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Harmonic geometric structures: the general theory and the cases of almost complex and almost contact structures

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Abstract. We try and describe here the general scheme of homogeneous fibre bundles and the harmonicity equations for their sections and how it applies to geometric structures arising from a reduction of the structure group. Among G-structures, we particularly pay attention to almost complex and almost contact structures and the relevant notions of harmonic sections and harmonic maps. We then review some of the more salient properties and results in these two instances. In the last section, we detail the proof of harmonicity for the class nearly cosymplectic.

Keywords: Harmonic section, harmonic map, harmonic unit vector field, almost complex structure, almost contact structure, nearly cosymplectic structure.

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1 Harmonic maps, harmonic sections

The theory of harmonic maps offers a framework to determine best members of a homotopy class. It has a prehistory in an unpublished 1954 MIT report of J. H. Sampson and a conference proceedings article by J. Eells in 1958 [17], describing the space of smooth maps between smooth manifolds as an infinite dimensional smooth manifold (see testimonies at the 1993 Smalefest [32]). But the real starting point is the seminal [22] where any map into a negatively curved manifold is continuously deformed into a harmonic representative, i.e. a critical point of the Dirichlet energy

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \, v_g.$$

This brought together under the same banner classic topics like harmonic functions, geodesics, holomorphic maps (at least between Kähler manifolds) and, decisively, minimal isometric immersions (cf. [18–20] for classic surveys).

In the late 70's and early 80's, some of the attention turned toward more geometric environments, probably to offset the hard analytical problems which had appeared. For nigh submersions, this has led, through conundrums, to harmonic morphisms, but we are here interested in a sort of dual case with sections of fibre bundles, beginning with vector fields. Here one must stress how the geometrical significance of vector fields, in contrast with mere mappings from M to TM, opens up the possibility of a second, weaker, notion of harmonicity, since admissible variations could be restricted to sections. For this variant, critical points of the energy are called harmonic vector fields or, more generally, harmonic sections.

Unfortunately, once the tangent bundle (a manifold in its own rights, twice the original dimension) has been equipped with its simplest Riemannian metric, due to Sasaki, it easily turns out, at least for compact spaces, that both versions of harmonicity imply parallel, with serious topological consequences ([8]).

Faced with the dilemma of relaxing the above set-up to allow a glimpse of hope of a solution, one could modify the Sasaki metric on TM (and then uncork a philosophical debate on the virtues of the replacements), opt for another functional (volume, bi-energy), restrict to unit vector fields (done with a modicum of success [37, 43]) or, more radical, change the bundle. This will be our course.

2 Homogeneous fibre bundles

Following C. M. Wood in [44], where details can be found, we describe the general construction of homogeneous fibre bundles, though one could read this section with the orthogonal group and the orthonormal frame bundle in mind.

Let G be a Lie group and $\xi : Q \to M$ a principal G-bundle. For a Lie subgroup H of G, we consider the orbit space $N = Q/H$ so that the H-orbit map $\zeta : Q \to N$ is a principal H-bundle and $\pi : N \to M$ is a fibre bundle with fibre G/H. Then $\xi = \pi \circ \zeta$ and π is isomorphic to the associated bundle $Q \times_G G/H$.

To go further, we need to assume that G/H is reductive, i.e. the Lie algebras satisfy $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathrm{Ad}_G(H)\mathfrak{m} \subset \mathfrak{m}$. The manifold M will be equipped with a Riemannian metric g and G/H with a G-invariant Riemannian metric, which, in other terms, is an $\mathrm{Ad}_G(H)$-invariant positive-definite inner-product $\langle , \rangle$ on $\mathfrak{m}$. On the principal G-bundle $\xi : Q \to M$ we will have a connection with its $\mathfrak{g}$-valued connection form denoted by ω. Then we can split the tangent bundle of $N = Q/H$ into its horizontal $\mathcal{H}$ and vertical $\mathcal{V}$ sub-bundles with

$$\mathcal{V} = \ker d\pi = d\zeta(\ker d\xi), \quad \mathcal{H} = d\zeta(\ker \omega).$$

With this decomposition, we construct a Sasaki-type Riemannian metric h on N by horizontally lifting the metric g on $\mathcal{H}$ and on the vertical sub-bundle, its orthogonal complement, use $\langle,\rangle$ as metric. Then $\pi : (N,h) \to (M,g)$ is a Riemannian submersion with totally geodesic fibres.

