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SEPARATION AXIOMS BY SIMPLE EXTENSIONS (*)

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Sunto. *Nella prima parte studieremo, per* $i = 0,1,2,2\frac{1}{2}$, *gli spazi topologici, che chiameremo* QT_i, *che verificano gli assiomi di separazione* T_j *per* $j < i$ *e che hanno un'estensione semplice* T_i.

Nella seconda parte prenderemo in considerazione topologie, su un sostegno fissato S, *che sono* T_j, $j < i$, *e non-*T_i *massimali* (MNT_i).

Daremo alcuni esempi e proveremo alcune proprietà delle topologie QT_i *e delle topolgie* MNT_i.

1. Given a topological space (S,τ) and a subset $X \subseteq S$, we shall denote by $\tau(X) = \{A \cup (B \cap X)/A,B\in\tau\}$ the simple extension of τ by X.

We shall denote by clX(intX) the closure (the interior) of X in (S,τ) and by $cl_{\tau'}X$ ($int_{\tau'}X$) the closure (the interior) of X with respect to any other topology τ' on S. $\mathscr{T}(x)$ ($\tau(x)$) will mean the family of (open) neighbourhoods of $x\in S$ in the topology τ . We shall call fundamental neighbourhood system of x in the topological space

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(S,τ) a basis of the filter $\mathcal{F}(x)$.

If $X \subseteq S$, CX will be the complement of X in S while we shall denote by $C_Y X$ or $Y-X$ the complement of X in Y when $X \subseteq Y \subseteq S$.

The topological definitions we need follow W.J. Thron's Topological structures [6] except that we shall denote by $T_{2\frac{1}{2}}$ the T_{2a} axiom.

Let R be a topological property which is preserved under expansions, i.e. such that (S,τ') is R whenever (S,τ) is and $\tau' \supseteq \tau$.

Definition 1. We shall call a topological space (S,τ) quasi-R if there exists a subset $X \subseteq S$ which determines a R simple extension $(S,\tau(X))$.

Of course an expansion of a quasi-R topology on a set S is a quasi-R topology on S too, because $\tau(X) \subseteq \tau'(X)$ if $\tau \subseteq \tau'$ are topologies on S and $X \subseteq S$.

Now we are going to investigate the existence and some properties of quasi-T_i spaces, $i = 0,1,2,2\frac{1}{2}$, providing examples and stating relations with separation axioms $T_0,T_1,T_2,T_{2\frac{1}{2}}$.

The following examples show the existence of non-quasi-T_0 and of quasi-T_i spaces.

Example 1. The indiscrete topology ω on a set S with three or more points is not quasi-T_0: trivially $\omega(X) = \{\emptyset,X,S\}$ fails to be T_0 if $X \subseteq S$.

Example 2. Let $S = \{x,x',y,y'\}$ and $\tau=\{\emptyset,\{x,x'\},\{y,y'\},S\}$. (S,τ) is not a T_0 space but $\tau' = \tau(\{x,y\})$ is a T_0 topology on S, so (S,τ) is quasi-T_0.

Furthermore $\tau'(\{x',y'\})$ is the discrete topology on S and then (S,τ') is quasi-T_i, $i = 1,2,2\frac{1}{2}$.

PROPOSITION 1. *If* (S,τ) *is quasi*-T_1 *then* (S,τ) *is* T_0.

Proof. Let $\tau' = \tau(X)$ be a T_1 simple extension of τ; trivially $\{A\cap X/A \in \tau(a)\}$ is a fundamental neighbourhood system of a point $a \in X$ while $\tau(b)$ is a fundamental neighbourhood system of a point $b \notin X$.

Now let $x \neq y$ be two points of S; if $x\notin X$, an open neighbourhood $U \in \tau(x)$ exists such that $y\notin U$; if $x,y \in X$, $A\cap X \in \tau'(x)$ and $y \notin A\cap X$, then $A \in \tau(x)$ and $y \notin A$.

While every quasi-T_i space, $i = 1,2,2\frac{1}{2}$, is T_0, example 2 shows that a quasi-T_i space is not necessarily T_1 even for $i > 0$.

