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Lorentzian manifolds with transitive conformal group

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Abstract. We study pseudo-Riemanniasn manifolds (M, g) with transitive group of conformal transformation which is essential, i.e. does not preserves any metric conformal to g. All such manifolds of Lorentz signature with non exact isotropy representation of the stability subalgebra are described. A construction of essential conformally homogeneous manifolds with exact isotropy represenatation is given. Using spinor formalism, we prove that it provides all 4-dimensional non conformally flat Lorentzian 4-dimensional manifolds with transitive essentially conformal group.

Keywords: Pseudo-Riemannian conformal structure, conformal group, pseudo-Riemannian conformally homogeneous manifolds, Fefferman space, conformal transformations of Lorentz manifolds.

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1 Introduction

It is well know that any Riemannian manifold which admits an essential group of conformal transformations is conformally equivalent to the standard sphere or the Euclidean space. It is the Lichnerowicz conjecture, proved in compact case by M. Obata and J. Ferrand , and in general case in [A], [A2], [Fer],[F]. On the other hand, there are many examples of pseudo-Riemannian (in particular Lorentzian) manifolds with essential conformal group. Ch. Frances [F], [F1] constructed first examples of conformally essential compact Lorentzian manifolds, M.N. Podoksenov [P] found examples of essential conformally homogeneous Lorentzian manifolds. A local description of Lorentzian manifolds with essential group of homotheties was given in [A1].

Our aim is to study essential conformally homogeneous pseudo-Riemannian manifolds $(M = G/H, g)$, i.e. manifolds with transitive group G of conformal

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transformstions which does not preserves any metric from the conformal class $c = [g]$. We split all such conformal manifolds $(M = G/H, c)$ into two types:

A. Manifolds with non-exact isotropy representation

$$j : \mathfrak{h} \to \mathfrak{co}(V),\, V = \mathfrak{g}/\mathfrak{h} \simeq T_o M$$

of the stability subalgebra $\mathfrak{h}$.

B. Manifolds with exact isotropy representation j.

We give a classification of conformally homogeneous Lorentzian manifolds of type A in any dimension and classification of non conformally flat manifolds of type B in dimension 4.

We will assume that the transitive conformal group G and the stability subgroup H are connected and we identify the pseudo-orthogonal Lie algebra $\mathfrak{so}_{k,\ell} = \mathfrak{so}(V)$ with the space $\Lambda^2 V$ of bivectors.

2 Conformally homogeneous manifolds and associated graded Lie algebra

Let $(M = G/H), g)$ be a conformally homogeneous pseudo-Riemannian manifold of signature $(k, \ell) = (- \cdots -, + \cdots +)$ and $j : H \to CO(V)$ (resp., $j : \mathfrak{h} \to \mathfrak{co}(V)$) the isotropy representation of the stability subgroup H (resp, stability subalgebra $\mathfrak{h}$) of the point $o = eH \in M$ in the tangent space $V = T_o M$. There is a filtration

$$\mathfrak{g}_{-1} = \mathfrak{g} \supset \mathfrak{g}_0 = \mathfrak{h} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 = 0$$

where $\mathfrak{g}_1 := \ker j$. The associated transitive graded Lie algebra is

$$\bar{\mathfrak{g}} := \mathrm{gr}(\mathfrak{g}) = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 = V + \mathfrak{g}^0 + \mathfrak{g}^1 \tag{1}$$

where $V = \mathfrak{g}/\mathfrak{h}$, $\mathfrak{g}^0 := \mathfrak{h}/\mathfrak{g}_1 = j(\mathfrak{h})$ and $\mathfrak{g}^1 = \mathfrak{g}_1 = \ker j$. Transitivity means that $[X, V] = 0$ for $X \in \mathfrak{g}^0 + \mathfrak{g}^1$ implies $X = 0$.

2.1 Example: Standard flat model

Let $\mathbb{R}^{k+1,\ell+1}$ be a pseudo-Euclidean vector space. The projectivisation $S^{k,\ell} = P\mathbb{R}_0^{k+1,\ell+1} \subset P\mathbb{R}^{k+1,\ell+1}$ of the isotropic cone $\mathbb{R}_0^{k+1,\ell+1} \subset \mathbb{R}^{k+1,\ell+1}$ carries a conformally flat conformal structure $[g_{st}]$ of signature (k, ℓ). Moreover, $S^{k,\ell}$ is a conformally homogeneous manifold $S^{k,\ell} = G/H = SO_{k+1,\ell+1}/\mathrm{Sim}(V)$ of the pseudo-orthogonal group $G = SO_{k+1,\ell+1}$ of type A and the stability subgroup H is identified with the group of similarities $\mathrm{Sim}(V) = \mathbb{R}^+ \cdot SO(V) \cdot V$ of the pseudo-Euclidean vector space $V = \mathbb{R}^{k,\ell}$ via stereographic projection.