To understand the tangent bundle TN, and especially the vertical sub-bundle, we construct an isomorphism between $\mathcal{V}$ and $\mathfrak{m}_Q$, the vector bundle associated to ζ with fibre $\mathfrak{m}$. Let $q \in Q$ and $a \in \mathfrak{m}$, then $t \mapsto q.\exp ta$ is a curve on Q whose tangent vector at Q is denoted by $a^*(q) \in T_qQ$. Since $\zeta(q) \in N$, it admits a fibre in $\mathfrak{m}_Q$ and $q \bullet a$ will denote the vector $a \in \mathfrak{m}$ in this fibre. Then $d\zeta(a^*(q))$ is in the sub-bundle $\mathcal{V}$, since image by $d\zeta$ of a vector tangent to a fibre of $Q \to M$, and the following map is a vector bundle isomorphism, the canonical isomorphism:

$$I : \mathcal{V} \to \mathfrak{m}_Q : \quad d\zeta(a^*(q)) \mapsto q \bullet a.$$

To deal with the connection form, we split ω into the sum of its $\mathfrak{h}$- and $\mathfrak{m}$-components: $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{m}}$. Then $\omega_{\mathfrak{m}}$ is an H-equivariant and ζ-horizontal $\mathfrak{m}$-valued one-form on Q, from which we define an $\mathfrak{m}_Q$-valued one-form ϕ on N, called the homogeneous connection form:

$$\phi(d\zeta(E)) = q \bullet \omega_{\mathfrak{m}}(E), \quad \forall E \in T_qQ,$$

so that

$$\phi|_{\mathcal{V}} = I \quad \text{ and } \ker \phi = \mathcal{H},$$

and the metric on N can be described as:

$$h = \pi^* g + \langle \phi, \phi \rangle,$$

so I becomes an isometry.

This description should remind the reader of the case of the bi-tangent bundle of a Riemannian manifold.

The $\mathfrak{m}$-component of the curvature form Ω of ω projects to an $\mathfrak{m}_Q$-valued two-form Φ on N, called the homogeneous curvature form. As to $\omega_{\mathfrak{h}}$, it allows the construction of a covariant derivative ∇^c on $\mathfrak{m}_Q$ and, in turn, an exterior differential operator d^c on $\mathfrak{m}_Q$-valued differential forms on N.

Let U denote the $\mathfrak{m}$-valued symmetric bilinear form on $\mathfrak{m}$:

$$\langle U(a,b), c\rangle = \langle [c,a]_{\mathfrak{m}}, b\rangle + \langle a, [c,b]_{\mathfrak{m}}\rangle.$$

Recall that U vanishes if and only if G/H is naturally reductive. More results on the differential geometry of homogeneous fibre bundles can be found in [44].

Assume now the existence of a reduction of the structure group G of $\xi : Q \to M$, to a sub-group H, i.e. there exists an H-sub-bundle $\xi' : Q' \to M$ (see also

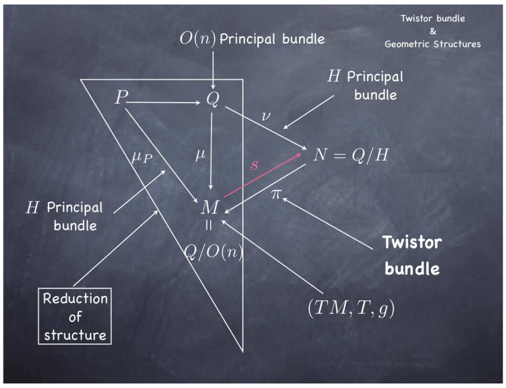

Figure 1. Reduction diagram

S. S. Chern's take on G-structures in [8]). The manifold $\zeta(Q') \subset N$ is transverse to the fibres of $\pi : N \to M$, since $T\zeta(Q')$ will be contained in $\mathcal{H}$. Moreover, as the fibres of Q' are copies of H, mapping Q' onto N by ζ will identify each fibre into a single point.

This allows a one-one correspondence between reductions of the structure group to H and sections of $\pi : N \to M$, as given a section σ, Q' can be reconstructed as $\zeta^{-1}(\sigma(M))$.

Pulling back by σ the homogeneous connection form gives the vertical component of the differential of σ, crucial to the computation of the vertical tension field:

$$\sigma^*\phi = \phi \circ d\sigma = I \circ d^v\sigma.$$

and [44, Theorem 3.2]

$$I(\tau^v(\sigma)) = -\delta(\sigma^*\phi) + \tfrac{1}{2}\operatorname{trace}(\sigma^*\phi^*U),$$

where δ is the coderivative for $\sigma^*\mathfrak{m}_Q$-valued differential forms on M relative to the σ-pull-back of ∇^c.