Definition 2. Let $i,j = 0,1,2,2\frac{1}{2}$. We say that (S,τ) is QT_i if it is quasi-T_i and it is T_j if $j < i$.

Trivially (S,τ) is QT_0 (QT_1) iff it is quasi-T_0 (quasi-T_1).

If we denote by T_i (QT_i), $i = 0,1,2,2\frac{1}{2}$, the class of T_i (QT_i) topological spaces, then the following proper inclusions hold

$$QT_0 \supset T_0 \supset QT_1 \supset T_1 \supset QT_2 \supset T_2 \supset QT_{2\frac{1}{2}} \supset T_{2\frac{1}{2}} .$$

Example 2 proves the existence of non-T_0 QT_0 spaces and non-T_1 QT_1 spaces. Further examples provide us the truth of other strict inclusions.

Example 3. If $S = \{a,b,c\}$ and $\tau = \{\emptyset,\{a\},\{a,b\},S\}$, then (S,τ) is a T_0 but not a QT_1 space.

Example 4. We remark that the particular point topology and the escludedpoint topology on a set S belong to QT_1 but not to T_1.

In the first case we obtain a T_1 simple extension by the complement of the particular point; in the second case we obtain the same result by the set containing the only excluded point.

Nevertheless non-T_1 QT_1 spaces having non-discrete T_1 simple extensions exist: is fact let us consider an infinete set S, take two distinct points $a \neq a$ in S and call $A \subseteq S$ open iff CA if finite and $a \in A \Longrightarrow a' \in A$; the simple extension of such a topology on S by the subset $C\{a'\}$ is then the cofinite topology on S.

Example 5. Let τ be the cofinite topology on the infinite set S. We shall prove that (S,τ) is a non-QT_2 T_1 space.

If $X \subseteq S$, $X \notin \tau$ (i.e. S-X is infinite) and $\tau' = \tau(X)$ let us take two distinct points $x \neq y$ in S-X and two open neighbourhoods $U \in \tau'(x)$, $U = A \cup (B \cap X)$ and $V \in \tau'(y)$, $V = A' \cup (B' \cap X)$; we have $x \in A \in \tau$ and $y \in A' \in \tau$ so $U \cap V \supseteq A \cap A' \neq \emptyset$ and $\tau(X)$ is not a T_2 topology.

Example 6. (Modified Fort Space, see [4]). Let S be the union of an infinite set N with a set having only two distinct points $x \neq y$ not belonging to N; consider the topology τ on S whose open sets are the cofinite sets in S and the subsets of N.

Trivially (S,τ) is a T_1 but not a T_2 spaces; furthermore we see that $\tau(\{x,y\})$ is the discrete topology and consequently (S,τ) belong to QT_2.

Example 7. (Relatively Prime integer topology, see [4]). Let S=Z be the set of positive integers and, if $a,b \in S$, $(a,b) = 1$ let us consider $U_a(b)=\{b+na \in S/n\in Z\}$; the family $\mathcal{B} = \{U_a(b)/a,b\in S,(a,b)= = 1\}$ is a basis of a topology τ on S.

It is well-know that (S,τ) is a T_2 but not a $T_{2\frac{1}{2}}$ space; we shall now prove that τ is not a $QT_{2\frac{1}{2}}$ topology on S.

Let $\tau' = \tau(X)$ be a simple extension of τ . We recall that the family $\{U_p(x) \cap X/(p,x)=1, p\in Z^+\}$ is a fundamental neighbourhood system of $x \in X$ in τ' as well as $\{U_p(y)/(p,y) = 1,\ p \in Z^+\}$ is a fundamental neighbourhood system of $y \in S-X$ in τ'.

First we suppose that for each $x\in X$ a positive integer h exists such that $hx \in S-X$, and we consider two distinct points $x \neq y$ in S-X and two fundamental neighbourhoods $U_p(x)$, $U_p(y)$ of x,y respectively. Then we have

$$cl_{\tau'}(U_p(x)) = cl(U_p(x)) \cap (CX \cup cl(U_p(x) \cap X))$$

$$cl_{\tau'}(U_q(y)) = cl(U_q(y)) \cap (CX \cup Cl(U_q(y) \cap (X)).$$

As the closure in τ of $U_p(x)$ (of $U_q(y)$) contains all multiples of p (of q) in S and CX contains at least one multiple of $[p,q]$, we have $cl_{\tau'}U_p(x) \cap cl_{\tau'}U_q(y) \neq \emptyset$ and consequently $\tau(X)$ is not a T_2 topology on S.