The associated graded Lie algebra is

$$\mathrm{gr}(\mathfrak{so}_{k+1,\ell+1}) \simeq \mathfrak{so}_{k+1,\ell+1} = V + \mathfrak{co}(V) + V^*, \tag{2}$$

where $V^* = \mathfrak{co}(V)^{(1)} = \{T^\xi, [T^\xi, X] = T^\xi_X = \xi(X)\mathrm{id} + X \wedge \xi\}$ is the first prolongation of $\mathfrak{co}(V)$ and $X \wedge \xi := X \otimes \xi - g^{-1}\xi \otimes gX \in \mathfrak{co}(V)$.

In the case of Riemannian signature $(k,\ell) = (0,n)$, the standard conformal manifold is the conformal sphere $M = S^n = SO_{1,n+1}/\mathrm{Sim}(\mathbb{R}^n)$

2.2 Embedding of $\mathrm{gr}(\mathfrak{g}) = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1$ into $\mathfrak{so}_{k+1,\ell+1}$

For any conformally homogeneous manifold $(M = G/H, [g])$, the associated graded Lie algebra $\bar{\mathfrak{g}}$ has natural embedding into the graded Lie algebra $\mathfrak{so}_{k+1,\ell+1} = V + \mathfrak{so}(V) + V^*$ as a graded subalgebra.
In particular, the conformal structure c in V induces a (may be, degenerate) conformal structure in $\mathfrak{g}^1 \subset V^*$.

The commutative subalgebra $\mathfrak{g}^1$ is a $\mathfrak{g}^0$-invariant subspace of the first prolongation $(\mathfrak{g}^0)^{(1)}$ and can be written as $\mathfrak{g}^1 = T^{V_1^*} \subset T^{V^*}$ such that $T^{V_1^*} \subset \mathrm{Hom}(V, \mathfrak{g}^0)$. In particular, if $\mathfrak{g}^0 \subset \mathfrak{so}(V)$ then $\mathfrak{g}^1 = 0$.

2.3 Subalgebras $\mathfrak{h} = \mathfrak{g}^0 \subset \mathfrak{co}(V)$ with non trivial prolongation

Definition 2.1. A decomposition

$$V = P + E + Q$$

of a pseudo-Euclidean vector space is called **standard** if $P, Q = P^*$ are isotropic k-dimensional subspaces such that $P + Q$ is a non-degenerate subspace and E is the orthogonal complement to $P + Q$.

We set $(P \wedge Q)^0 = \{B \in P \wedge Q, \mathrm{tr}B = 0\} = \{\mathrm{diag}(A, -A^t), A \in \mathfrak{sl}_k(\mathbb{R})\} \simeq \mathfrak{sl}(P) \simeq \mathfrak{sl}(Q)$.

Proposition 2.2. Let $\mathfrak{g}^0$ be a proper subalgebra of the conformal linear Lie algebra $\mathfrak{co}(V)$, $V = \mathbb{R}^{k,\ell}$ with non-trivial first prolongation $\mathfrak{h}^{(1)} \subset T^{V^*}$. Then there is a standard decomposition $V = P + E + Q$ such that $(\mathfrak{h})^{(1)} = T^{g \circ P}$. Moreover, if $k = 1$, $V = \mathbb{R}p + E + \mathbb{R}q$, then

$$\mathfrak{g}^0_{min} := \mathbb{R}(\mathrm{id} - p \wedge q) + p \wedge E \subset \mathfrak{g}^0 \subset \mathfrak{g}^0_{max} := \mathfrak{g}^0_{min} + \mathfrak{so}(E).$$

If $k > 1$, then

$$\gg^0_{min} := \mathbb{R}I + (P \wedge Q)^0 + P \wedge (P + E) \subset \mathfrak{h} = \gg^0 \subset \gg^0_{max} := \gg^0_{min} + \mathfrak{so}(E).$$

where $I = k\mathrm{id} + \mathrm{diag}(-\mathrm{id}, 0, \mathrm{id}) \in \mathfrak{gl}(P + E + Q)$.

The proof follows from

Lemma 2.3. If the first prolongation of a subalgebra $\mathfrak{g}^0 \subset \mathfrak{co}(V)$ contains a non degenerate element T^ξ, $g^{-1}(\xi,\xi) \neq 0$, then $\mathfrak{g}^0 = \mathfrak{co}(V)$.

Corollary 2.4. Let $(M = G/H, c)$ be a conformally homogeneous manifold. If the kernel $\mathfrak{g}_1$ of the isotropy representation contains a non-isotropic element T^ξ then, up to a covering, M is conformally equivalent to the standard conformal model $(S^{k,\ell}, g_{st})$. In particular, any Riemannian conformally homogeneous manifold with a non-exact isotropy representation is conformally equivalent to the conformal sphere.

3 Conformally homogeneous Lorentz manifolds of type A

3.1 Conformally flat conformally homogeneous manifolds associated with graded subalgebra of $\mathfrak{so}_{k+1,\ell+1}$

Let $\mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 = V + \mathfrak{g}^0 + \mathfrak{g}^1$ be a graded subalgebra of the graded Lie algebra

$$\mathfrak{so}_{k+1,\ell+1} = V + \mathfrak{co}(V) + V^*.$$

Assume that $\mathfrak{g}^1 \neq 0$ and denote by G the simply connected Lie group associated with $\mathfrak{g}$ and by H the connected subgroup generated by the subalgebra $\mathfrak{h} = \mathfrak{g}^0 + \mathfrak{g}^1$. We assume that H is a closed subgroup.