Moreover, since [44, Theorem 3.4]

$$2g(\pi_*\nabla d\sigma(X,Y),Z) = \langle\sigma^*\phi \otimes \sigma^*\Phi\rangle(X,Y,Z) + \langle\sigma^*\phi \otimes \sigma^*\Phi\rangle(Y,X,Z),$$

a harmonic section σ is a harmonic map if and only if

$$\langle\sigma^*\phi \otimes \sigma^*\Phi\rangle = 0.$$

Finally, C. M. Wood proved in 1990 an Eells-Sampson-type existence theorem for harmonic sections in [40], which extended a result of Donaldson on twisted harmonic maps [14].

3 Almost complex structures

The most accomplished example so far is taking $G = \mathrm{SO}(2n)$ and $H = \mathrm{U}(n)$ on an even-dimensional manifold. We then recover the classical twistor space $\pi : N \to M$. Then, and the results on this page first appeared in [41, 42], the Lie algebra $\mathfrak{h}$ is the space of anti-symmetric matrices commuting with

$$J_0 = \begin{pmatrix} \mathbb{O}_n & -\mathbb{I}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{pmatrix}$$

and $\mathfrak{m}$ the space of anti-symmetric matrices anti-commuting with J_0. On the vector bundle $\pi^*TM \to N$, there is a universal almost Hermitian structure $\mathcal{J}$ defined at $y \in N$ by $\mathcal{J}_y \in \mathrm{End}(T_{\pi(y)}M)$ given, with respect to any orthonormal frame of $\zeta^{-1}(y)$, by J_0.

An element β of $\mathfrak{g}$ decomposes into its $\mathfrak{h}$- and $\mathfrak{m}$-components:

$$\tfrac{1}{2}J_0[\beta, J_0] - \tfrac{1}{2}J_0\{\beta, J_0\},$$

where $\{A, B\} = AB + BA$ is the anti-commutator.

The energy functional is

$$E(\sigma) = \frac{\dim(M)}{2} + \frac{1}{2}\int_M \frac{1}{4}|\nabla J|^2 \, v_g,$$

so that Kähler structures are absolute minimisers and the corresponding Euler-Lagrange equation is:

$$I(\tau^v(\sigma)) = \frac{1}{4}[\nabla^*\nabla J, J].$$

So J is a harmonic section if and only if it commutes with its rough Laplacian and a harmonic map if it moreover satisfies

$$g(R(E_i, Z)J, \nabla_{E_i} J) = 0, \quad \forall Z \in TM.$$

The first examples of harmonic sections, but also of harmonic maps, have been nearly Kähler structures [41], in part because of the curvature identities of Gray [29, 30]. This forms a rather large and well-known class of spaces, with Dubruille [16] showing that the only homogeneous examples of dimension six are $\mathrm{G}_2/\mathrm{SU}(3)$, $\mathrm{SU}(3)/\mathbb{S}^1 \times \mathbb{S}^1$, $\mathrm{Sp}(2)/\mathbb{S}^1 \times \mathrm{Sp}(1)$ and $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\mathrm{Sp}(1)$, and the study of the space of nearly Kähler structures, still in dimension six, in terms of certain $(1,1)$-eigenforms of a Laplace operator [34]. On the other hand, $(1,2)$-symplectic structures are harmonic sections if and only if the Ricci* operator is symmetric, where

$$\mathrm{Ricci}^*(X, Y) = \langle R(X, E_i)JE_i, Y\rangle,$$

for an orthonormal frame $\{E_i\}$.

The Calabi-Eckmann complex structure on the product of odd-dimensional spheres and the Abbena-Thurston almost-Kähler structure are harmonic sections (but not critical for the volume functional) [42]. Finally, we must mention the impressive work of Davidov and Muskarov [10, 11], where they show the harmonicity, as sections or maps, of the Atiyah-Hitchin-Singer and Eells-Salamon almost Hermitian structures on the twistor space of an oriented Riemannian four-manifold, a real mise en abyme.

These two almost complex structures, J_1 and J_2, are defined by decomposing the tangent space to the twistor space into an horizontal subspace $\mathcal{H}$ isomorphic to the tangent space of the base manifold, on which both J_1 and J_2 are defined as the horizontal lift of the point of the twistor space (which is, by definition, an almost complex structure), while on the vertical subspace J_1 is taken to be the complex structure defined by vector product with the base point of the twistor space, while J_2 is the conjugate. Note that [1] shows that J_1 is integrable if and only if M is self-dual and J_2 plays an important role in harmonic map theory [21].

