Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **37** (2017) suppl. 1, 161–186. 10.1112/blms/23.4.372

Inequalities for algebraic Casorati curvatures and their applications

Mukut Mani Tripathi[i]
Department of Mathematics
Institute of Science
Banaras Hindu University
Email: mmtripathi66@yahoo.com

Received: 20-07-2016; accepted: 17-10-2016.

Abstract. The notion of different kind of algebraic Casorati curvatures are introduced. Some results expressing basic Casorati inequalities for algebraic Casorati curvatures are presented. Equality cases are also discussed. As a simple application, basic Casorati inequalities for different δ-Casorati curvatures for Riemannian submanifolds are presented. Further applying these results, Casorati inequalities for Riemannian submanifolds of real space forms are obtained. Finally, some problems are presented for further studies.

Keywords: Casorati curvature, algebraic Casorati curvature, Casorati inequalities, real space form

MSC 2010 classification: 53B20

Introduction

Felice Casorati was one of the great Italian mathematicians best known for the Casorati-Weierstrass theorem in complex analysis. He was born in Pavia on December 17, 1835 and his soul departed on September 11, 1890 in Casteggio. Before his departure, in 1889, Casorati [8] defined a curvature for a regular surface in Euclidean 3-space which turns out to be the normalized sum of the squared principal curvatures. In [9], the author says that he could not check the paper [8] before printing, and advices readers to rather use a subsequent paper [10]. This curvature is now well known as the Casorati curvature. Several geometers believe that Casorati preferred this curvature over the traditional Gaussian curvature because the Casorati curvature vanishes for a surface in Euclidean 3-space if and only if both Euler normal curvatures (or principal curvatures) of the surface vanish simultaneously and thus corresponds better with the common intuition of curvature. For a hypersurface of a Riemannian manifold the Casorati curvature is defined to be the normalized sum of the squared principal normal curvatures of the hypersurface, and in general, the

[i]This work is dedicated to Felice Casorati

Casorati curvature of a submanifold of a Riemannian manifold is defined to be the normalized squared length of the second fundamental form [22]. Geometrical meaning and the importance of the Casorati curvature, discussed by several geometers, can be visualized in several research/survey papers including [19], [23], [24], [28], [30], [33], [34], [46], [56] and [57].

The paper is organized as follows. In section 1, some preliminaries regarding curvature like tensors are presented. In section 2, given an n-dimensional Riemannian manifold (M, g), a Riemannian vector bundle (B, g_B) over M, a B-valued symmetric $(1, 2)$-tensor field ζ and a (curvature-like) tensor field T satisfying the algebraic Gauss equation, we introduce the notion of different kind of algebraic Casorati curvatures $\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(n-1)$, $\delta_{\mathcal{C}^{T,\zeta}}(n-1)$, $\delta_{\mathcal{C}^{T,\zeta}}(r; n-1)$, $\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(r; n-1)$, which in special cases of Riemannian submanifolds reduce to already known δ-Casorati curvatures. In section 3, first we prove a useful Lemma regarding a constrained extremum problem. Then we present results expressing basic Casorati inequalities for algebraic Casorati curvatures. Equality cases are also discussed. After this, application parts begin. In section 4, we obtain basic Casorati inequalities for Casorati curvatures $\delta(r; n-1)$, $\widehat{\delta}(r; n-1)$, $\delta(n-1)$, $\widehat{\delta}(n-1)$ for Riemannian submanifolds. In section 5, we further apply these results to obtain Casorati inequalities for Riemannian submanifolds of real space forms with very short proofs. Finally, in section 6, we present some problems for further studies.

1 Curvature like tensor

In 1967, R.S. Kulkarni introduced the notion of a *curvature structure* (cf. [35, §8 of Chapter 1], [36]), which is now widely known as a *curvature-like tensor* (*field*). Let (M, g) be an n-dimensional Riemannian manifold. Let T be a curvature-like tensor so that it satisfies the following properties

$$T(X, Y, Z, W) = -T(Y, X, Z, W), \tag{1}$$

$$T(X, Y, Z, W) = T(Z, W, X, Y), \tag{2}$$

$$T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W) = 0 \tag{3}$$

for all vector fields X, Y, Z and W on M. For a curvature-like tensor field T, the T-*sectional curvature* associated with a 2-plane section Π_2 spanned by orthonormal vectors X and Y at $p \in M$, is given by [6]

$$K_T(\Pi_2) = K_T(X \wedge Y) = T(X, Y, Y, X).$$

Let $\{e_1, e_2, \ldots, e_n\}$ be any orthonormal basis of T_pM. The *T-Ricci tensor* S_T is defined by

$$S_T(X,Y) = \sum_{j=1}^{n} T\left(e_j, X, Y, e_j\right), \qquad X, Y \in T_pM.$$

The *T-Ricci curvature* is given by

$$\mathrm{Ric}_T(X) = S_T(X,X), \qquad X \in T_pM.$$

The *T-scalar curvature* is given by [6]

$$\tau_T(p) = \sum_{1\le i<j\le n} T\left(e_i, e_j, e_j, e_i\right). \tag{4}$$

Now, let Π_k be a k-plane section of T_pM and X a unit vector in Π_k. If $k = n$ then $\Pi_n = T_pM$; and if $k = 2$ then Π_2 is a plane section of T_pM. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of Π_k. Then we define the *T-k-Ricci curvature* of Π_k at e_i, $i \in \{1, \ldots, k\}$, denoted $(\mathrm{Ric}_T)_{\Pi_k}(e_i)$, by

$$(\mathrm{Ric}_T)_{\Pi_k}(e_i) = \sum_{j=1, j\ne i}^{k} K_T(e_i \wedge e_j). \tag{5}$$

We note that a *T-n-Ricci curvature* $(\mathrm{Ric}_T)_{T_pM}(e_i)$ is the usual *T-Ricci curvature* of e_i, denoted $\mathrm{Ric}_T(e_i)$. The *T-k-scalar curvature* $\tau_T(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau_T(\Pi_k) = \sum_{1\le i<j\le k} K_T(e_i \wedge e_j). \tag{6}$$

We note that

$$\tau_T(\Pi_k) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1, j\ne i}^{k} K_T(e_i \wedge e_j) = \frac{1}{2}\sum_{i=1}^{k}(\mathrm{Ric}_T)_{\Pi_k}(e_i). \tag{7}$$

The T-scalar curvature of M at p is identical with the T-n-scalar curvature of the tangent space T_pM of M at p, that is, $\tau_T(p) = \tau_T(T_pM)$. If Π_2 is a 2-plane section, $\tau_T(\Pi_2)$ is nothing but the T-sectional curvature $K_T(\Pi_2)$ of Π_2. The *T-k-normalized scalar curvature* of a k-plane section Π_k at p is defined as

$$(\tau_T)_{\mathrm{Nor}}(\Pi_k) = \frac{2}{k(k-1)}\,\tau_T(\Pi_k).$$

The *T-normalized scalar curvature* at p is defined as

$$(\tau_T)_{\mathrm{Nor}}(p) = (\tau_T)_{\mathrm{Nor}}(T_pM) = \frac{2}{n(n-1)}\,\tau_T(p).$$