Theorem 3.1. The homogeneous manifold $M = G/H$ with the natural conformal structure defined by the $j(H)$-invariant conformal structure in V is a conformally homogeneous manifold of type A. The commutative subgroup generated by commutative subalgebra V has open orbit in M and the manifold M is conformally flat.

Note that, in general, the filtered Lie algebra $\mathfrak{g}$ of a conformally homogeneous manifold is non isomorphic to the associated graded Lie algebra $\bar{\mathfrak{g}}$. In the next section we give an example.

3.2 The standard gradation of $\mathfrak{su}_{k+1,\ell+1}$ and the Fefferman space

Let $V = \mathbb{C}^{k+1,\ell+1} = V^1 + V^0 + V^{-1} = \mathbb{C}e_+ + V^0 + \mathbb{C}e_-$ be a gradation of the complex vector space V. We fix a Hermitian form

$$V \ni Z = ue_+ + z + ve_- = (u,z,v) \mapsto h(Z,Z) = \bar{u}v + \bar{v}u + h^0(z,z)$$

of complex signature $(k+1, \ell+1)$ where $h^0(z,z) = \bar{z}^t \mathbb{E}_{k,\ell} z$ is the Hermitian form in V^0 of complex signature (k,ℓ) with the Gram matrix $\mathbb{E}_{k,\ell} =$

$\mathrm{diag}(-1,\cdots,-1,1,\cdots,1)$. This gradation induces a depth 2 gradation of the special unitary Lie algebra $\mathfrak{g} = \mathfrak{su}_{k+1,\ell+1} = \mathfrak{su}(V) = \mathfrak{aut}(V,h)$ which may be written as

$$\mathfrak{g} \quad = \mathfrak{g}^{-2} \quad + \mathfrak{g}^{-1} \quad + \mathfrak{g}^{0} \quad + \mathfrak{g}^{1} \quad + \mathfrak{g}^{2}$$

Note that this gradation is the ad_D-eigenspace decomposition for $D = \mathrm{diag}(1,0,-1) = e_+ \wedge_J e_-$ where we use notation $x \wedge_J y = x \wedge y + ix \wedge iy$.

In matrix notation, the gradation is given by

$$\mathfrak{su}_{k+1,\ell+1} = \begin{pmatrix} \gg^{0} & \gg^{1} & \gg^{2} \\ \gg^{-1} & \gg^{0} & \gg^{1} \\ \gg^{-2} & \gg^{-1} & \gg^{0} \end{pmatrix} = \left\{ \begin{pmatrix} \lambda + i\mu & -w^* & i\beta \\ z & -\frac{2i\mu}{m}\mathrm{id} + B & w \\ i\alpha & -z^* & -\lambda + i\mu \end{pmatrix} \right\}$$

where $B \in \mathfrak{su}_{k,\ell}$, $z, w \in V^0 = \mathbb{C}^{k,\ell}$, $z^* := \bar{z}^t$, $\alpha, \beta, \lambda, \mu \in \mathbb{R}$, $m = k + \ell$.

An element $L \in \mathfrak{su}_{k+1,\ell+1}$ can be written as

$$L \quad = \alpha Q \quad + E_z \quad + \mu P \quad + \lambda D \quad + B \quad + \hat{E}_w \quad + \beta T$$

where $D = \mathrm{diag}(1,0,-1) = e_+ \wedge_J e_-$ is the grading element,

$$\begin{array}{lll} Q & = e_- \wedge_J e_- \in \mathfrak{g}^{-2}, & T = e_+ \wedge_J e_+ \in \mathfrak{g}^2 \\ E_z & = z \wedge_J e_- \in \mathfrak{g}^{-1}, & \hat{E}_w = w \wedge_J p \in \mathfrak{g}^1 \\ P & = ip \wedge_J q - \frac{2i}{m}\mathrm{id}_{V^0} = i\mathrm{diag}(1, -\frac{2}{m}\mathrm{id}, 1) \in \mathfrak{g}^0 & \end{array}$$

Denote by $\mathcal{P} = G^0 \cdot G^+$ the parabolic subgroup of $G = SU_{k+1,\ell+1}$ generated by the non-negatively graded subalgebra $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^+ = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$.