By re-writing the equations in terms of the fundamental two-form, [11] shows that J_1 is a harmonic section if and only if (M, g) is self-dual, while, for J_2, (M, g) must moreover have constant scalar curvature. Besides, they both are harmonic maps exactly when (M, g) is either self-dual and Einstein or self-dual and locally the product of an open interval of $\mathbb{R}$ and a three-dimensional manifold of constant curvature.

In the more usual sense, they also give conditions for the harmonicity of integrable and almost Kähler structures on four-manifolds. This yields many new examples (cf. [10, 12]): the complex structure on primary Kodaira surfaces is a harmonic map mapping into its twistor space, left-invariant almost Kähler four-dimensional Lie groups also define harmonic maps, while Inoue surfaces of type S^0 admit locally conformal Kähler metrics with a complex structure which is a harmonic section but not an harmonic map.

4 Almost contact structures

An almost contact structure is given by the pairing of a unit vector field ξ and a $(1, 1)$-tensor θ on a, necessarily odd-dimensional, Riemannian manifold related by the equation:

$$\theta^2 = -\operatorname{Id} + \eta \otimes \xi, \tag{1}$$

where $\eta(\xi) = 1$ and metric compatibility is taken as a blanket assumption. The origin of this definition goes back to John W. Gray [31] and the best source of examples is [3], but perhaps the most useful example to have in mind is the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. Note that warped-products of almost complex manifolds with a line can be equipped with almost contact structures.

The distribution orthogonal to the ξ-direction will be denoted by $\mathcal{F}$ and we will use the symbol J for the restriction of θ to $\mathcal{F}$. The induced connection and curvature on $\mathcal{F}$ will be denoted by $\bar{\nabla}$ and $\bar{R}$.

While there is no integrability comparable to the complex case, contact structures are defined with the condition $\eta \wedge (d\eta)^n \neq 0$, i.e. the contact subbundle $\mathcal{F}$ is as far from being integrable as possible [3].

Almost contact structures have been studied through the lens of the harmonicity of the unit vector field ξ in [35, 36] and the reader can refer to [27] and [15]

Recently, Casals, Pancholi and Presas proved in [7] that, in dimension five, an almost contact structure on a closed manifold, whose existence is only obstructed by the third integral Steifel-Whitney class, can always be homotopically deformed into a contact structure. This was already known to Geiges for simply-connected manifolds [25] and spin closed manifolds with $\pi_1 = \mathbb{Z}_2$ [26]. By Gromov's h-principle, this is also valid in higher dimensions for open manifolds.

We summarise [38] and refer to this article for further details.

The general approach of harmonicity via reduction of the structure group applies to almost contact structures if one chooses the groups $G = \mathrm{SO}(2n+1)$ and $H = \mathrm{U}(n)$, included as

$$A + \mathrm{i}B \mapsto \begin{pmatrix} A & -B & 0 \\ B & A & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

Let

$$\phi_0 = \begin{pmatrix} \mathbb{O}_n & -\mathbb{I}_n & 0 \\ \mathbb{I}_n & \mathbb{O}_n & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

in the Lie algebra $\mathfrak{g}$ of G, then $H = \{A \in G : A\phi_0 A^{-1} = \phi_0\}$ and its Lie algebra is $\mathfrak{h} = \{a \in \mathfrak{g} : [a, \phi_0] = 0\}$. The orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, with respect to the Killing form, splits naturally into $\mathfrak{m}_1$ and $\mathfrak{m}_2$:

$$\mathfrak{m}_1 = \{a \in \mathfrak{g} : \{a, \phi_0\} = 0\}, \quad \mathfrak{m}_2 = \{\{a, \eta_0 \otimes \xi_0\} : a \in \mathfrak{g}\},$$

where $\xi_0 = (0, \dots, 0, 1) \in \mathbb{R}^{2n+1}$ and η_0 is the dual of ξ_0. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ is an $\mathrm{Ad}(H)$-invariant splitting with:

$$[\mathfrak{h}, \mathfrak{m}_1] \subset \mathfrak{m}_1, \quad [\mathfrak{h}, \mathfrak{m}_2] \subset \mathfrak{m}_2$$

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{h} \oplus \mathfrak{m}_1, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2.$$

If $a \in \mathfrak{g}$ then $a = a_{\mathfrak{h}} + a_{\mathfrak{m}_1} + a_{\mathfrak{m}_2}$ with

$$a_{\mathfrak{h}} = -\tfrac{1}{2}(\phi_0\{a, \phi_0\} + a \circ (\eta_0 \otimes \xi_0)),$$

$$a_{\mathfrak{m}_1} = \tfrac{1}{2}(\phi_0[a, \phi_0] - a \circ (\eta_0 \otimes \xi_0)), \quad a_{\mathfrak{m}_2} = \{a, \eta_0 \otimes \xi_0\}.$$

