only if around each point there exists a local coordinate system such that all Christoffel symbols vanish.*

However, no result is known if the torsion tensor does not vanish. Thus, our main porpoise is provided one new result in this direction. But we are only going to work on two-dimensional manifolds. The reason is that there are many locally homogeneous affine structures on 2-dimensional manifolds although two-dimensional locally homogeneous Riemannian manifolds are those with constant curvature. Moreover, it is remarkable that a seemingly easy problem to classify all locally homogeneous *torsion-less* connections in the plane domains was solved in 2004 in [10] (direct method) and in [7] (group-theoretical method). Unfortunately, no relation between both classifications was given. See also the previous partial results in [5] and [6]. Furthermore, the essential relationship between the classifications given in [7] and [10] and the classification of all locally homogeneous affine connections with *arbitrary torsion* in the plane domains have been obtained only recently in [1]. For dimension three, to make a classification seems to be a hard problem.

The original result by the author and O. Kowalski [1] was the following:

2 Theorem (Classification Theorem)**.** *Let ∇ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold M. Then, either ∇ is locally a Levi-Civita connection of the unit sphere or, in a neighborhood $\mathcal{U}$ of each point $m \in M$, there is a system (u, v) of local coordinates and constants a, b, c, d, e, f, g, h such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:*

Type A

$$\nabla_{\partial_u}\partial_u = a\partial_u + b\partial_v, \quad \nabla_{\partial_u}\partial_v = c\partial_u + d\partial_v,$$

$$\nabla_{\partial_v}\partial_u = e\partial_u + f\partial_v, \quad \nabla_{\partial_v}\partial_v = g\partial_u + h\partial_v.$$

Type B

$$\nabla_{\partial_u}\partial_u = \tfrac{a\partial_u + b\partial_v}{u}, \quad \nabla_{\partial_u}\partial_v = \tfrac{c\partial_u + d\partial_v}{u},$$

$$\nabla_{\partial_v}\partial_u = \tfrac{e\partial_u + f\partial_v}{u}, \quad \nabla_{\partial_v}\partial_v = \tfrac{g\partial_u + h\partial_v}{u},$$

where not all a, b, c, d, e, f, g, h are zero.

It is clear from Theorem 1 that if the Ricci tensor, *Ric*, and the torsion tensor, T, vanish on (M^2, ∇), all Christoffel symbols can be made constant and the corresponding affine manifolds are of type A. Nonetheless, an example which shows that the last result (and, as a direct consequence, also Theorem 1) can not be extended to the flat connections with torsion has been given in [1]. More explicitly, they have proved that the one-parameter family of connections

$$\nabla_{\partial_u}\partial_u = \tfrac{-1}{2u}\partial_u, \quad \nabla_{\partial_u}\partial_v = -2eu\partial_u + \tfrac{1}{2u}\partial_v, \quad \nabla_{\partial_v}\partial_u = eu\partial_u + \tfrac{1}{2u}\partial_v,$$
$$\nabla_{\partial_v}\partial_v = -2e^2u^3\partial_u - eu\partial_v, \quad e \neq 0,$$

which are flat and with nonzero torsion are *not* of type A.

Now, we shall study the case $T \neq 0$, $\nabla T = 0$ and $Ric = 0$ on 2-dimensional affine manifolds.

In the next section, we shall recall some details and partial results about the proof of the Classification Theorem. This information is needed in the last section where we shall prove our main result.

3 Theorem (Main result)**.** *Let ∇ be a locally homogeneous flat affine connection on a 2-dimensional manifold M such that $T \neq 0$ and $\nabla T = 0$. Then, ∇ is of type A.*

2 Preliminaries

Let ∇ be an affine connection on a manifold M. It is locally homogeneous, if for each two points $x, y \in M$ there exists a neighborhood $\mathcal{U}$ of x, a neighborhood $\mathcal{V}$ of y and an affine transformation $\varphi : \mathcal{U} \to \mathcal{V}$ such that $\varphi(x) = y$. It means that φ is a (local) diffeomorphism such that

$$\nabla_{\varphi_* X}\varphi_* Y = \varphi_*(\nabla_X Y)$$

holds for every vector fields X, Y defined in $\mathcal{U}$.