Then the flag manifold $Fl = G/\mathcal{P} = SU_{k+1,\ell+1}/G^0 \cdot G^+$ is the projectivization of the cone of isotropic complex lines in $\mathbb{C}^{k+1,\ell+1}$. It is diffeomorphic to the sphere if $k = 0$ and it has a natural invariant CR structure. The Fefferman space is defined as the manifold F of real isotropic lines. The group $SU_{k+1,\ell+1}$ acts transitively on $F = SU_{k+1,\ell+1}/H$ with the stability subgroup $H = \mathbb{R}^+ \cdot SU_{k\ell} \cdot G^+ \subset \mathcal{P} = \mathbb{C}^* \cdot SU_{k,\ell} \cdot G^+$. The Fefferman space is the total space of a natural equivariant S^1-fibration

$$F = SU_{k+1,\ell+1}/H = SU_{k+1,\ell+1}/\mathbb{R}^+ \cdot SU_{k+1.\ell+1} \cdot G^+ \to Fl = SU_{k+1,\ell+1}/\mathcal{P}$$

over the flag manifold. The Hermitian metric h of V induces an invariant conformal metric of signature $(2k+1, 2\ell+1))$ in $F = SU_{k+1,\ell+1}/H$.

The solvable non commutative Lie algebra

$$\mathfrak{l} = \mathbb{R}Q + E_{\mathbb{C}^{k,\ell}} + \mathbb{R}P = \left\{ \begin{pmatrix} i\mu & 0 & 0 \\ z & -\frac{2\mu}{n}\mathrm{id} & 0 \\ i\alpha & -z^* & i\mu \end{pmatrix} \right\}$$

generate the subgroup L which has an open orbit in F. We identify $\mathfrak{l}$ with the tangent space T_0F. Then the isotropy representation of the stability subalgebra

$$\mathfrak{h} = \mathbb{R}D + \mathfrak{su}_{\mathfrak{k},\ell} + E_{V^0} + RT$$

is given by

$$\begin{array}{lll} j(D): & \alpha Q + E_z + \mu P & \to 2\alpha Q - E_z + 0 \\ j(C): & \alpha Q + E_z + \mu P & \to E_{Cz} \\ j(\hat{E}_w): & \alpha Q + E_z + \mu P & \to 0 + \alpha E_{iw} + \rho(w,z)P, \end{array}$$

where $C \in \mathfrak{su}_{k,\ell}$, $w^*z = \mathrm{Re}(w^*z) + \mathrm{Im}(w^*z)i = w \cdot z - \rho(w,z)i$

Note that

$$[T, E_z] = \hat{E}_{iz}, \quad [T, Q] = -D, \quad [T, P] = 0,$$

and that $\mathfrak{su}_{k,\ell}$ acts by the tautological representation on $E_{\mathbb{C}^{k,\ell}}$ and $\hat{E}_{\mathbb{C}^{k,\ell}}$. The Fefferman space is an example of conformally homogeneous manifolds of type A, such that the associated filtered Lie algebra $\mathfrak{g}$ is not isomorphic to the graded Lie algebra $\mathrm{gr}(\mathfrak{g})$. Moreover, we have

Theorem 3.2. Let $(M = G/H, c)$ be a homogeneous conformally Lorentzian manifold of type A such that the isotropy algebra $j(\mathfrak{h})$ is a proper subalgebra of $\mathfrak{co}(V)$. If the Lie algebra $\mathfrak{g}$ is not isomorphic to the associated graded Lie algebra $\mathrm{gr}(\mathfrak{g})$, than M is conformally isomorphic to the Fefferman space $F = SU_{1,m+1}/H$ with conformal metric of signature $(1, 2m+1)$.

3.3 Sketch of the proof of Theorem 3.2

3.3.1 Step 1.

The graded Lie algebra $\mathrm{gr}(\mathfrak{g}) = \bar{\mathfrak{g}}$ associated with $M = G/H$ has the form

$$\mathrm{gr}(\mathfrak{g}) = \bar{\mathfrak{g}} = \bar{V} + (\mathbb{R}\bar{D} + \bar{p} \wedge \bar{E} + \bar{\mathfrak{k}} + \mathbb{R}T^{g \circ p}), \;\; \bar{\mathfrak{k}} \subset \mathfrak{so}(\bar{E}) \tag{3}$$

where $\bar{V} = \mathbb{R}\bar{p} + \bar{E} + \mathbb{R}\bar{q}$ is the standard decompositon of the Minkowski vector space with $g(\bar{p}, \bar{q}) = 1$, $\bar{D} := [\bar{q}, T^{g \circ \bar{p}}] = -T^{g \circ \bar{p}}_{\bar{q}} = -\mathrm{id} + \bar{p} \wedge \bar{q}$.

The element $\bar{D}$ defines a depth two gradation

$$\begin{array}{cccccccc} \mathrm{gr}(\mathfrak{g}) = & \mathbb{R}\bar{q}+ & \bar{E}+ & \mathbb{R}\bar{p}+ & \mathbb{R}\bar{D}+ & \bar{\mathfrak{k}}+ & \bar{p}\wedge\bar{E}+ & \mathbb{R}T \\ \mathrm{ad}_{\bar{D}} & -2 & -\mathrm{id} & 0 & 0 & 0 & \mathrm{id} & 2 \end{array}$$

Note that a complementary subspace V to $\mathfrak{h}$ and a complementary subspace $\mathfrak{g}^0$ to $\mathfrak{h}_1$ in $\mathfrak{h}$ defines a decomposition

$$\mathfrak{g} = V + \mathfrak{g}^0 + \mathfrak{h}_1 \tag{4}$$

of $\mathfrak{g}$, consistent with the filtration $\mathfrak{g} \supset \mathfrak{g}_1 = \mathfrak{h} \supset \mathfrak{g}_1 = \mathfrak{h}_1$ and an isomorphism of the graded vector spaces $\mathfrak{g}$ with

$$\mathrm{gr}(\mathfrak{g}) = \bar{V} + \bar{\mathfrak{g}}^1 + \bar{\mathfrak{g}}^2.$$

We will identify these spaces.