Now we suppose that each positive integer multiple of x is in X for some x in X. We consider such an x and we observe that if $k \in Z^+$, $p \in Z^+$ and $(p,kx) = 1$, then

$$U_p(kx) \cap X \supseteq \{(k-mp)x \in S/m \in Z\} = A_{pk} .$$

It is easily seen that $\{ptx/t\in Z^+\} \subseteq clA_{pk}$; in fact given any integer a prime with ptx we have

$$U_a(ptx) \cap A_{pk} \neq \emptyset \iff \exists n,m \in Z \text{ such that } an+pxm+(pt-k)x = 0.$$

This is surely true since a and px are relatively prime.

If we take $h \neq k$ in Z^+ and p,q in Z^+ prime with kx, hx respectively then the closures

$$cl_{\tau'}(U_p(kx) \cap X) = cl(U_p(kx) \cap X) \supseteq A_{pk} U\{ptx \in S/t\in Z\}$$

$$cl_{\tau'}(U_q(hx) \cap X) = cl(U_q(hx) \cap X) \supseteq A_{qh} U\{qrx \in S/r \in Z\}$$

have a non-empty intersection and $\tau(X)$ is not $T_{2\frac{1}{2}}$ in this case too.

We remark that also the coarser prime integer topology (see [4] again) is a T_2 but not a $QT_{2\frac{1}{2}}$ topology.

Example 8. The Double Origin Topology, the Semplified Arens Square and the Minimal Hausdorff Topology considered in [4] provide examples of $QT_{2\frac{1}{2}}$ spaces. We give a detailed description of the third case only.

Let us consider the topological product of $A=\{1,2,3,..,\omega,...,-3,-2,-1\}$ linearly ordered with the interval topology and the discrete topological space (Z^+,α); let (S,τ) be obtained from such topological product by adding two ideal points a and -a whose fundamental neighbourhoods are of the kind

$$\{a\} \cup \{(i,j)/i < \omega,\ j > n\} \quad \text{and} \quad \{-a\} \cup \{(i,j)/i > \omega,\ j > m\}$$

respectively.

(S,τ) is a minimal Hausdorff non-$T_{2\frac{1}{2}}$ space while the simple extension of τ by $X = \{a\} \cup \{(i,j)/\ i \leq \omega,\ j > 0\}$ is a $T_{2\frac{1}{2}}$ topology on S.

In order to give a characterization of QT_1 topological spaces we shall denote by T the set of those points which are in the closure of some other distinct point in the space (S,τ) i.e.

$$T = \bigcup_{x\in S} ((cl\{x\}) - \{x\}).$$

Trvially $(S,\tau) \in T_1$ iff $T = \emptyset$.

LEMMA 1. $X \subseteq S,\ (S,\tau(X)) \in T_1 \Longrightarrow T \subseteq X$.

Proof. If there were two distinct points $x \neq y$ such that $y \in cl\{x\}$ and $y \notin X$, then $\tau(y)$ would be a fundamental neighbourhood system of y in $\tau' = \tau(X)$ and it would follow from $y \in cl\{x\}$ that every neighbourhood in $\tau'(y)$ contains x.

PROPOSITION 2. $(S,\tau) \in QT_1 \Longleftrightarrow (T,\tau_{|T}) \in T_1$.

Proof. $\Longrightarrow$) Let X be a subset of S such that $(S,\tau(X)) \in T_1$; it follows from lemma 1 that $T \subseteq X$ and consequently $\tau_{|T} = \tau(X)_{|T}$; so $(X,\tau_{|T}) \in T_1$.

$\Longleftarrow$) It is easily seet that $(S,\tau') \in T_1$, if $\tau' = \tau(T)$.