Now, let us recall the following criterion describing affine Killing vector fields. An affine Killing vector field X is characterized by the equation:

$$[X, \nabla_Y Z] - \nabla_Y[X, Z] - \nabla_{[X,Y]}Z = 0 \tag{1}$$

which has to be satisfied for arbitrary vector fields Y, Z (see Proposition 2.2 in Chapter VI of [4]). The following assertion is standard:

4 Proposition. *A smooth connection ∇ on M is locally homogeneous if and only if it admits, in a neighborhood of each point $p \in M$, at least two linearly independent affine Killing vector fields.*

¿From now on, we assume that M is 2-dimensional. We choose a fixed coordinate domain $\mathcal{U}(u, v) \subset M$ and we express a vector field X in the form $X = a(u, v)\partial_u + b(u, v)\partial_v$. Then, for a connection ∇ with arbitrary torsion in $\mathcal{U}(u, v)$, we put

$$\begin{aligned} \nabla_{\partial_u}\partial_u &= A(u,v)\partial_u + B(u,v)\partial_v, & \nabla_{\partial_u}\partial_v &= C(u,v)\partial_u + D(u,v)\partial_v, \\ \nabla_{\partial_v}\partial_u &= E(u,v)\partial_u + F(u,v)\partial_v, & \nabla_{\partial_v}\partial_v &= G(u,v)\partial_u + H(u,v)\partial_v. \end{aligned} \tag{2}$$

In the following, we will often denote the functions $a(u,v)$, $b(u,v)$, $A(u,v)$, $B(u,v)$, $C(u,v)$, $D(u,v)$, $E(u,v)$, $F(u,v)$, $G(u,v)$, $H(u,v)$ by a, b, A, B, C, D, E, F, G, H respectively, if there is no risk of confusion.

Writing the formula (1) in local coordinates, we find that any affine Killing vector field X must satisfy eight basic equations. We shall write these equations in the simplified notation:

$$\begin{array}{ll}
1) & a_{uu} + Aa_u - Ba_v + (C+E)b_u + A_u a + A_v b = 0,\\
2) & b_{uu} + 2Ba_u + (F+D-A)b_u - Bb_v + B_u a + B_v b = 0,\\
3) & a_{uv} + (A-D)a_v + Gb_u + Cb_v + C_u a + C_v b = 0,\\
4) & b_{uv} + Da_u + Ba_v + (H-C)b_u + D_u a + D_v b = 0,\\
5) & a_{uv} + (A-F)a_v + Gb_u + Eb_v + E_u a + E_v b = 0,\\
6) & b_{uv} + Fa_u + Ba_v + (H-E)b_u + F_u a + F_v b = 0,\\
7) & a_{vv} - Ga_u + (C+E-H)a_v + 2Gb_v + G_u a + G_v b = 0,\\
8) & b_{vv} + (D+F)a_v - Gb_u + Hb_v + H_u a + H_v b = 0.
\end{array} \tag{3}$$

Moreover, after some direct calculations, we obtain the following formulas for the Ricci tensor:

$$\begin{aligned}
Ric(\partial_u,\partial_u) &= B_v - F_u + B(H-E) + F(A-D),\\
Ric(\partial_u,\partial_v) &= D_v - H_u + CF - BG,\\
Ric(\partial_v,\partial_u) &= E_u - A_v + CF - BG,\\
Ric(\partial_v,\partial_v) &= G_u - C_v + C(H-E) + G(A-D).
\end{aligned} \tag{4}$$

Furthermore, we obtain the following equations for vanishing of the covariant derivative of the torsion tensor by a straightforward computation.

$$\begin{aligned}
&DE - CF + C_u - E_u = 0,\\
&(C-E)B + A(F-D) + D_u - F_u = 0,\\
&(C-E)H + G(F-D) - C_v + E_v = 0,\\
&DE - CF - D_v + F_v = 0.
\end{aligned} \tag{5}$$

Now we are going to recall some results obtained in [1] to make our exposition self-contained.

In [1] the authors have classified all locally homogeneous affine connections with arbitrary torsion in the plane domains from the group-theoretical point of view. This means that they always started with a specific *transitive* Lie algebra $\mathfrak{g}$ of vector fields from the list of P. J. Olver [9] and they were looking for all affine connections with arbitrary torsion for which, in the same domain and with respect to the same local coordinates, $\mathfrak{g}$ is the *full* algebra of affine Killing vector fields. Such connections are called *corresponding* to $\mathfrak{g}$. It happened quite often that the given Lie algebra of vector fields did not admit any invariant affine connection or it only admitted torsion-less invariant affine connections.