3.3.2 Step 2

We can chose the decomposition (4) of the Lie algebra $\mathfrak{g}$ such that the endomorphism ad_D defines a depth two gradation as follows

$$\begin{array}{cccccccc} \mathfrak{g} = & (\mathbb{R}q+ & E+ & \mathbb{R}p)+ & (\mathbb{R}D+ & \mathfrak{k}+ & p \wedge E)+ & \mathbb{R}T \\ \mathrm{ad}_D & -2 & -\mathrm{id} & 0 & 0 & 0 & \mathrm{id} & 2 \end{array}$$

Then $V = \mathbb{R}q + E + \mathbb{R}p$ is a subalgebra, which defines a subgroup of G with open orbit.The assumptions implies that V is not commutative subalgebra.

3.4 Step 3

Analyzing Jacobi identity we prove that $\gg$ is of the following form

$$\begin{array}{ccccccccccccccc} \mathfrak{g} & = & \mathbb{R}q & + & E & + & \mathbb{R}p & + & \mathbb{R}D & + & \mathfrak{k} & + & p \wedge E & + & \mathbb{R}T \\ D & = & -2 & + & -\mathrm{id} & + & 0 & + & 0 & + & 0 & + & \mathrm{id} & + & 2 \\ \mathrm{ad}_p & = & 0 & + & A & + & 0 & + & 0 & + & 0 & + & A & + & 0 \\ \mathrm{ad}_k & = & 0 & + & C & + & 0 & + & 0 & + & \mathrm{ad}_C & + & C & + & 0 \end{array}. \tag{5}$$

Here $k \in \mathfrak{k}$ and $C = \mathrm{ad}_k|_E \in \mathfrak{so}(\mathfrak{E})$.

Moreover, $[e, e'] = 2\rho(e, e')q = 2 < Je, e' > q$ where $J \in \mathfrak{so}(E)$, $[\mathrm{ad}_{\mathfrak{k}}, J] = 0$ and the following relations hold

$$\mathrm{ad}_T : \begin{cases} q \to -D \\ e \to -p \wedge e, \; e \in E \\ p \to 0 \\ D \to -2T \\ \mathfrak{k} + p \wedge E \to 0, \end{cases} \qquad \mathrm{ad}_{p \wedge e} : \begin{cases} q \to -e \\ e' \to < e, e' > p+ \\ + < Je, e' > D + K_{e,e'} \\ p \to -p \wedge Ae \\ D \to -p \wedge e \\ \mathrm{ad}_{\mathfrak{k}} \ni C \to -p \wedge \mathrm{ad}_C e, \\ p \wedge e' \to 2 < Je, e' > T. \end{cases}$$

Here $K_{e.e'} \in \mathfrak{k}$ is a $\mathfrak{k}$-valued symmetric bilinear form on E. The remaining Jacobi may be written as

$$\begin{array}{clcl} (*) & K_{e,e'}e'' - E_{e,e''}e' & = & -2 < Je', e'' > e + < Je, e' > e'' - \\ & & & < Je, e'' > e' - < e, e' > Ae'' + < e, e'' > Ae', \\ (**) & K_{Ae,e'} + K_{e,Ae'} & = & 0, \\ (***) & C(K_{e,e'}) & = & K_{Ce,e'} + K_{e,Ce'},\ C = \mathrm{ad}_k,\ k \in \mathfrak{k}. \end{array}$$

3.4.1 Step 4

The unique solution of (*) is

$$K_{e,e'} = Je \wedge e' - e \wedge Je' + < e, e' > (J - A).$$

The equation (**) implies that either $J = 0$ or J and $A = \lambda J$ are proportional non degenerate skew-symmetric endomorphism and $J^2 = -p\mathrm{id}$ for $p > 0$. Rescaling metric, we may assume that $J^2 = -1$. In the case $J = 0$, one can show that $\mathfrak{g}$ is isomorphic to $\mathrm{gr}\,\mathfrak{g}$. The equation (***) shows that $\mathrm{ad}_{\mathfrak{k}}|_E \subset \mathfrak{u}(E) = \{C \in \mathfrak{so}(E),\ [C, J] = 0\}$. Then one can check that $\gg \simeq \mathfrak{su}_{1,m+1}$ where $n := \dim M = 2(m + 2)$ and $M \simeq F = SU_{1,n+1}/\mathbb{R}^+ \cdot SU_n \cdot Heis(\mathbb{C}^n)$.