If T is replaced by the Riemann curvature tensor R, then T-sectional curvature K_T, T-Ricci tensor S_T, T-Ricci curvature Ric_T, T-scalar curvature τ_T, T-normalized scalar curvature $(\tau_T)_{\mathrm{Nor}}$, T-k-Ricci curvature $(\mathrm{Ric}_T)_{\Pi_k}$, T-k-scalar curvature $\tau_T(\Pi_k)$, T-k-normalized scalar curvature $(\tau_T)_{\mathrm{Nor}}(\Pi_k)$ and T-normalized scalar curvature $(\tau_T)_{\mathrm{Nor}}$ become the sectional curvature K, the Ricci tensor S, the Ricci curvature Ric, the scalar curvature τ, the normalized scalar curvature τ_{Nor}, k-Ricci curvature Ric_{Π_k}, k-scalar curvature $\tau(\Pi_k)$, k-normalized scalar curvature $\tau_{\mathrm{Nor}}(\Pi_k)$ and normalized scalar curvature τ_{Nor}, respectively.

2 Algebraic Casorati curvatures

Let (M, g) be an n-dimensional submanifold of an m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. The equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) \;\; &= \;\; \widetilde{R}(X,Y,Z,W) + \widetilde{g}\left(\sigma(Y,Z),\sigma(X,W)\right) \\ &\quad - \widetilde{g}\left(\sigma(X,Z),\sigma(Y,W)\right) \end{aligned} \tag{8}$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M, respectively and σ is the second fundamental form of the immersion of M in $\widetilde{M}$. The Ricci-Kühn equation is given by

$$R^{\perp}(X,Y,N,V) = \widetilde{R}(X,Y,N,V) + g\left([A_N, A_V]\, X, Y\right) \tag{9}$$

for all $X, Y \in TM$ and for all $N, V \in T^{\perp}M$, where

$$R^{\perp}(X,Y)N = \nabla^{\perp}_X \nabla^{\perp}_Y N - \nabla^{\perp}_Y \nabla^{\perp}_X N - \nabla^{\perp}_{[X,Y]} N,$$

$$[A_N, A_V] = A_N A_V - A_V A_N,$$

with $\nabla^{\perp}$ being the induced normal connection in the normal bundle $T^{\perp}M$ and A_N being the shape operator in the direction N.

Let M be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$. A point $p \in M$ is said to be an *invariantly quasi-umbilical point* if there exist $m-n$ mutually orthogonal unit normal vectors $N_{n+1}, \ldots, N_m$ such that the shape operators with respect to all directions N_α have an eigenvalue of multiplicity $n-1$ and that for each N_α the distinguished eigendirection is the same. The submanifold is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point. For details, we refer to [4].

Let (M, g) be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_α belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^\perp M$. We let

$$\sigma_{ij}^\alpha = \widetilde{g}\left(\sigma\left(e_i, e_j\right), e_\alpha\right), \quad i, j \in \{1, \ldots, n\}, \quad \alpha \in \{n+1, \ldots, m\}.$$

Then, the squared mean curvature of the submanifold M in $\widetilde{M}$ is defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} \sigma_{ii}^\alpha \right)^2,$$

and the squared norm of second fundamental form σ is

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} \widetilde{g}\left(\sigma\left(e_i, e_j\right), \sigma\left(e_i, e_j\right)\right).$$

Let K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$, respectively. In view of (8), we have

$$K_{ij} = \widetilde{K}_{ij} + \sum_{\alpha=n+1}^{m} \left(\sigma_{ii}^\alpha \sigma_{jj}^\alpha - (\sigma_{ij}^\alpha)^2 \right). \tag{10}$$

From (10) it follows that

$$2\tau(p) = 2\widetilde{\tau}\left(T_pM\right) + n^2 \|H\|^2 - \|\sigma\|^2, \tag{11}$$

where

$$\widetilde{\tau}\left(T_pM\right) = \sum_{1 \leq i < j \leq n} \widetilde{K}_{ij}$$

denotes the scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$. From (11), it immediately follows that

$$\tau_{\mathrm{Nor}}(p) = \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right) + \frac{n}{n-1} \|H\|^2 - \frac{1}{n(n-1)} \|\sigma\|^2, \tag{12}$$

where

$$\tau_{\mathrm{Nor}}(p) = \frac{2\tau(p)}{n(n-1)}, \qquad \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right) = \frac{2\widetilde{\tau}\left(T_pM\right)}{n(n-1)}. \tag{13}$$

The *Casorati curvature* $\mathcal{C}$ [22] of the Riemannian submanifold M is defined to be the normalized squared length of the second fundamental form σ, that is,

$$\mathcal{C} = \frac{1}{n} \|\sigma\|^2 = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \left(\sigma_{ij}^\alpha\right)^2. \tag{14}$$

For a k-dimensional subspace Π_k of T_pM, $k \geq 2$ spanned by $\{e_1, \ldots, e_k\}$, the Casorati curvature $\mathcal{C}(\Pi_k)$ of the subspace Π_k is defined to be [21]

$$\mathcal{C}(\Pi_k) = \frac{1}{k} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{k} \left(\sigma_{ij}^{\alpha}\right)^2 .$$

The *normalized δ-Casorati curvatures* $\widehat{\delta}_{\mathcal{C}}(n-1)$, $\delta'_{\mathcal{C}}(n-1)$ of a Riemannian submanifold M are given by [21]

$$[\widehat{\delta}_{\mathcal{C}}(n-1)]_p = 2\,\mathcal{C}_p - \frac{2n-1}{2n} \sup\left\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \tag{15}$$

$$[\delta'_{\mathcal{C}}(n-1)]_p = \frac{1}{2}\,\mathcal{C}_p + \frac{n+1}{2n(n-1)} \inf\left\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}. \tag{16}$$

In [21], the authors denoted $\delta'_{\mathcal{C}}(n-1)$ by $\delta_{\mathcal{C}}(n-1)$. The (modified) *normalized δ-Casorati curvatures* $\delta_{\mathcal{C}}(n-1)$ of the Riemannian submanifold M is given by ([39], [63])

$$[\delta_{\mathcal{C}}(n-1)]_p = \frac{1}{2}\,\mathcal{C}_p + \frac{n+1}{2n} \inf\left\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}. \tag{17}$$