Finally, they proved some simple algebraic lemmas which enabled to decide very easily if a connection corresponding to a Lie algebra $\mathfrak{g}$ had, in some local coordinate system (u', v'), Christoffel symbols of type A, or of type B, respectively. In such a case it said shortly that such a connection is of type A, or of type B, respectively. Due to those lemmas, the whole procedure depend only on the structure of the algebra $\mathfrak{g}$.

Based on the computations, they illustrated the essential relationship between the classifications given in [7] and [10]. Moreover, they proved that, for some Lie algebras $\mathfrak{g}$, all connections corresponding to such a $\mathfrak{g}$ are simultaneously of type A and of type B. These facts can be easily checked in the table that they used to summarize their results. Now, we shall reproduce it to make easier the understanding of the main result's proof.

This table is a refinement of the tables 1 and 6 from [9] completed by additional information. In each case, or subcase, they got a Lie algebra of vector fields given by its generators. They were looking for all locally homogeneous connections which (in the same local coordinates) are *corresponding* to a Lie algebra in question. Moreover, T denotes the torsion tensor, "VCS" means that all Christoffel symbols vanish with respect to the *given* coordinates (u, v) and the label "flat" means that the Ricci tensor vanishes. In the column $T \neq 0$, "in general" means that the torsion tensor is different from zero except some special cases.

Properties of connections associated with the (refined) Olver list.

Case	***Generators***	***Remarks***	***Type A***	***Type B***	***T≠ 0***
1.1	∂_v, $v\partial_v - u\partial_u$, $v^2\partial_v - 2uv\partial_u$.		No	Yes	In general
1.2	∂_v, $v\partial_v - u\partial_u$, $v^2\partial_v - (2uv+1)\partial_u$.	∇ is the Levi-Civita connection of a Lorentzian metric with constant curvature.	No	Yes	Never
1.3	∂_v, $v\partial_v$, $u\partial_u$, $v^2\partial_v - uv\partial_u$.		Yes (Flat)	Yes	Never

Case	***Generators***	***Remarks***	***Type A***	***Type B***	$\boldsymbol{T\neq 0}$
1.4	∂_v, $v\partial_v$, $v^2\partial_v$, ∂_u, $u\partial_u$, $u^2\partial_u$.	No corresponding invariant affine connection.			
1.5	∂_v, $\eta_1(v)\partial_u, \dots, \eta_k(v)\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	The functions $\eta_1(v)$,..., $\eta_k(v)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u] = 0$.			
1.5 a)	∂_u, ∂_v.	$k = 1$	Yes	No	In general
1.5 b)	∂_v, $\mathrm{e}^v\partial_u$.	$k = 1$	No	Yes	In general
1.5 c)	∂_u, ∂_v, $\mathrm{e}^v\partial_u$.	$k = 2$	Yes	Yes	In general
1.5 d)	∂_u, ∂_v, $v\partial_u$.	$k = 2$	Yes	No	In general
1.5 e)	∂_v, $\mathrm{e}^{\alpha v}\partial_u$, $\mathrm{e}^{\beta v}\partial_u$, $\alpha \neq \beta$, $\alpha, \beta \neq 0$.	This case becomes equivalent to the case 1.6 e')			
1.5 f)	∂_v, $\mathrm{e}^{\alpha v}\partial_u$, $v\mathrm{e}^{\alpha v}\partial_u$, $\alpha \neq 0$.	This case becomes equivalent to the case 1.6 f')			
1.5 g)	$\mathrm{e}^{\alpha v}\cos(\beta v)\partial_u$, $\mathrm{e}^{\alpha v}\sin(\beta v)\partial_u$, ∂_v, $\beta \neq 0$.	This case becomes equivalent to the case 1.6 g')			
1.5 h)	$\eta_1(v)\partial_u, \dots,$ $\eta_k(v)\partial_u$, ∂_v, $k > 2$.	No corresponding invariant affine connection.			
1.6	∂_v, $u\partial_u$, $\eta_1(v)\partial_u, \dots, \eta_k(v)\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.	The functions $\eta_1(v)$,..., $\eta_k(v)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u] = 0$.			
1.6 a')	∂_u, ∂_v $u\partial_u$.	$k = 1$	Yes	No	In general
1.6 b')	∂_v, $\mathrm{e}^v\partial_u$ $u\partial_u$.	$k = 1$	Yes	Yes	In general
1.6 c')	∂_u, ∂_v, $\mathrm{e}^v\partial_u$, $u\partial_u$.	$k = 2$	Yes	Yes	In general