3.5 The curvature of the Fefferman space and the Cahen-Wallach symmetric spaces

Recall that all indecomposable Lorentzian symmetric spaces are exhausted by the spaces of constant curvature and the Cahen-Wallach symmetric spaces $CW_S^{1,n-1}$ Let

$$V = \mathbb{R}^{1,n-1} = \mathbb{R}q + E + \mathbb{R}p$$

be the standard decomposition of the Minkowski space and e_i an orthonormal basis of E. Then the contravariant curvature tensor R_S of the Cahen-Wallach space is given by

$$R_S = \sum_{i=1}^{n-2} q \wedge Se_i \vee q \wedge e_i.$$

It defines a Lie algebra with a symmetric decomposition

$$\mathfrak{g} = \mathfrak{h} + V = q \wedge E + V \subset \mathfrak{so}(V) + V$$

with the Lie bracket $[x, y] = -R(x, y) \in \mathfrak{h} = q \wedge E,\ x, y \in V$. The Cahen-Wallach space $CW_S^{1,n-1} = G/H$ the symmetric manifold associated with this symmetric decomposition. It is conformally flat if and only if $S = \lambda\mathrm{id}$, see [G].

Theorem 3.3. For any point x of the Fefferman space $(F,[g])$ there is a metric $g \in [g]$ whose contravariant curvature tensor at x coincides with the curvature tensor of the conformally flat Cahen-Wallach space. In particular, the Fefferman space is conformally flat.

4 Petrov classification of Weyl tensors

4.1 Spinor formalism

To describe 4-dimensional Lorentzian conformally homogeneous manifolds of type B, we recall a spinor description of Weyl tensor of a Lorentzian 4-manifold.

Let $\mathbb{S}$ be the complex 2-space with the symplectic form $\omega = e_- \wedge e_+$ where e_+, e_- is a symplectic basis of $\mathbb{S}$ and we identify $\mathbb{S}$ with the dual space $\mathbb{S}^*$. $\omega(e_+, e_-) = 1$ which is identified with the dual space The associated standard basis $E_- = E_{21}$, $E_0 = E_{11} - E_{22}, E_+ = E_{12}$ of the unimodular Lie algebra $\mathfrak{sl}_2(C)$ defines a gradation

$$\mathfrak{sl}_2(C) = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 = \mathbb{C}E_- + \mathbb{C}(E_0) + \mathbb{C}E_+.$$

The space $\mathbb{S} \otimes \bar{\mathbb{S}}$ of Hermitian bilinear forms has the basis $e_i \otimes \bar{e_j}$, $i, j \in \{+,-\}$ where $\bar{e}_+, \bar{e}_-$ is the basis of the conjugated vector space $\bar{\mathbb{S}} = \bar{\mathbb{C}}^2 = \{\bar{z_+}\bar{e}_+ + \bar{z_-}\bar{e_-}\}$. If $j : a \otimes \bar{b} \mapsto (a \otimes \bar{b})^* = b \otimes \bar{a}$ is the Hermitian conjugation, then the fix point space $V = (\mathbb{S} \otimes \bar{\mathbb{S}})^j$ of j is the space of Hermitian symmetric matrices.

We may write

$$V = \quad \{X = uE_{1\bar{1}} + (zE_{1\bar{2}} + \bar{z}E_{2\bar{1}}) + vE_{2\bar{2}}\} = \Big\{ \begin{pmatrix} u & z \\ \bar{z} & v \end{pmatrix}, u, v \in \mathbb{R}, z \in \mathbb{C} \Big\}.$$

We set $p = 2E_{1\bar{1}}$, $q = 2E_{2\bar{2}}, E_z = zE_{1\bar{2}} + \bar{z}E_{2\bar{1}}$ such that $E = \{E_z, z \in \mathbb{C}\} \simeq \mathbb{C}$ and denote by

$$V = V^{-1} + V^0 + V^1 = \mathbb{R}q + E + \mathbb{R}p$$

the associated gradation of V. The determinant defines the Lorentz metric in V :

$$g(X, X) = X \cdot X = \det X = uv - z\bar{z} = uv - x^2 - y^2,\ z = x + iy$$

such that

$$p^2 := p \cdot p = q^2 = 0,\ p \cdot q = 2,\ e_1^2 = e_i^2 = -1, e_1 \cdot e_i = 0, (\mathbb{R}p + \mathbb{R}q) \perp E$$

where $e_1 := E_1, e_i = E_i$.
For $X, Y \in V$ we denote by $X \wedge Y : Z \mapsto < Y, Z > X - < X, Z > Y$ the

associated endomorphism from $\mathfrak{so}(V)$. The group $SL(\mathbb{S})$ acts isometrically in V by

$$\varphi : SL(\mathbb{S}) \ni A \mapsto \phi(A) : X \mapsto AXA^*.$$

The associated isomorphism of Lie algebras $\mathfrak{sl}(\mathbb{S})$ and $\mathfrak{so}(V)$ is given by

$$\begin{array}{ll} \varphi(E_0) = 2p \wedge q & \varphi(iE_0) = 2e_1 \wedge e_i, \\ \varphi(E_+) = \sqrt{2}e_1 \wedge p & \varphi(iE_+) = -\sqrt{2}e_i \wedge p, \\ \varphi(E_-) = \sqrt{2}e_1 \wedge q & \varphi(iE_-) = -\sqrt{2}e_i \wedge q. \end{array}$$