It should be noted that the normalized δ-Casorati curvatures $\widehat{\delta}_{\mathcal{C}}(n-1)$, $\delta'_{\mathcal{C}}(n-1)$ and $\delta_{\mathcal{C}}(n-1)$ vanish trivially for $n = 2$ [63]. In [39], the authors pointed out that the coefficient $\frac{n+1}{2n(n-1)}$ in (16) was inappropriate and therefore they modified the coefficient from $\frac{n+1}{2n(n-1)}$ to $\frac{n+1}{2n}$ in the definition of $\delta'_{\mathcal{C}}(n-1)$ to obtain the definition of $\delta_{\mathcal{C}}(n-1)$ (see also [40]). For a positive real number $r \neq n(n-1)$, letting

$$\mathrm{a}(r) = \frac{1}{nr}(n-1)(n+r)\left(n^2 - n - r\right),$$

the *normalized δ-Casorati curvatures* $\delta_{\mathcal{C}}(r; n-1)$ and $\widehat{\delta}_{\mathcal{C}}(r; n-1)$ of a Riemannian submanifold M are given by [22]

$$[\delta_{\mathcal{C}}(r; n-1)]_p = r\,\mathcal{C}_p + \mathrm{a}(r) \inf\left\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \tag{18}$$

if $0 < r < n(n-1)$, and

$$[\widehat{\delta}_{\mathcal{C}}(r; n-1)]_p = r\,\mathcal{C}_p + \mathrm{a}(r) \sup\left\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \tag{19}$$

if $n(n-1) < r$, respectively.

In [39] the normalized δ-Casorati curvatures $\widehat{\delta}_{\mathcal{C}}(r;n-1)$ and $\delta_{\mathcal{C}}(r;n-1)$ are called as the generalized normalized δ-Casorati curvatures $\widehat{\delta}_{\mathcal{C}}(r;n-1)$ and $\delta_{\mathcal{C}}(r;n-1)$, respectively. We see that [40]

$$[\delta_{\mathcal{C}}(n-1)]_p = \frac{1}{n(n-1)}\left[\delta_{\mathcal{C}}\left(\frac{n(n-1)}{2};n-1\right)\right]_p, \tag{20}$$

$$\left[\widehat{\delta}_{\mathcal{C}}(n-1)\right]_p = \frac{1}{n(n-1)}\left[\widehat{\delta}_{\mathcal{C}}\left(2n(n-1);n-1\right)\right]_p \tag{21}$$

for all $p \in M$.

Let (M,g) be an n-dimensional Riemannian manifold and (B,g_B) a Riemannian vector bundle over M. If ζ is a B-valued symmetric $(1,2)$-tensor field and T a $(0,4)$-tensor field on M such that

$$T(X,Y,Z,W) = g_B(\zeta(X,W),\zeta(Y,Z)) - g_B(\zeta(X,Z),\zeta(Y,W)) \tag{22}$$

for all vector fields X,Y,Z,W on M, then the equation (22) is said to be an *algebraic Gauss equation* [15]. Every $(0,4)$-tensor field T on M, which satisfies (22), becomes a curvature-like tensor.

A typical example of an algebraic Gauss equation is given for a submanifold M of an Euclidean space, if B is the normal bundle, ζ the second fundamental form and T the curvature tensor. Some nice situations, in which such T and ζ satisfying an algebraic Gauss equation exist, are Lagrangian and Kaehlerian slant submanifolds of complex space forms and C-totally real submanifolds of Sasakian space forms.

Now, let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of the tangent space T_pM and e_α belong to an orthonormal basis $\{e_{n+1},\ldots,e_m\}$ of the Riemannian vector bundle (B,g_B) over M at p. We put

$$\zeta_{ij}^{\alpha} = g_B\left(\zeta\left(e_i,e_j\right),e_\alpha\right), \quad \|\zeta\|^2 = \sum_{i,j=1}^{n} g_B\left(\zeta\left(e_i,e_j\right),\zeta\left(e_i,e_j\right)\right),$$

$$\operatorname{trace}\zeta = \sum_{i=1}^{n}\zeta\left(e_i,e_i\right), \quad \|\operatorname{trace}\zeta\|^2 = g_B(\operatorname{trace}\zeta,\operatorname{trace}\zeta).$$

Motivated by the definitions given in [21], [22] and [39] we give the following definitions.

Definition 1. Let (M,g) be an n-dimensional Riemannian manifold, (B,g_B) a Riemannian vector bundle over M, ζ a B-valued symmetric $(1,2)$-tensor field on M, and T a curvature-like tensor field satisfying the algebraic Gauss equation (22). Then the *algebraic Casorati curvature* $\mathcal{C}^{T,\zeta}$ with respect to T and the Riemannian vector bundle (B, g_B) over M is defined to be

$$\mathcal{C}^{T,\zeta} = \frac{1}{n}\,\|\zeta\|^2 = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(\zeta_{ij}^{\alpha}\right)^2. \tag{23}$$

For a k-dimensional subspace Π_k of T_pM, $k \geq 2$, spanned by $\{e_1, \ldots, e_k\}$, the *algebraic Casorati curvature* $\mathcal{C}^{T,\zeta}(\Pi_k)$ of the subspace Π_k is defined to be

$$\mathcal{C}^{T,\zeta}(\Pi_k) = \frac{1}{k}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{k}\left(\zeta_{ij}^{\alpha}\right)^2. \tag{24}$$

We note that

$$\mathcal{C}_p^{T,\zeta} = \mathcal{C}^{T,\zeta}(T_pM), \qquad p \in M.$$

Definition 2. Let (M, g) be an n-dimensional Riemannian manifold, (B, g_B) a Riemannian vector bundle over M, ζ a B-valued symmetric $(1, 2)$-tensor field on M, and T a curvature-like tensor field satisfying the algebraic Gauss equation (22). Then we define the following *algebraic Casorati curvatures* $\delta_{\mathcal{C}^{T,\zeta}}(n-1)$ and $\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(n-1)$ by

$$\begin{aligned}&[\delta_{\mathcal{C}^{T,\zeta}}(n-1)]_p \\ =\;& \frac{1}{2}\,\mathcal{C}_p^{T,\zeta} + \frac{n+1}{2n}\inf\left\{\mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\},\end{aligned} \tag{25}$$

$$\begin{aligned}&[\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(n-1)]_p \\ =\;& 2\,\mathcal{C}_p^{T,\zeta} - \frac{2n-1}{2n}\sup\{\mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}.\end{aligned} \tag{26}$$

Definition 3. Let (M, g) be an n-dimensional Riemannian manifold, (B, g_B) a Riemannian vector bundle over M, ζ a B-valued symmetric $(1, 2)$-tensor field on M, and T a curvature-like tensor field satisfying the algebraic Gauss equation (22). For a positive real number $r \neq n(n-1)$, let

$$\mathrm{a}(r) = \frac{1}{nr}(n-1)\,(n+r)\left(n^2 - n - r\right)$$

and define the *algebraic Casorati curvatures* $\delta_{\mathcal{C}^{T,\zeta}}(r; n-1)$ and $\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(r; n-1)$ by

$$\begin{aligned} & [\delta_{\mathcal{C}^{T,\zeta}}(r; n-1)]_p \\ = \; & r\,\mathcal{C}_p^{T,\zeta} + \mathrm{a}(r) \inf \left\{ \mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM \right\} \end{aligned} \tag{27}$$

if $0 < r < n(n-1)$, and

$$\begin{aligned} & [\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(r; n-1)]_p \\ = \; & r\,\mathcal{C}_p^{T,\zeta} + \mathrm{a}(r) \sup \left\{ \mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM \right\} \end{aligned} \tag{28}$$

if $n(n-1) < r$.