Crucial to obtain the harmonicity equations in terms of tensors are the "hat" isomorphisms:

$$\mathfrak{h} \oplus \mathfrak{m}_1 \to \mathrm{Skew}(\mathbb{R}^{2n}) : a \mapsto \hat{a} = a|_{\mathbb{R}^{2n}}$$

$$\mathfrak{m}_2 \to \mathbb{R}^{2n} : a_2 \mapsto \hat{a}_2 = a_2(\xi_0).$$

Let $\mu : P \to M$ be the principal G-bundle of oriented orthonormal frames over the Riemannian manifold (M, g) and $\nu : P \to P/H = N$ a principal H-bundle, with $\pi : N \to M$, naturally isomorphic to the homogeneous G/H-bundle associated to μ. The pull-back bundle $\mathcal{E}$ by π of the tangent bundle (though [38] actually deals with the more general case of Riemannian vector bundles) admits an induced universal almost contact structure defined by $\boldsymbol{\xi}$ and $\mathbf{\Phi}$, coming from ξ_0 and ϕ_0.

Each H-module $\mathfrak{h}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are fibres of vector bundles $\boldsymbol{\mathfrak{H}}$, $\boldsymbol{\mathfrak{M}}_1$ and $\boldsymbol{\mathfrak{M}}_2$, associated to $\nu : P \to N$ and the "hat" isomorphisms induce isomorphisms of these bundles with the sub-bundles of $\mathrm{Skew}\,\boldsymbol{\mathcal{F}} \to N$ which commute and anti-commute with $\mathbf{\Phi}$ (restricted to $\boldsymbol{\mathcal{F}}$) and with $\boldsymbol{\mathcal{F}}$. Then, the homogeneous connection form, cf. [38], is the $\boldsymbol{\mathfrak{M}}_1 \oplus \boldsymbol{\mathfrak{M}}_2$-valued one-form θ on N obtained by projecting the $(\mathfrak{m}_1 + \mathfrak{m}_2)$-component of the connection form $\omega \in \Omega^1(P, \mathfrak{g})$. Then its $\boldsymbol{\mathfrak{M}}_1$- and $\boldsymbol{\mathfrak{M}}_2$-components are [38, Lemma 2.1]:

$$\theta_1 = \tfrac{1}{2}\mathbf{\Phi} \circ (\nabla\mathbf{\Phi})_1; \quad \theta_2 = [\mathbf{\Phi}, (\nabla\mathbf{\Phi})_2],$$

and applying the "hat" isomorphisms yields [38, Proposition 2.2]:

$$\hat{\theta}_1 = \tfrac{1}{2}\mathbf{J}\bar{\nabla}\mathbf{J}; \quad \hat{\theta}_2 = \nabla\boldsymbol{\xi}.$$

On the other hand, the $\mathfrak{h}$-component of ω is a connection form in $\nu : P \to N$, and induces a canonical connection ∇^c on the associated bundles $\boldsymbol{\mathfrak{M}}_1, \boldsymbol{\mathfrak{M}}_2 \to N$. If $\alpha_i \in \boldsymbol{\mathfrak{M}}_i$ $(i = 1, 2)$

$$(\nabla^c\alpha_1)\hat{} = \tfrac{1}{2}\mathbf{J}[\bar{\nabla}\hat{\alpha}_1, \mathbf{J}]; \quad (\nabla^c\alpha_2)\hat{} = \bar{\nabla}\hat{\alpha}_2 - \tfrac{1}{2}\mathbf{J}\bar{\nabla}\mathbf{J}(\hat{\alpha}_2). \tag{2}$$

From these considerations, one can derive the harmonicity equations of a section $\sigma : M \to N$. The corresponding almost contact structure on TM is obtained by pulling back the universal structure $(\mathbf{\Phi}, \boldsymbol{\xi}, \boldsymbol{\mathcal{F}}, \mathbf{J})$. Similarly, the bundles $\boldsymbol{\mathfrak{M}}_1$ and $\boldsymbol{\mathfrak{M}}_2$ can be pulled back to obtain sub-bundles $\mathfrak{M}_i$ $(i = 1, 2)$ of $\mathrm{Skew}\,TM \to M$

which anti-commute with J and swap $\mathcal{F}$ and $\langle\xi\rangle$. Then, ψ, the pull-back by σ of the homogeneous connection form θ, splits into $\mathfrak{M}_1$- and $\mathfrak{M}_2$-components which, after applying the "hat" isomorphisms, yield

$$\hat{\psi}_1 = \tfrac{1}{2} J\bar{\nabla}J \quad \text{ and } \hat{\psi}_2 = \nabla\xi.$$