4.2 Spinor description of the space $\mathcal{R}_0(V)$ of Weyl tensors

Recall that the space of Weyl tensors is defined by

$$\mathcal{R}_0(V) = \{W \in \mathrm{Hom}(\Lambda^2 V, \mathfrak{so}(V)), \mathrm{cycl}\, W(X \wedge Y)Z = 0,$$

$$\mathrm{tr}(X \to W(X, \cdot)X) = 0, \forall X, Y, Z \in V\}.$$

Recall that $\Lambda^2 V \simeq \mathfrak{sl}_2(\mathbb{C}) \simeq \mathbb{C}^3$ where the complex structure in $\Lambda^2 V$ is defined by Hodge star operator. Note that $V^{\mathbb{C}} = \mathbb{S} \otimes \bar{\mathbb{S}}$ and $\Lambda^2 V^{\mathbb{C}} = S^2\mathbb{S} \otimes \bar{\omega} + \omega \otimes S^2(\bar{\mathbb{S}})$ where $\omega, \bar{\omega}$ are symplectic forms in $\mathbb{S}$ and $\bar{\mathbb{S}}$.
We denote by $S^2_0(\Lambda^2(V))$ the 5-dimensional complex space of trace free symmetric complex endomorphisms of the complex space $\Lambda^2(V) = \mathbb{C}^3$.

Theorem 4.1. (A. Petrov, R. Penrose) There exists a natural isomorphisms of $\mathfrak{sl}_2(\mathbb{C})$- modules

$$\mathcal{R}_0(V) \simeq S^2_0(\Lambda^2(V)) = S^4(\mathbb{S}^*).$$

The covariant form $g \circ W$ of the Weyl tensor W associated with symmetric 4-form φ is given by

$$W_\varphi = \varphi \otimes \bar{\omega}^2 + \omega^2 \otimes \bar{\varphi}.$$

4.3 Petrov classification of Weyl tensors

Since any symmetric 4-form $\phi \in S^4(\mathbb{S})$ can be factorized as $\phi = \alpha\beta\gamma\delta$ into a product of four linear forms $\alpha, \beta, \gamma, \delta$, we get the following classification of Weyl tensors:
Type (4) or (N) $\phi = \alpha^4$; Type (31) or (III) $\phi = \alpha^3\beta$;
Type (22) or (D) $\phi = \alpha^2\beta^2$; Type (211) or (II) $\phi = \alpha^2\beta\gamma$;
Type (1111) or (I) $\phi = \alpha\beta\gamma\delta$,
where $\alpha, \beta, \gamma, \delta$ are different linear forms in $\mathbb{S}$. Each linear form α in spinor space $\mathbb{S}$ up to a scaling is defined by its kernel $\alpha = 0$ which is a point in to projective line $\mathbb{C}P^1 = S^2$. So up to a complex factor, the 4-form ϕ is determined by 4 points on the conformal sphere. For a symmetric 4-form ϕ we denote by

$\mathfrak{aut}(\phi)$ (respectively,$\mathfrak{conf}(\phi)$) the Lie subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ which preserves ϕ (respectively, preserves ϕ up to a complex factor).

Proposition 4.2. i) $\mathfrak{conf}(\phi) = 0$ for a form of types (1111), (211);
ii) $\mathfrak{aut}(\phi) = 0$ for a form of types different from $(2,2)$ and (4);
iii) $\mathfrak{conf}(\phi) = \mathbb{C} = \mathbb{C}E_0$ for a form of types (31);
iv) $\mathfrak{conf}(\phi) = \mathfrak{aut}(\phi) = \mathbb{C}E_0$ for type (22) ;
v) $\mathfrak{conf}(\phi) = \mathbb{C}E_0 + \mathbb{C}E_+$, $\mathfrak{aut}(\phi) = \mathbb{C}E_+$ for type (4).
In particular, only the form of type (31) and (4) admits a conformal transformation which is not an authomorphism.

Proof. There exists unique conformal transformation of the sphere which transform three different points into another three different points. This implies the first claim. If $\phi = \alpha^4$, then the stabilizer of ϕ in $\mathfrak{sl}_2(\mathbb{C})$ is the same as the stabilizer of the 1-form α. We may assume that $\alpha = e^*_- = (0,1)$. Then $\mathfrak{aut}(\phi) = \mathbb{C}E_+$ and $\mathfrak{conf}(\phi) = \mathbb{C}E_0 + \mathbb{C}E_+$. If $\phi = \alpha^2\beta^2$ or $\alpha^3\beta$, we may assume that α, β are basic 1-forms and then the stabilizer of $\mathbb{C}\phi$ will be the diagonal subalgebra. In the first case it preserves ϕ. QED

5 Conformally homogeneous manifolds of type B

In this section we describe a class of conformally homogeneous pseudo-Riemannian manifolds of type B and prove that all 4-dimensional conformally homogeneous non conformally flat manifold belong to this class.