Case	*Generators*	*Remarks*	*Type A*	*Type B*	$T\neq 0$
1.6 d')	∂_u, ∂_v, $v\partial_u$, $u\partial_u$.	$k=2$	Yes	No	In general
1.6 e')	∂_v, $\mathrm{e}^{\alpha v}\partial_u$, $\mathrm{e}^{\beta v}\partial_u$, $u\partial_u$, $\alpha\neq\beta$, $\alpha,\beta\neq 0$.	$k=2$	Yes	Yes	In general
1.6 f')	∂_v, $\mathrm{e}^{\alpha v}\partial_u$, $v\mathrm{e}^{\alpha v}\partial_u$, $u\partial_u$, $\alpha\neq 0$.	$k=2$	Yes	Yes	In general
1.6 g')	$\mathrm{e}^{\alpha v}\cos(\beta v)\partial_u$, $\mathrm{e}^{\alpha v}\sin(\beta v)\partial_u$, ∂_v, $u\partial_u$, $\beta\neq 0$.	$k=2$	Yes	No	In general
1.6 h')	$\eta_1(v)\partial_u,\ldots,$ $\eta_k(v)\partial_u$, ∂_v, $u\partial_u$, $k>2$.	No corresponding invariant affine connection.			
1.7	∂_u, ∂_v, $v\partial_v+\alpha u\partial_u$, $v\partial_u,\ldots,$ $v^{k-1}\partial_u$, $k\in\mathbb{Z}$, $k\geq 1$.	$k=1$, $\alpha=0$	Yes	Yes	In general
		$k=1$, $\alpha=1/2,2$ or $k=2$, $\alpha=2$	Yes (Flat)	Yes	Never
		$k=1$, $\alpha\neq 0,1/2,2$ or $k=2$, $\alpha\neq 2$	Yes (VCS)	Yes	Never
		$k>2$	No corresponding invariant affine connection.		
1.8	∂_u, ∂_v, $v\partial_v+(ku+v^k)\partial_u$, $v\partial_u,\ldots,$ $v^{k-1}\partial_u$, $k\in\mathbb{Z}$, $k\geq 1$.	$k=1$	Yes (VCS)	Yes	Never
		$k\geq 2$	No corresponding invariant affine connection.		
1.9	∂_u, ∂_v, $v\partial_v$, $u\partial_u$, $v\partial_u,\ldots,$ $v^{k-1}\partial_u$, $k\in\mathbb{Z}$, $k\geq 1$.	$k=1,2$	Yes (VCS)	Yes	Never
		$k>2$	No corresponding invariant affine connection.		