Applying the pull-back of Equation (2) to ψ_1 and ψ_2, we obtain

$$\begin{aligned}&(\nabla^c\psi_1)\hat{} = \tfrac{1}{4}[J, \bar{\nabla}^2 J];\\ &(\nabla^c\psi_2)\hat{} = \nabla^2\xi + \langle\nabla\xi\otimes\nabla\xi\rangle\xi - \tfrac{1}{2}J\bar{\nabla}J\otimes\nabla\xi.\end{aligned}$$

The vertical tension field of σ is

$$\tau^v(\sigma) = \operatorname{trace}\nabla^v d^v\sigma$$

and it vanishes if and only if σ is a harmonic section. From [44, Equation (3.2)], the canonical vector bundle isomorphism $I : \mathcal{V} = \ker d\pi \to \boldsymbol{\mathfrak{M}}_1 \oplus \boldsymbol{\mathfrak{M}}_2$ sends $d^v\sigma$ to $\psi = \sigma^*\theta$ and

$$I\circ\tau^v(\sigma) = \nabla^c_{E_i}\psi(E_i).$$

Projecting onto $\hat{\mathfrak{M}}_1$ and $\hat{\mathfrak{M}}_2$ gives the two harmonic section equations [38, Theorem 3.2]:

$$\begin{aligned}&(I_1\tau^v(\sigma))\hat{} = \tfrac{1}{4}[\bar{\nabla}^*\bar{\nabla}J, J];\\ &(I_2\tau^v(\sigma))\hat{} = -\nabla^*\nabla\xi + |\nabla\xi|^2\xi - \tfrac{1}{2}J\operatorname{trace}\bar{\nabla}J\otimes\nabla\xi.\end{aligned}$$

For the third equation, which harmonic sections have to satisfy in order to be harmonic maps, we use [44] to obtain:

$$\tfrac{1}{4}\langle R(E_i, X), J\bar{\nabla}_{E_i}J\rangle + \langle R(E_i, X)\xi, \nabla_{E_i}\xi\rangle = 0.$$

Alternatives of the above equations, based on the intrinsic torsion, were given by Gonzalez-Davila and Martin-Cabrera (cf. [28]), in their study of the harmonicity of the classes of Chinea and Gonzalez-Davila [9].

The energy functional of an almost contact structure can be computed to be:

$$E(\sigma) = \frac{\dim(M)}{2} + \frac{1}{2}\int_M \frac{1}{4}|\bar{\nabla}J|^2 + |\nabla\xi|^2\, v_g$$

and since variations are to be taken among unit vector fields and $(1,1)$-tensors related by (1), this could be seen as a constrained variational problem.

Examples are rather numerous, even counting out the systematic approach of [28]. The canonical almost contact structure of a hypersurface of a Kähler manifold is harmonic if the Reeb vector field is harmonic and for the unit sphere it is also a harmonic map, as for Sasakian manifolds. Of the hyperspheres of the nearly Kähler $\mathbb{S}^6$, only the equator defines a harmonic section, which is also a harmonic map. This easily extends to nearly cosymplectic manifolds with parallel characteristic vector field and the general case is the subject of the next section.

More detailed results, especially on trans-Sasakian manifolds and hypersurfaces of nearly Kähler manifolds can be found in [38], while the case of contact structures was studied in [39], where one will also find a comparison of the harmonicity of the total space (almost contact) and the base manifold (almost Hermitian) of a Boothby-Wang fibration, later extended to a warped-product construction.

5 Nearly cosymplectic structures

The class of nearly cosymplectic structures was introduced by Blair in 1971 in [2] and Blair and Showers in 1974 [4] as contact counterparts of nearly Kähler manifolds. Defined by the condition that θ is Killing, i.e. $\nabla\theta$ is anti-symmetric, one immediate consequence is that the Reeb vector field ξ is Killing.

The best-known example is the equator sphere $\mathbb{S}^5$ in $\mathbb{S}^6$, as an hypersurface of a nearly Kähler manifold with second fundamental form proportional to $\eta\otimes\eta$.