Proposition 5.1. Let $M = G/H$ be a conformally homogeneous manifold of type B. Then the isotropy Lie algebra $j(\mathfrak{h}) \subset \mathfrak{co}(V)$, $V = T_0M$ has a decomposition

$$j(\mathfrak{h}) = \mathbb{R}D + \mathfrak{l},$$

where $\mathfrak{l} \subset \mathfrak{so}(V)$ is an ideal of $\mathfrak{h}$ an the endomorphism $D = \mathrm{id} + C$, $C \in \mathfrak{so}(V)$ is a non trivial homothety.

Proof. Indeed,assume that $j(\mathfrak{h}) \subset \mathfrak{so}(V)$. Then the isotropy group $j(H)$ preserves a metric g_0 in the tangent space $V = T_0M$ which can be extended by left translations to G-invariant metric g on the homogeneous space $M = G/H$. Hence, the conformal group G is not essential. QED

5.1 A construction of pseudo-Riemannian conformally homogeneous manifold of type B

Let $V = \mathbb{R}q + E + \mathbb{R}p$ be a standard decomposition of a pseudo-Euclidean vector space $(V, g =< .,. >)$ of signature (k,ℓ). The homothety $D = \mathrm{id} + q \wedge p \in$

$\mathfrak{co}(V)$. defines a gradation $V = \mathbb{R}p + E + \mathbb{R}q = V^0 + V^1 + V^2$. A non-degenerate 2-form $\omega(x,y)$ in E defines the structure of the Heisenberg Lie algebra with the center $\mathbb{R}q$ and the bracket $[x,y] = \omega(x,y)q$, $x,y \in E$ in $\mathfrak{heis}(E) = E + \mathbb{R}q$. Moreover, an endomorphism $A \in \mathrm{End}(E)$ with

$$(A \cdot \omega)(x,y) := \omega(Ax,y) + \omega(x,Ay) = \lambda\omega(x,y)$$

is a derivation of this algebra and defines the structure of a graded Lie algebra

$$V = V^0 + V^1 + V^2 = \mathrm{R}p + \mathfrak{heis}(E),$$

such that $\mathrm{ad}_p q = \lambda q$, $\mathrm{ad}_p|_E = A$, with the grading element $D = \mathrm{id} + q \wedge p$. Denote by G the Lie group generated by the Lie algebra $\gg = \mathbb{R}D + V$ and by H the closed subgroup generated by the subalgebra $\mathbb{R}D$.

Proposition 5.2. The metric g in V defines an invariant pseudo-Riemannian conformal structure in the manifold $M = M(\lambda, \omega, A) = G/H$. The manifold M is a conformally homogeneous manifold of type B.
The curvature operator of the manifold M is given by

$$R_{pq} = R_{qx} = 0,\ R_{px} = (A^a A^s - A^s A - JA^s)x \wedge q,\ x \in E$$

where $g^{-1} \circ \omega = 2J$ and $A^a = \frac{1}{2}(A + A^t)$, $A^s = \frac{1}{2}(A - A^t)$ are skew-symmetric and symmetric parts of A. In particular, in general the manifold M is not conformally flat.

5.2 Classification of Lorentzian 4-dimensional conformally homogeneous manifolds of type B

Theorem 5.3. Any conformally homogeneous 4-dimensional Lorentzian manifold of type B which is not conformally flat is conformally isometric to a manifold $M(\lambda, \omega, A)$.

The proof is based on

Lemma 5.4. If a conformally homogeneous Lorentzian 4-manifold $M^4 = G/H$ of type B is not conformally flat, then the isotropy Lie algebra contains the homothety $D = \mathrm{id} + q \wedge p$ with respect to an approprite standard decomposition $V = \mathbb{R}p + E + \mathbb{R}q$ of the tangent space $V = T_o M$.

Proof. Let $D = \mathrm{id} + C \in j(\mathfrak{h})$ be a non trivial homothetic endomorphism, $C \in \mathfrak{so}(V)$. By assumption,the Weyl tensor $W \neq 0$. Since $\mathrm{id} \cdot W = -2$ and $D \cdot W = (\mathrm{id} + C) \cdot W = 0$, $C \cdot W = 2W$. Then $C \cdot \phi = 2\phi$, where $\phi \in S^4(\mathbb{S}^2)$ is 4-form which represents W . Proposition 4.2 shows that the 4-form ϕ has Pertov type (4) or (31) and $C = -\frac{1}{2}E_0 + bE_- \in \mathfrak{sl}_2(\mathbb{C})$. Changing the basis, we may assume that $b = 0$. Then $\varphi^{-1}(C) = q \wedge p \in \mathfrak{so}(V)$ and $D = \mathrm{id} + q \wedge p$. QED

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