Remark 1. Let (M, g) be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Let the Riemannian vector bundle (B, g_B) over M be replaced by the normal bundle $T^{\perp}M$, and the B-valued symmetric $(1,2)$-tensor field ζ be replaced by the second fundamental form of immersion σ. Then the algebraic Casorati curvature $\mathcal{C}^{T,\zeta}$ becomes the *Casorati curvature* $\mathcal{C}$ of the Riemannian submanifold M given by (14). The algebraic Casorati curvatures $\delta_{\mathcal{C}^{T,\zeta}}(n-1)$ and $\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(n-1)$ become *normalized δ-Casorati curvatures* $\delta_{\mathcal{C}}(n-1)$ and $\widehat{\delta}_{\mathcal{C}}(n-1)$ of the Riemannian submanifold M given by (17) and (15), respectively. Finally, algebraic Casorati curvatures $\delta_{\mathcal{C}^{T,\zeta}}(r; n-1)$ and $\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(r; n-1)$ become normalized δ-Casorati curvatures $\delta_{\mathcal{C}}(r; n-1)$ and $\widehat{\delta}_{\mathcal{C}}(r; n-1)$ of the Riemannian submanifold M given by (18) and (19), respectively.

Now, we present the following useful Lemma.

Lemma 1. *Let (M, g) be an n-dimensional Riemannian manifold, (B, g_B) a Riemannian vector bundle over M and ζ a B-valued symmetric $(1,2)$-tensor field. Let T be a curvature-like tensor field satisfying the algebraic Gauss equation (22). Then*

$$n\mathcal{C}^{T,\zeta} - \|\operatorname{trace}\zeta\|^2 = -\,2\tau_T. \tag{29}$$

Proof. Let $p \in M$, the set $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_α belong to an orthonormal basis $\{e_{n+1}, \dots, e_m\}$ of the Riemannian vector bundle (B, g_B) over M at p. From (22), we get

$$(K_T)_{ij} = T\,(e_i, e_j, e_j, e_i) = \sum_{\alpha=n+1}^{m} \left(\zeta_{ii}^{\alpha}\zeta_{jj}^{\alpha} - (\zeta_{ij}^{\alpha})^2 \right), \tag{30}$$

which implies that

$$2\tau_T = \|\operatorname{trace}\zeta\|^2 - \|\zeta\|^2 = \|\operatorname{trace}\zeta\|^2 - n\mathcal{C}^{T,\zeta}. \tag{31}$$

This gives (29). ■

3 Basic Casorati inequalities

We begin with the following two Lemmas:

Lemma 2. ([18, Theorem 21.4, p. 425]) *Let $\Upsilon \subset \mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. Then a C^2 function $f : \Upsilon \to \mathbb{R}$ is a convex function on the open convex set Υ if and only if for each $x \in \Upsilon$, the Hessian of f at x, denoted* $(\mathrm{Hess} f)_x$, *is a positive semidefinite matrix.*

Lemma 3. ([18, Corollary 21.2, p. 429]) *Let $\Upsilon \subset \mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. Let $f : \Upsilon \to \mathbb{R}$ be a C^1 convex function with a point $x_0 \in \Upsilon$ such that* $\mathrm{grad} f\,(x_0) = 0$, *then the point x_0 is a global minimizer of f over Υ.*

For application purposes, we prove the following

Lemma 4. *Let*

$$\Upsilon = \left\{ \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n : x^1 + \cdots + x^n = k \right\}$$

be a hyperplane of $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ a quadratic form given by

$$f\left(x^1, \ldots, x^n\right) = a \sum_{i=1}^{n-1} \left(x^i\right)^2 + b\left(x^n\right)^2 - 2 \sum_{1 \le i < j \le n} x^i x^j, \qquad a > 0,\ b > 0. \tag{32}$$

Then the constrained extremum problem

$$\min_{(x^1, \ldots, x^n) \in \Upsilon} f \tag{33}$$

has a global solution given by

$$\begin{cases} x^1 = x^2 = \cdots = x^{n-1} = \dfrac{k}{a+1}, \\ x^n = \dfrac{k}{b+1} = \dfrac{n-1}{b}\left(\dfrac{k}{a+1}\right) = (a-n+2)\,\dfrac{k}{a+1}, \end{cases} \tag{34}$$

provided that

$$b = \frac{n-1}{a-n+2}. \tag{35}$$

Proof. First we note that the set Υ is an open convex set in $\mathbb{R}^n$ and the function f is a C^∞ function (and hence a C^2 function). Now we compute the matrix for the Hessian Hess$f$ of the function f. The partial derivatives of the function f are

$$\begin{cases} \dfrac{\partial f}{\partial x^i} = 2\,(a+1)\,x^i - 2\displaystyle\sum_{\ell=1}^{n} x^\ell, \quad i \in \{1,\dots,n-1\}, \\ \dfrac{\partial f}{\partial x^n} = 2\,(b+1)\,x^n - 2\displaystyle\sum_{\ell=1}^{n} x^\ell. \end{cases} \tag{36}$$

From (36), we have

$$\begin{cases} \dfrac{\partial^2 f}{\partial\,(x^i)^2} = 2a, \quad i \in \{1,\dots,n-1\}, \\ \dfrac{\partial^2 f}{\partial x^i \partial x^j} = -\,2, \quad i,j \in \{1,\dots,n-1\}, \\ \dfrac{\partial^2 f}{\partial x^i \partial x^n} = -\,2, \quad i \in \{1,\dots,n-1\}, \\ \dfrac{\partial^2 f}{\partial\,(x^n)^2} = 2b. \end{cases} \tag{37}$$

Thus, in the standard frame of $\mathbb{R}^n$, the Hess$f$ has the matrix given by

$$2\begin{pmatrix} a & -\,1 & \cdots & -\,1 & -\,1 \\ -\,1 & a & \cdots & -\,1 & -\,1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\,1 & -\,1 & \cdots & a & -\,1 \\ -\,1 & -\,1 & \cdots & -\,1 & b \end{pmatrix}.$$

We note that for any $X = \left(X^1,\dots,X^n\right) \in T_x\Upsilon$, $x \in \Upsilon$, it follows that $\sum_{\ell=1}^{n} X^\ell = 0$. Consequently, for any $X = \left(X^1,\dots,X^n\right) \in T_x\Upsilon$, $x \in \Upsilon$ we have

$$\mathrm{Hess}f\,(X,X) \geq 0.$$

Thus, for each $x \in \Upsilon$, the Hessian $(\mathrm{Hess}f)_x$ of f at x is positive semidefinite. In view of Lemma 2, this implies that the C^2 function f is a convex function on the open convex set Υ.