However more examples can be deduced from the work of Cappelleti-Montano and Dileo [6], as they proved that, in dimension five, any nearly Sasakian manifold also admits a nearly cosymplectic structure, necessarily Einstein, and every Einstein-Sasaki manifold admits a one-parameter family of nearly cosymplectic structures. Moreover, in higher dimensions, De Nicola, Dileo and Yudin [13] proved that a nearly cosymplectic manifold is locally isometric to the Riemannian product of a nearly Kähler manifold with either a nearly cosymplectic five-manifold or a line.

Combined with recent results on nearly Kähler manifolds (cf. Section 3), this allows a large number of examples, including co-Kähler, called cosymplectic by Blair.

Note also the results of Endo [24] on the first Betti number of nearly cosymplectic manifolds.

The objective of this section is to give an outline of the proof appearing in [33] that nearly cosymplectic structures are harmonic maps. The strategy is inspired by [41] where the same conclusion was reached for nearly Kähler structures. First, one has to rewrite the equations, two for the harmonicity of sections plus one for maps, in terms of curvature. Then, as Gray did in [29, 30], take second covariant derivatives of the fundamental equation:

$$\theta^2 = -\operatorname{Id} + \eta\otimes\xi,$$

and, in a time-honoured fashion, make curvature terms appear, in various combinations. Plugging these formulas into the equations of harmonicity leads to the result.

The first of the two harmonic section equations is, in its initial form, given by:

$$[\bar{\nabla}^*\bar{\nabla}J, J] = 0, \tag{3}$$

and, using repeatedly the anti-symmetry of $\nabla\theta$, it can be re-written as:

$$\mathrm{Ricci}^*(\theta X, \theta Y) = \mathrm{Ricci}^*(X, Y), \quad \forall X, Y \in \mathcal{F}.$$

This expression is also valid for the case of nearly Kähler structures, which should not be so surprising as it does not involve the vector field ξ and the Cartesian product of a nearly Kähler manifold and a circle is nearly cosymplectic. On the other hand, unless ξ is parallel, the second harmonic section equation is, in general, given by:

$$\nabla^*\nabla\xi - |\nabla\xi|^2 = -\tfrac{1}{2}J \circ \mathrm{trace}(\bar{\nabla}J \otimes \nabla\xi), \tag{4}$$

and, for nearly cosymplectic structures, can be re-worked into

$$\nabla^*\nabla\xi - |\nabla\xi|^2\xi = -\tfrac{1}{2}[R(F_i, \theta F_i), \theta]\xi,$$

where $\{F_i\}_{i=1,\dots,2n}$ is a local orthonormal frame of the $\mathcal{F}$-distribution.

Following Gray in taking second derivatives of the defining formula of almost contact structures yields the relation:

$$|(\nabla_X\theta)(Y)|^2 + g^2(Y, \nabla_X\xi) = -R(X, Y, X, Y) + R(X, Y, \theta X, \theta Y), \quad \forall X, Y \in TM, \tag{5}$$

which, after refinement and polarisation extends to

$$R(W, X, Y, Z) - R(\theta W, \theta X, \theta Y, \theta Z) = \tfrac{1}{3}[A(W, X, Y, Z) - B(W, X, Y, Z)], \tag{6}$$

where

$$\begin{aligned} A(W, X, Y, Z) = \tfrac{1}{2}[&T(W+Y, Z+X) - T(W+Y, X) + T(W, X) + T(Y, X) \\ &- T(W, Z+X) - T(W+Y, Z) + T(W, Z) + T(Y, Z) \\ &- T(Y, Z+X)], \end{aligned}$$

$$\begin{aligned} B(W, X, Y, Z) = \tfrac{1}{2}[&T(W+Z, X+Y) - T(W, X+Y) - T(Z, X+Y) \\ &- T(W+Z, X) + T(W, X) + T(Z, X) - T(W+Z, Y) \\ &+ T(W, Y) + T(Z, Y)] \end{aligned}$$

and

$$\begin{aligned}T(X,Y) = &-2g(Y,\xi)g(\theta(\nabla_X\theta)(Y),\nabla_X\xi) + 2g(X,\xi)g(\theta(\nabla_X\theta)(Y),\nabla_Y\xi)\\ &- 2g(X,\xi)g(Y,\xi)g(\nabla_X\xi,\nabla_Y\xi) + g^2(Y,\xi)|\nabla_X\xi|^2 + g^2(X,\xi)|\nabla_Y\xi|^2\\ &+ g(Y,\xi)R(\theta X,\theta Y,X,\xi) - g(X,\xi)R(\theta X,\theta Y,Y,\xi).\end{aligned}$$

This formula should be compared with Endo's in [24], much simpler and, hopefully, irrelevant to our case.