First we consider $a,b \in T$, $a \neq b$; as each open set in $\tau_{|T}$ belongs to τ' and $\tau_{|T}$ is a T_1 topology, then $U \in \tau'(a)$ and $V \in \tau'(b)$ exist such that $b \notin U$ and $a \notin V$.

If $a \in T$ and $b \in S-T$, we have $b \notin (cl\{y\}) - \{y\}$ for each $y \in S$;

in particular $b \notin (\mathrm{cl}\{a\})-\{a\}$ and, of course, $b\notin \mathrm{cl}\{a\}$; $U\in\tau(b) \subseteq \tau'(b)$ exists such that $a\notin U$. On the other hand we can condider $A \in\tau(a)$ and the open set $(A\cap T) \in\tau'$ is an open neighbourhood of a in τ' which does not contain b.

The case $a,b \in S-T$, $a\neq b$, is trivial.

Remarks. If the set T defined above is a non-trivial open set in (S,τ), then (S,τ) cannot be a QT_1 space.

The set T in the spaces of example 4 is closed. This is not true for all spaces. For instance let us consider an infinite set S and fix $a\in S$; let the open sets in τ be the cofinite subsets of S containing a. (S,τ) is a T_0 space and its subspace $T = S - \{a\}$ is a T_1 space; (S,τ) is then a QT_1 space and $T = S - \{a\}$ is not a closed subset.

PROPOSITION 3. *If* (S,τ) *is a* QT_0 *regular space which has a* T_0 *regular simple extension, then* (S,τ) *is a* T_2 *space.*

Proof. Let $(S,\tau(X))$ be a T_0 regular simple extension; then $(S,\tau(X))$ is a T_2 space and consequently $(S,\tau) \in QT_2$. If follows from proposition 1 that (S,τ) must be T_0 and the assertion is true.

Remark. The existence of a T_0 regular simple extension is an essential hypothesis in proposition 3; the space $S = \{a,b\}$ with the indiscrete topology is a QT_0 regular non-T_2 space.

2. We shall denote by $\mathscr{L}(S)$ the complete lattice of topologies on a given set S under set inclusion. The least element is the indiscrete topology and the greatest element is the discrete topology.

In agreement with [2] we shall call antiatom of $\mathscr{L}(S)$ any topology on S which is coarser than itself and the discrete topology only.

It is well known that the antiatioms are the ultratopologies already described in [2].

Definition 3. If R is a topological property which is preserved under expansions, we shall say that (S,τ) is a maximal non-R (MNR) space if it is not a R space but every proper expansion of τ is a R topology.

Remarks. (S,τ) is MNR iff τ is not R but $\tau(X)$ is R for each $X \notin \tau$. Furthermore it is obvious that a MNR space is a quasi-R space. Finally we remark that the spaces given in examples 2,4,6,8 are QT_i but are not MNT_i spaces, $i=0,1,2,2\frac{1}{2}$ respectively.

MNT_i spaces, $I=0,1,2,3$, were considered by Thomas in [5]; here we shall give more detailed characterizations and properties of MNT_1 and MNT_2 spaces and a proof that no T_2 $MNT_{2\frac{1}{2}}$ space exists.

If follows trivially from proposition 1 that every MNT_i, $i>0$, space is a T_0 space. Furthermore the following results are easily seen to be true and can be found in [2] and [5].

Let (S,τ) be a non-T_1 topological space. Then (S,τ) is MNT_1 iff (S,τ) is an antiatom in $\mathscr{L}(S)$.

If (S,τ) is MNT_1 then (S,τ) is MNT_2, $MNT_{2\frac{1}{2}}$, MNT_3.

Every regular antiatom in $\mathscr{L}(S)$ is a T_1 space.

Thomas proved in [5] that every non-T_0 (non-T_1, non-T_2) topology is coarser than a MNT_0 (MNT_1, MNT_2) topology; moreover each T_1 non-T_2 topology is included in a MNT_2 topology which is a MNT_3 topology too.

We shall prove a characterization of T_1 MNT_2 spaces and obtain as an immediate consequence (example 9) the construction of a T_1 MNT_2 space already given by Thomas in [5] theorem 1.