For an optimal solution $\left(x^1,\dots,x^n\right)$ of the problem (33), the vector grad$f$ is normal to Υ, equivalently, it is collinear with the vector $(1,1,\dots,1)$. From

(36), for a critical point $x = \left(x^1, \ldots, x^n\right)$ of the function f we have

$$\begin{cases} (a+1)\, x^i - \sum_{\ell=1}^n x^\ell = 0, & i \in \{1, \ldots, n-1\}, \\ (b+1)\, x^n - \sum_{\ell=1}^n x^\ell = 0. \end{cases} \tag{38}$$

From (38), it follows that a critical point $\left(x^1, \ldots, x^{n-1}, x^n\right)$ of the function f has the form

$$x^1 = \cdots = x^{n-1} = t, \quad x^n = \frac{n-1}{b} t. \tag{39}$$

Since

$$x^1 + x^2 + \cdots + x^n = k,$$

in view of (39), a critical point $\left(x^1, \ldots, x^n\right)$ of the considered problem is given by (34). Solving one of the following three relations appearing in (34)

$$\frac{k}{b+1} = \frac{n-1}{b} \left(\frac{k}{a+1}\right) = (a-n+2)\, \frac{k}{a+1},$$

we get the equivalent relation given by (35). Consequently, in view of Lemma 3, the point $\left(x^1, \ldots, x^n\right)$ given by (34) is a global minimum point. Inserting (34) into (32) we have $f\left(x^1, \ldots, x^n\right) = 0$. ■

Now, we present the following Theorem, involving the Casorati inequalities for algebraic Casorati curvatures $\delta_{\mathcal{C}^{T,\zeta}}\,(r; n-1)$ and $\widehat{\delta}_{\mathcal{C}^{T,\zeta}}\,(r; n-1)$.

Theorem 1. *Let* (M, g) *be an* n*-dimensional Riemannian manifold,* (B, g_B) *a Riemannian vector bundle over* M *and* ζ *a* B*-valued symmetric* $(1,2)$*-tensor field. Let* T *be a curvature-like tensor field satisfying the algebraic Gauss equation* (22)*. Then*

$$(\tau_T)_{\mathrm{Nor}}\,(p) \leq \frac{1}{n(n-1)}\, [\delta_{\mathcal{C}^{T,\zeta}}(r; n-1)]_p\,, \qquad 0 < r < n\,(n-1)\,, \tag{40}$$

$$(\tau_T)_{\mathrm{Nor}}\,(p) \leq \frac{1}{n(n-1)} [\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(r; n-1)]_p, \qquad n(n-1) < r. \tag{41}$$

If

$$\inf\{\mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}$$
$$\big(\text{resp. } \sup\{\mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}\big)$$

is attained by a hyperplane Π_{n-1} *of* T_pM*,* $p \in M$*, then the equality sign holds in* (40) (*resp.* (41)) *if and only if with respect to a suitable orthonormal tangent frame* $\{e_1, ..., e_n\}$ *and a suitable orthonormal frame* $\{e_{n+1}, ..., e_m\}$ *of the Riemann vector bundle* (B, g_B)*, the components of* ζ *satisfy*

$$\zeta^\alpha_{ij} = 0 \qquad i, j \in \{1, \ldots, n\},\ \ i \neq j \ \ \alpha \in \{n+1, \ldots, m\}, \tag{42}$$

$$\zeta^\alpha_{11} = \zeta^\alpha_{22} = \cdots = \zeta^\alpha_{n-1\,n-1} = \frac{r}{n(n-1)}\, \zeta^\alpha_{nn} \qquad \alpha \in \{n+1, \ldots, m\}. \tag{43}$$

Proof. Let $p \in M$ and the set $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_α belong to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the Riemannian vector bundle (B, g_B) over M at p. We consider the following function

$$\mathcal{P} = r\mathcal{C}^{T,\zeta} + \mathrm{a}(r)\mathcal{C}^{T,\zeta}(\Pi_{n-1}) - 2\tau_T(p), \tag{44}$$

where Π_{n-1} is a hyperplane of T_pM. In view of (29), the relation (44) becomes

$$\mathcal{P} = (n+r)\,\mathcal{C}^{T,\zeta} + \mathrm{a}(r)\mathcal{C}^{T,\zeta}(\Pi_{n-1}) - \|\mathrm{trace}\,\zeta\|^2. \tag{45}$$

Without loss of generality, assume that the hyperplane Π_{n-1} is spanned by $e_1, \ldots, e_{n-1}$. Then from (45) it follows that

$$\mathcal{P} = \frac{n+r}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} (\zeta_{ij}^\alpha)^2 + \frac{\mathrm{a}(r)}{n-1} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n-1} (\zeta_{ij}^\alpha)^2 - \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} \zeta_{ii}^\alpha \right)^2. \tag{46}$$

The function $\mathcal{P}$ is a quadratic polynomial in the components of the tensor ζ and can be written as

$$\begin{aligned} \mathcal{P} \;&=\; \sum_{\alpha=n+1}^{m} \left\{ 2\left(\frac{r}{n} + \frac{\mathrm{a}(r)}{n-1} + 1\right) \sum_{1\le i<j\le n-1} (\zeta_{ij}^\alpha)^2 + 2\left(\frac{r}{n}+1\right) \sum_{i=1}^{n-1} (\zeta_{in}^\alpha)^2 \right. \\ &\quad \left. + \left(\frac{r}{n} + \frac{\mathrm{a}(r)}{n-1}\right) \sum_{i=1}^{n-1} (\zeta_{ii}^\alpha)^2 + \frac{r}{n} (\zeta_{nn}^\alpha)^2 - 2 \sum_{1\le i<j\le n} \zeta_{ii}^\alpha \zeta_{jj}^\alpha \right\} \\ &\ge\; \sum_{\alpha=n+1}^{m} \left\{ \left(\frac{r}{n} + \frac{\mathrm{a}(r)}{n-1}\right) \sum_{i=1}^{n-1} (\zeta_{ii}^\alpha)^2 + \frac{r}{n} (\zeta_{nn}^\alpha)^2 - 2 \sum_{1\le i<j\le n} \zeta_{ii}^\alpha \zeta_{jj}^\alpha \right\}. \end{aligned} \tag{47}$$