Case	***Generators***	***Remarks***	***Type A***	***Type B***	$\boldsymbol{T \neq 0}$
1.10	∂_v, ∂_u, $v\partial_u$, …, $v^{k-1}\partial_u$, $2v\partial_v + (k-1)u\partial_u$, $v^2\partial_v + (k-1)uv\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.		No corresponding invariant affine connection.		
1.11	∂_v, ∂_u, $v\partial_u$, …, $v^{k-1}\partial_u$, $v\partial_v$, $u\partial_u$, $v^2\partial_v + (k-1)uv\partial_u$, $k \in \mathbb{Z}$, $k \geq 1$.		No corresponding invariant affine connection.		
2.1	∂_v, ∂_u, $\alpha(v\partial_v + u\partial_u)$ $+u\partial_v - v\partial_u$.		Yes (VCS)	Yes	Never
2.2	∂_v, $v\partial_v + u\partial_u$, $(v^2 - u^2)\partial_v + 2uv\partial_u$.	∇ is the Levi-Civita Connection of the hyperbolic plane.	No	Yes	Never
2.3	$u\partial_v - v\partial_u$, $(1 + v^2 - u^2)\partial_v + 2uv\partial_u$, $2uv\partial_v + (1 - v^2 + u^2)\partial_u$.	∇ is the Levi-Civita Connection of the sphere.	No	No	Never
2.4	∂_v, ∂_u, $v\partial_v + u\partial_u$, $u\partial_v - v\partial_u$.		Yes (VCS)	Yes	Never
2.5	∂_v, ∂_u, $v\partial_v - u\partial_u$, $u\partial_v$, $v\partial_u$.		Yes (VCS)	Yes	Never
2.6	∂_v, ∂_u, $v\partial_v$, $u\partial_u$, $u\partial_v$, $v\partial_u$.		Yes (VCS)	Yes	Never
2.7	∂_v, ∂_u, $v\partial_v + u\partial_u$, $u\partial_v - v\partial_u$, $(v^2 - u^2)\partial_v + 2uv\partial_u$, $2uv\partial_v + (u^2 - v^2)\partial_u$.		No corresponding invariant affine connection.		
2.8	∂_v, ∂_u, $v\partial_v$, $u\partial_u$, $u\partial_v$, $v\partial_u$, $v^2\partial_v + uv\partial_u$, $uv\partial_v + u^2\partial_u$.		No corresponding invariant affine connection.		

Finally, we only summarize in detail the information obtained in [1] about the cases 1.1, 1.5 b), 1.5 c) and 1.6 e') of the refined Olver list.

5 Proposition. *The Lie algebras from the cases 1.1, 1.5 b), 1.5 c) and 1.6 e') of the refined Olver list produce just the following homogeneous connections:*

a) For Case 1.1, the Christoffel symbols are given by

$$\begin{aligned} A(u) = \frac{-1}{2u}, \quad B(u) = 0, \quad C(u) = cu, \quad D(u) = \frac{1}{2u}, \\ E(u) = eu, \quad F(u) = \frac{1}{2u}, \quad G(u) = gu^3, \quad H(u) = (c+e)u, \end{aligned} \tag{6}$$

with three arbitrary parameters c, e, g. Here the torsion tensor, T, is not zero if and only if $c \neq e$ and the Ricci tensor, Ric, is zero if and only if $g = -2e^2$, $c = -2e$.

b) For Case 1.5 b), the Christoffel symbols are given by

$$\begin{aligned} A(u) = C_1u + C_2, \quad B(u) = C_1, \quad H(u) = C_1u^2 - (C_3 + C_5)u + C_7, \\ C(u) = -\,C_1u^2 + (C_3 - C_2)u + C_4, \quad D(u) = -C_1u + C_3, \\ E(u) = -\,C_1u^2 + (C_5 - C_2)u + C_6, \quad F(u) = -C_1u + C_5, \\ G(u) = C_1u^3 + (C_2 - C_3 - C_5)u^2 + (C_7 - C_4 - C_6 - 1)u + C_8, \end{aligned} \tag{7}$$

where $C_1, \ldots, C_8$ are constant parameters. Moreover $T \neq 0$ if and only if $C_3 \neq C_5$ or $C_4 \neq C_6$.

c) For Case 1.5 c), the Christoffel symbols are given by

$$\begin{aligned} A = B = D = F = 0, \quad C(u) = 2c_1, \quad E(u) = 2c_2, \\ H(u) = 1 + 2(c_1 + c_2), \quad G(u) = c_3, \end{aligned} \tag{8}$$

where c_1, c_2, c_3 are constants. Moreover $T \neq 0$ if and only if $c_1 \neq c_2$.

d) For Case 1.6 c'), the Christoffel symbols are given by

$$\begin{aligned} A = B = D = F = 0, \quad C(u) = -\alpha - \beta + 2(c_3 - c_2), \\ E(u) = 2c_2, \quad H(u) = 2c_3, \quad G(u) = \alpha\beta u, \end{aligned} \tag{9}$$

where c_2, c_3, $\alpha \neq 0$, $\beta \neq 0$ are constants and $\alpha \neq \beta$. Moreover, $T \neq 0$ if and only if $2c_3 - 4c_2 - \alpha - \beta \neq 0$.