Since $T(X,Y) = 0$ for vectors in $\mathcal{F}$, one can deduce that

$$R(\theta W,\theta X,\theta Y,\theta Z) = R(W,X,Y,Z),$$

and therefore that $\mathrm{Ricci}^*(\theta X,\theta Y) = \mathrm{Ricci}^*(X,Y)$, so that the first harmonic section equation is satisfied.

Judicious choices of vectors in (6), $W = F_i, X = \theta F_i, Y = \xi, Z = \theta W$ and then $W = \xi, X = Y = F_i, W \in \mathcal{F}$, will show that, first the vector field ξ must be a unit harmonic vector field i.e.:

$$\nabla^*\nabla\xi - |\nabla\xi|^2 = 0,$$

and, second, that the right-hand side of (4) must vanish. The section must therefore be harmonic and, as by-product, so is ξ, as a unit vector field.

To establish that the third equation is also satisfied and be able to conclude that the section is indeed a harmonic map, we need to apply Equation (5) and combine it with the Bianchi identity to obtain

$$\begin{aligned}R(Y,X,W,Z) - R(Y,X,\theta W,\theta Z) &= -g((\nabla_W\theta)(Z),(\nabla_Y\theta)(X)) \qquad (7)\\ &\quad + g(Y,\nabla_X\xi)g(Z,\nabla_W\xi), \quad \forall X,Y,Z,W \in \mathcal{F}. \qquad (8)\end{aligned}$$

Using this formula, we are able to show that for $X \in \mathcal{F}$

$$g((\bar{\nabla}_{E_i}J)(F_j),[R(E_i,X),\theta]F_j) = 0,$$

where $\{F_i\}_{i=1,\ldots,2n}$ is an orthonormal basis of $\mathcal{F}$ and

$$\{E_i\}_{i=1,\ldots,2n+1} = \{F_i\}_{i=1,\ldots,2n} \cup \{\xi\},$$

while going back to (6) we can establish the ξ-counterpart i.e.:

$$g([R(E_i,\xi),\theta]F_j,(\bar{\nabla}_{E_i}J)(F_j)) = 0, \qquad (9)$$

and conclude that the first term in the harmonic map equation vanishes for all tangent vectors.

Remark 1. One may notice a slight discrepancy in [33] on the curvature term of this equation. First, wrongly stated with the induced curvature of the $\mathcal{F}$-distribution in the introduction and Section 4 of [33], the correct curvature is used in the crucial Proposition 4.2 and Proposition 4.3. Compare with [38, Theorem 3.4] where it is formulated slightly differently but is clear as to the required curvature. An easy direct computation links the two conditions.

For the second term of the harmonic map equation, we first derive, from (5) and the Bianchi identity, an expression for $g(R(W,Z)Y,\xi)$ and repeated use of (5) and (6) for various vectors yields the equation:

$$\begin{aligned}\tfrac{3}{2}g(R(W,Z)Y,\xi) = \tfrac{18}{7}g(\nabla_Y\xi,\theta(\nabla_Z\theta)(W)) + \tfrac{9}{7}g(\nabla_W\xi,\theta(\nabla_Z\theta)(Y))\\ - \tfrac{9}{7}g(\nabla_Z\xi,\theta(\nabla_W\theta)(Y)).\end{aligned}$$

If we take $W = F_i, Y = \nabla_{F_i}\xi$ and $Z = X \in \mathcal{F}$, this formula shows that the second term of the harmonic map equation vanishes for all vector in $\mathcal{F}$, while for the ξ-direction, we need to go back and exploit an intermediate equation

$$-2g((\nabla_W\theta)(Z),\theta\nabla_W\xi) = -2g(R(W,\xi)W,Z) + g(R(W,\xi)\theta W,\theta Z), \tag{10}$$

to conclude that the section corresponding to a nearly cosymplectic structure is a harmonic map.

Theorem 1. *Let (M,θ,ξ,η,g) be a nearly cosymplectic almost contact metric manifold then its corresponding section of the associated homogeneous fibre bundle is a harmonic map.*

This extends the results of [41]on the harmonicity of nearly Kähler structures and of [38] where the cases of $\mathbb{S}^5$ and nearly cosymplectic structures with parallel Reeb vector fields were established.

The next logical task would be to investigate other classes of almost contact structures, and work on normal almost contact structures is in progress, but, in general, there is no reason why they should behave like nearly cosymplectic and be harmonic without extra conditions.

Another interesting question would be the stability of such harmonic sections or maps, but for the moment this remains difficult to fathom and [41] and [5] are the only examples known to us in that direction.

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