For $\alpha = n+1, \ldots, m$, we consider a quadratic form

$$f_\alpha : \mathbb{R}^n \to \mathbb{R}$$

given by

$$f_\alpha\left(\zeta_{11}^\alpha, \ldots, \zeta_{nn}^\alpha\right) = \left(\frac{r}{n} + \frac{\mathrm{a}(r)}{n-1}\right) \sum_{i=1}^{n-1} (\zeta_{ii}^\alpha)^2 + \frac{r}{n} (\zeta_{nn}^\alpha)^2 - 2 \sum_{1\le i<j\le n} \zeta_{ii}^\alpha \zeta_{jj}^\alpha \tag{48}$$

and the constrained extremum problem

$$\min f_\alpha,$$

subject to the condition

$$\zeta_{11}^\alpha + \cdots + \zeta_{nn}^\alpha = k_\alpha,$$

where k_α is a real constant. Comparing (48) with (32), we see that

$$a = \left(\frac{r}{n} + \frac{\mathrm{a}(r)}{n-1}\right), \qquad b = \frac{r}{n},$$

which verifies the relation

$$b = \frac{n-1}{a-n+2}$$

of (35). Thus applying Lemma 4, we see that the critical point

$$\zeta^{\mathrm{c}} = \left(\zeta_{11}^\alpha, \zeta_{22}^\alpha, \ldots, \zeta_{n-1\,n-1}^\alpha, \zeta_{nn}^\alpha\right)$$

given by

$$\zeta_{11}^\alpha = \zeta_{22}^\alpha = \cdots = \zeta_{n-1\,n-1}^\alpha = \frac{r}{(n-1)\,(n+r)}\,k_\alpha, \quad \zeta_{nn}^\alpha = \frac{n}{n+r}\,k_\alpha \tag{49}$$

is a global minimum point. Inserting (49) into (48) we have $f_\alpha(\zeta^{\mathrm{c}}) = 0$. Hence we have

$$\mathcal{P} \geq 0, \tag{50}$$

which in view of (44) gives

$$\frac{2\tau_T(p)}{n(n-1)} \leq \frac{r}{n(n-1)}\,\mathcal{C}_p^{T,\zeta} + \frac{\mathrm{a}(r)}{n(n-1)}\,\mathcal{C}^{T,\zeta}(\Pi_{n-1}) \tag{51}$$

for every tangent hyperplane Π_{n-1} of T_pM.

If $0 < r < n(n-1)$, then $\mathrm{a}(r) > 0$ and taking the infimum over all the tangent hyperplanes Π_{n-1} of T_pM, the relation (51) gives the inequality (40). If $n(n-1) < r$, then $\mathrm{a}(r) < 0$, and taking the supremum over all the tangent hyperplanes Π_{n-1} of T_pM, the relation (51) gives the inequality (41).

Suppose that

$$\inf\{\mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}$$
$$\big(\text{resp. } \sup\{\mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}\big)$$

is attained by a hyperplane Π_{n-1} spanned by $e_1, \ldots, e_{n-1}$. Then the equality sign holds in (40) (resp. (41)) if and only if we have the equality in all the previous inequalities. Thus the equality sign is true in the inequality (40) (resp. (41)) if and only if the relations (42) and (43) are true. ■

Now, we have the following two results.

Theorem 2. *Let (M, g) be an n-dimensional Riemannian manifold, (B, g_B) a Riemannian vector bundle over M and ζ a B-valued symmetric $(1, 2)$-tensor field. Let T be a curvature-like tensor field satisfying the algebraic Gauss equation (22). Then the T-normalized scalar curvature $(\tau_T)_{\mathrm{Nor}}$ is bounded above by the algebraic Casorati curvature $\delta_{\mathcal{C}^{T,\zeta}}(n-1)$ given by (25), that is,*

$$(\tau_T)_{\mathrm{Nor}}(p) \le \left[\delta_{\mathcal{C}^{T,\zeta}}(n-1)\right]_p. \tag{52}$$

If

$$\inf\{\mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}$$

is attained by a hyperplane Π_{n-1} of T_pM, then the equality sign holds in (52) if and only if with respect to suitable orthonormal tangent frame $\{e_1, ..., e_n\}$ and orthonormal frame $\{e_{n+1}, ..., e_m\}$, the components of ζ satisfy

$$\zeta_{ij}^{\alpha} = 0 \qquad i, j \in \{1, \dots, n\}, \ \ i \neq j \ \ \alpha \in \{n+1, \dots, m\}, \tag{53}$$

$$\zeta_{11}^{\alpha} = \zeta_{22}^{\alpha} = \cdots = \zeta_{n-1\,n-1}^{\alpha} = \frac{1}{2}\,\zeta_{nn}^{\alpha}, \qquad \alpha \in \{n+1, \dots, m\}. \tag{54}$$

Proof. Using

$$\left[\delta_{\mathcal{C}^{T,\zeta}}(n-1)\right]_p = \frac{1}{n(n-1)}\left[\delta_{\mathcal{C}^{T,\zeta}}\left(\frac{n(n-1)}{2}; n-1\right)\right]_p \tag{55}$$

in (40), we get (52). Taking $2r = n(n-1)$ in (43) we get (54). ∎

Theorem 3. *Let (M, g) be an n-dimensional Riemannian manifold, (B, g_B) a Riemannian vector bundle over M and ζ a B-valued symmetric $(1, 2)$-tensor field. Let T be a curvature-like tensor field satisfying the algebraic Gauss equation (22). Then the T-normalized scalar curvature $(\tau_T)_{\mathrm{Nor}}$ is bounded above by the algebraic Casorati curvature $\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(n-1)$, that is,*

$$(\tau_T)_{\mathrm{Nor}}(p) \le [\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(n-1)]_p. \tag{56}$$

If

$$\sup\{\mathcal{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}$$

is attained by a hyperplane Π_{n-1} of T_pM, then the equality sign in (56) is true if and only if with respect to a suitable orthonormal tangent frame $\{e_1, ..., e_n\}$ and a suitable orthonormal frame $\{e_{n+1}, ..., e_m\}$ of the Riemann vector bundle (B, g_B), the components of ζ satisfy

$$\zeta_{ij}^{\alpha} = 0, \qquad i, j \in \{1, \dots, n\}, \quad i \neq j, \quad \alpha \in \{n+1, \dots, m\}, \tag{57}$$

$$\zeta_{11}^{\alpha} = \zeta_{22}^{\alpha} = \cdots = \zeta_{n-1\,n-1}^{\alpha} = 2\,\zeta_{nn}^{\alpha}, \qquad \alpha \in \{n+1, \dots, m\}. \tag{58}$$