3 Proof of the main result

The only cases from the refined Olver list such that the torsion tensor T could be nonzero and the connections corresponding to the Lie algebra are not of type A, are the cases 1.1 and 1.5 b).

In the particular case a) of Proposition 5 (Case 1.1 of the refined Olver list), the conditions $T = 0$ and $\nabla T = 0$ are equivalent. This is a straightforward computation using (6) and (5). Thus, this case is in contradiction with the assumptions of Theorem 3 and we can omit it.

Now, it remains to start from Case 1.5 b) of the refined Olver list. According to Proposition 5, part b), all such connections are described by the formula (7). Because we shall put additional geometric conditions on these connections, we must admit also occurrence of *non-corresponding* connections to Case 1.5 b) given by the formula (7). What we are going to show is that the connections satisfying assumptions of Theorem 3 are *corresponding* either to Case 1.5 c) or to Case 1.6 e'). In both cases, the corresponding Lie algebras give connections of type A.

First, we shall found the family of connections such that $T \neq 0$, $\nabla T = 0$ and $Ric = 0$. From Proposition 5 b) we know that $T \neq 0$ if and only if $C_3 \neq C_5$ or $C_4 \neq C_6$. From (7) and (5) we know that $\nabla T = 0$ if and only if

$$\begin{aligned} &i) C_3C_6 - C_4C_5 = 0,\\ &ii) C_2(-C_3 + C_5) + C_1(C_4 - C_6) = 0,\\ &iii)(C_3 - C_5 + 2(C_3C_6 - C_4C_5)) = 0,\\ &iv)(C_4 - C_6)C_7 + (C_5 - C_3)C_8 = 0. \end{aligned} \tag{10}$$

From (7) and (4) we know that $Ric = 0$ if and only if

$$\begin{aligned} &v)(C_2 - C_3)C_5 + C_1(1 - C_6 + C_7) = 0,\\ &vi) C_3 + C_5 + C_4C_5 - C_1C_8 = 0,\\ &vii) - C_2 + C_5 + C_4C_5 - C_1C_8 = 0,\\ &viii) - 1 - C_4 - C_6 - C_4C_6 + C_7 + C_4C_7 + C_2C_8 - C_3C_8 = 0. \end{aligned} \tag{11}$$

Substituting $i)$ in $iii)$ we obtain $C_3 = C_5$. Then, the condition $\nabla T = 0$ and the equation $i)$ give $C_4 \neq C_6$ and $C_3 = C_5 = 0$. Consequently, from $ii)$, $iv)$ and $vii)$ we obtain $C_1 = C_2 = C_7 = 0$. Now, the equation $viii)$ gives $C_4 = -1$ (or, equivalently, $C_6 = -1$). Therefore, we obtain the following subfamily of connections:

$$\begin{aligned} A(u) = B(u) = D(u) = F(u) = H(u) = 0,\\ C(u) = -1, \quad E(u) = C_6, \quad G(u) = -C_6u + C_8, \end{aligned} \tag{12}$$

where C_6, C_8 are constant parameters and $C_6 \neq -1$.

If now $C_6 = 0$, it is clear that the Christoffel symbols are the same as in the formula (8), with the additional conditions $c_1 = -1/2$, $c_2 = 0$, which is a special subcase of the case c) in Proposition 5, i.e., a special subcase of Case 1.5 c) of the refined Olver list. These connections are of type A.

Finally, if $C_6 \neq 0$ we change the coordinate system in the following way: $u' = u - \frac{C_8}{C_6}$, $v' = v$. Then, in the new coordinate system, which we denote again by (u, v), we have still the same generators $\{\partial_v, \mathrm{e}^v\partial_u\}$. Moreover, the Christoffel symbols (12) get the new form

$$\begin{aligned} A(u) = B(u) = D(u) = F(u) = H(u) = 0, \\ C(u) = -1, \quad E(u) = C_6, \quad G(u) = -C_6 u, \end{aligned} \tag{13}$$

where $C_6 \in \mathbb{R} \setminus \{0, -1\}$. Now, it is easy to check from (3) that the connection given by (13) has two additional Killing vector fields $u\partial_u$ and $\mathrm{e}^{-vC_6}\partial_u$. Thus we found the equivalence with Case 1.6 e') of the refined Olver list. This concludes the proof.

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