Proof. Using

$$\left[\widehat{\delta}_{\mathcal{C}^{T,\zeta}}(n-1)\right]_p = \frac{1}{n(n-1)}\left[\widehat{\delta}_{\mathcal{C}^{T,\zeta}}\left(2n(n-1);n-1\right)\right]_p \tag{59}$$

in (41), we get (56). Taking $r = 2n(n-1)$ in (43) we get (58). ■

4 Casorati inequalities for Riemannian submanifolds

Theorem 4. *Let (M,g) be an n-dimensional Riemannian submanifold of m-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$. Then the generalized normalized δ-Casorati curvatures $\delta_{\mathcal{C}}(r;n-1)$ and $\widehat{\delta}_{\mathcal{C}}(r;n-1)$ satisfy*

$$\tau_{\mathrm{Nor}}(p) \leq \frac{1}{n(n-1)}\left[\delta_{\mathcal{C}}\left(r;n-1\right)\right]_p + \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right), \quad 0<r<n(n-1), \tag{60}$$

$$\tau_{\mathrm{Nor}}(p) \leq \frac{1}{n(n-1)}\left[\widehat{\delta}_{\mathcal{C}}(r;n-1)\right]_p + \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right), \quad n(n-1)<r. \tag{61}$$

If

$$\inf\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}$$
$$(\text{resp. } \sup\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\})$$

is attained by a hyperplane Π_{n-1} of T_pM, $p \in M$, then the equality sign holds in (60) (*resp.* (61)) *for all $p \in M$ if and only if (M,g) is an invariantly quasi-umbilical submanifold with trivial normal connection in $(\widetilde{M},\widetilde{g})$, such that with respect to suitable tangent orthonormal frame $\{e_1,...,e_n\}$ and normal orthonormal frame $\{e_{n+1},...,e_m\}$, the shape operators $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1,...,m\}$, take the following forms:*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \dfrac{n(n-1)}{r}a \end{pmatrix}, \ A_{n+2} = \cdots = A_m = 0. \tag{62}$$

Proof. Let (M,g) be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$. Let the Riemannian vector bundle (B,g_B) over M be replaced by the normal bundle $T^{\perp}M$, and the B-valued symmetric $(1,2)$-tensor field ζ be replaced by the second fundamental form of immersion σ. In (22), we set

$$T\left(X,Y,Z,W\right) = R(X,Y,Z,W) - \widetilde{R}(X,Y,Z,W)$$

with R the Riemann curvature tensor on M and $\zeta = \sigma$. Then we see that

$$(\tau_T)_{\mathrm{Nor}}(p) = \tau_{\mathrm{Nor}}(p) - \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right),$$

$$\delta_{\mathcal{C}^{T,\varsigma}}(r;n-1) = \delta_{\mathcal{C}}\left(r;n-1\right),$$

$$\widehat{\delta}_{\mathcal{C}^{T,\varsigma}}(r;n-1) = \widehat{\delta}_{\mathcal{C}}(r;n-1).$$

Using these facts in (40) and (41), we get (60) and (61), respectively.

The conditions of equality cases (42) and (43) become

$$\sigma_{ij}^{\alpha} = 0 \qquad i,j \in \{1,\ldots,n\},\ \ i \neq j\ \ \alpha \in \{n+1,\ldots,m\} \tag{63}$$

and

$$\sigma_{11}^{\alpha} = \sigma_{22}^{\alpha} = \cdots = \sigma_{n-1\,n-1}^{\alpha} = \frac{r}{n(n-1)}\,\sigma_{nn}^{\alpha}, \qquad \alpha \in \{n+1,\ldots,m\}, \tag{64}$$

respectively. Thus the equality sign holds in both the inequalities (60) and (61) if and only if (63) and (64) are true.

The interpretation of the relations (63) is that the shape operators with respect to all normal directions e_α commute, or equivalently, that the normal connection $\nabla^\perp$ is flat, or still, that the *normal curvature tensor* $R^\perp$, that is, the curvature tensor of the normal connection, is trivial. Furthermore, the interpretation of the relations (64) is that there exist $m-n$ mutually orthogonal unit normal vectors $\{e_{n+1},...,e_m\}$ such that the shape operators with respect to all directions e_α ($\alpha \in \{e_{n+1},...,e_m\}$) have an eigenvalue of multiplicity $n-1$ and that for each e_α the distinguished eigendirection is the same (namely e_n), that is, the submanifold is *invariantly quasi-umbilical* [4].

Thus from the relations (63) and (64), we conclude that the equality holds in (60) and/or (61) for all $p \in M$ if and only if the Riemannian submanifold M is invariantly quasi-umbilical with trivial normal connection $\nabla^\perp$ in $\widetilde{M}$, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the form given by (62). ■

Theorem 5. *Let (M,g) be an n-dimensional Riemannian submanifold of m-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$. Then the normalized δ-Casorati curvature $\delta_{\mathcal{C}}(n-1)$ satisfies*

$$\tau_{\mathrm{Nor}}(p) \leq [\delta_{\mathcal{C}}(n-1)]_p + \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right). \tag{65}$$

If $\inf\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}$

is attained by a hyperplane Π_{n-1} of T_pM, $p \in M$, then the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal *connection in $\widetilde{M}$, such that with respect to suitable orthonormal tangent frame $\{e_1, ..., e_n\}$ and normal orthonormal frame $\{e_{n+1}, ..., e_m\}$, the shape operators $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, ..., m\}$, take the following forms*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & ... & 0 & 0 \\ 0 & a & 0 & ... & 0 & 0 \\ 0 & 0 & a & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & a & 0 \\ 0 & 0 & 0 & ... & 0 & 2a \end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0. \tag{66}$$

Proof. Using (20) in (60), we get (65). Putting $2r = n(n-1)$ in (62) we get (66). ■

Theorem 6. *Let (M, g) be an n-dimensional Riemannian submanifold of m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then the normalized δ-Casorati curvature $\widehat{\delta}_{\mathcal{C}}(n-1)$ satisfies*

$$\tau_{\mathrm{Nor}}(p) \leq [\widehat{\delta}_{\mathcal{C}}(n-1)]_p + \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right). \tag{67}$$

If $$\sup\{\mathcal{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}$$

is attained by a hyperplane Π_{n-1} of T_pM, $p \in M$, then the equality sign holds if and only if (M, g) is an invariantly quasi-umbilical submanifold with trivial normal connection in $(\widetilde{M}, \widetilde{g})$, such that with respect to suitable orthonormal tangent frame $\{e_1, ..., e_n\}$ and normal orthonormal frame $\{e_{n+1}, ..., e_m\}$, the shape operators $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, ..., m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & ... & 0 & 0 \\ 0 & a & 0 & ... & 0 & 0 \\ 0 & 0 & a & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & a & 0 \\ 0 & 0 & 0 & ... & 0 & \frac{1}{2}a \end{pmatrix}, \ A_{n+2} = \cdots = A_m = 0. \tag{68}$$

Proof. Using (21) in (61), we get (67). Putting $r = 2n(n-1)$ in (62) we get (68). ■

5 Casorati inequalities for submanifolds of real space forms

An m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$ with constant sectional curvature c, denoted $\widetilde{M}(c)$, is called a *real space form*, and its Riemann curvature tensor $\widetilde{R}$ is then given by

$$\widetilde{R}(X,Y,Z,W) = c\left\{\widetilde{g}\,(Y,Z)\,\widetilde{g}\,(X,W) - \widetilde{g}\,(X,Z)\,\widetilde{g}\,(Y,W)\right\} \tag{69}$$

for all vector fields X,Y,Z,W on $\widetilde{M}$. The model spaces for real space forms are the Euclidean spaces ($c = 0$), the spheres ($c > 0$), and the hyperbolic spaces ($c < 0$). For an n-dimensional Riemannian submanifold (M,g) of a real space form $\widetilde{M}(c)$ it is easy to see that

$$\widetilde{\tau}_{\mathrm{Nor}}\,(T_pM) = c. \tag{70}$$

Theorem 7. [22, Theorem 2.1 and Corollary 3.1] *Let (M,g) be an n-dimensional Riemannian submanifold of m-dimensional real space form $\widetilde{M}(c)$. Then*

$$\tau_{\mathrm{Nor}}(p) \leq \frac{1}{n(n-1)}\,[\delta_{\mathcal{C}}(r;n-1)]_p + c, \quad 0 < r < n(n-1), \tag{71}$$

$$\tau_{\mathrm{Nor}}(p) \leq \frac{1}{n(n-1)}\,[\widehat{\delta}_{\mathcal{C}}(r;n-1)]_p + c, \quad n(n-1) < r. \tag{72}$$

The equality sign holds in (71) (*resp.* (72)) *for all $p \in M$ if and only if (M,g) is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{M}(c)$, such that with respect to suitable tangent orthonormal frame $\{e_1,...,e_n\}$ and normal orthonormal frame $\{e_{n+1},...,e_m\}$, the shape operators $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1,...,m\}$, take the forms given by* (62).

Proof. Using (70) in (60) and (61) we get (71) and (72), respectively. ■

Theorem 8. (Theorem 4.1, [63]) *Let (M,g) be an n-dimensional Riemannian submanifold of m-dimensional real space form $\widetilde{M}(c)$. Then the normalized δ-Casorati curvature $\delta_{\mathcal{C}}(n-1)$ satisfies*

$$\tau_{\mathrm{Nor}}(p) \leq [\delta_{\mathcal{C}}(n-1)]_p + c. \tag{73}$$

Moreover, the equality sign holds for all $p \in M$ if and only if (M,g) is an invariantly quasi-umbilical submanifold with trivial normal connection in $(\widetilde{M},\widetilde{g})$, such that with respect to suitable orthonormal tangent frame $\{e_1,...,e_n\}$ and normal orthonormal frame $\{e_{n+1},...,e_m\}$, the shape operators $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1,...,m\}$, take the forms given by (66).

Proof. Using (20) in (71), we get (73). ■

Theorem 9. (Theorem 1 and Corollary 3, [21]) *Let (M,g) be an n-dimensional Riemannian submanifold of m-dimensional real space form $\widetilde{M}(c)$. Then the normalized δ-Casorati curvature $\widehat{\delta}_{\mathcal{C}}(n-1)$ satisfies*

$$\tau_{\mathrm{Nor}}(p) \leq [\widehat{\delta}_{\mathcal{C}}(n-1)]_p + c. \tag{74}$$

Moreover, the equality sign holds for all $p \in M$ if and only if (M, g) is an invariantly quasi-umbilical submanifold with trivial normal connection in $(\widetilde{M}, \widetilde{g})$, such that with respect to suitable orthonormal tangent frame $\{e_1, ..., e_n\}$ and normal orthonormal frame $\{e_{n+1}, ..., e_m\}$, the shape operators $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, ..., m\}$, take the forms given by (68).

Proof. Using (21) in (72), we get (74). ■

6 Further studies

In this section, we present some problems. Similar problems can be formulated in those situations, where Riemman curvature tensor of the ambient manifold has some nice well known form.

Problem 6.1. Like in [14], to obtain Casorati inequalities for conformally flat submanifolds of a real space form.

Problem 6.2. Riemannian manifolds of quasi-constant curvature (cf. [5], [16], [26], [42], [58]) represent a good generalization of real space forms. To obtain Casorati inequalities for submanifolds of quasi-constant curvature manifolds. To study Casorati ideal submanifolds of quasi-constant curvature manifolds.

Problem 6.3. To obtain Casorati inequalities for submanifolds of generalized complex space forms (cf. [32], [45], [55], [51]).

Problem 6.4. Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Lagrangian [13] and Kaehlerian slant submanifolds [12] of a complex space form, if possible.

Problem 6.5. To obtain Casorati inequalities for different kind of submanifolds of locally conformal Kaehler space forms (cf. [25], [54]).

Problem 6.6. Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Lagrangian submanifolds of a locally conformal Kaehler space form (under some conditions), if possible.

Problem 6.7. To obtain Casorati inequalities for submanifolds of Kaehler manifolds of quasi constant holomorphic sectional curvatures (cf. [27], [2]).

Problem 6.8. To obtain Casorati inequalities for different kind of submanifolds of Bochner-Kaehler manifolds [17].

Problem 6.9. Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Lagrangian submanifolds of Bochner-Kaehler manifolds, if possible.

Problem 6.10. Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Lagrangian submanifolds of a quaternionic space form [31], if possible.

Problem 6.11. To obtain Casorati inequalities for different kind of submanifolds [52] of generalized (κ, μ) space forms [7] and in particular generalized Sasakian space forms [1] and Sasakian space forms.

Problem 6.12. Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Legendrian submanifolds of a Sasakian space form (cf. [50], [3]).

Problem 6.13. To obtain Casorati inequalities for different kind of submanifolds of different kind of manifolds equipped with a semi-symmetric metric connection (cf. [47], [59], [43]).

Problem 6.14. To obtain Casorati inequalities for centroaffine hypersurfaces [44].

Acknowledgements. The author is thankful to Professor Ugo Gianazza (gianazza@imati.cnr.it), Claudio Gnoli (claudio.gnoli@unipv.it), Claudia Olivati and Anna Bendiscioli from University of Pavia, Italy for their help in tracing the original papers of Felice Casorati ([8], [9]).

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