Next we realize that all T_1 MNT_2 topologies are the ones we describe in example 9.

PROPOSITION 4. *Let* (S,τ) *be a* T_1 *space.*
(S,τ) *is* MNT_2 *iff the following conditions hold*

1) *Only two distinct points* $x \neq y$ *exist in* S *such that each neighbourhood of* x *intersect each neighbourhood of* y.

2) *The subspace* $S - \{x,y\}$ *has the discrete topology.*

3) $X \subseteq S$, $X \notin \tau$, $x \in X$ $(y \in X) \Longrightarrow CX \cup \{y\} \in \tau$ $(CX \cup \{x\} \in \tau)$.

Proof. The given conditions are necessary.

Trivially distinct points $x \neq y$ exist which verify the first condition; if there were another point z, $z \neq x$ and $z \neq y$, such that the neighbourhoods of z and those of y intersect each other, then y and z would be non-separated points in $\tau(\{x\})$, which contradicts the hypothesis since $\{x\}$ is a non-open set.

Every point distinct from x and y must be open; otherwise x and y should be non-separated points in $\tau(\{a\})$, if $\{a\}$ were a non-open subset of S.

In order to verify the third condition, we first prove that if $X \subseteq S$ and $X \notin \tau$ then $CX \in \tau$.

Indeed let $X \subseteq S$ be neither open nor closed so that $\tau(X)$ and $\tau(CX)$ are T_2 topologies; by 1) and 2) X contains only one (say x) of the points x,y and consequently we can find $A \in \tau(x)$, $B \in \tau(y)$

such that $(A\cap X)\cap B=\emptyset$ i.e. $A\cap B\subseteq CX$. Then if $U\in\tau(x)$, $V\in\tau(y)$ we have $(U\cap V)\cap CX\neq\emptyset$: othewise we should have $U\cap V\subseteq X$ and the open neighbourhoods $(U\cap A)\in\tau(x)$, $(V\cap B)\in\tau(y)$ would separate x and y in τ. Eventually we obtain, from the assumption that X is neither open nor closed, that for every $U\in\tau(x)$ and $V\in\tau(y)$ one must have $U\cap(V\cap CX)=(U\cap V)\cap CX\neq\emptyset$ which contradicts the Hausdorff character of the topology $\tau(CX)$.

Finally we suppose that $X\subseteq S$, $X\notin\tau$, $x\in X$. If $y\notin X$ be condition 3) is trivial; if $y\in X$ and $CX\cup\{y\}\notin\tau$ we contradict the hypothesis by finding two disjoint neighbourhoods of x in τ in the following way: we consider $U\in\tau(x)$ and $V\in\tau(y)$ such that $U\cap X$ and V separate x and y in the T_2 topology $\tau(X)$; then $(U\cap V)\cap X=\emptyset$; since $(X-\{y\})\in\tau(x)$ we have $U'=(X-\{y\})\cap U\in\tau(x)$, $V\in\tau(y)$ and $U'\cap V=\emptyset$.

The conditions are sufficient.

By 1) (S,τ) cannot be a T_2 space.

Now let X be a non-open subset of S containing x; $CX\cup\{y\}$ is an open neighbounhood of y in τ wich does not intersect the open neighbourhood $X-\{y\}$ of x in $\tau(X)$; hence $\tau(X)$ is a T_2 topology.

Remark. The T_1 axiom is only requested to show that conditions 1), 2), 3) in proposition 4 are sufficient to have a MNT_2 space.

Example 9. Let $S=\{x,y\}\cup N$ where $x,y\notin N$ and N is an infinite set. Furthermore let ϕ be an ultrafilter on N such that $\bigcap_{F\in\phi}F=\emptyset$. Then $\tau=\{A\subseteq S/A-N\neq\emptyset\Longrightarrow A\cap N\in\phi\}$ is a T_1 but not a T_2 topology on S. The conditions of proposition 4 are easily verified; hence (S,τ) is a MNT_2 space.

COROLLARY 1. *The spaces described in example* 9 *are the only*

$T_1 MNT_2$ *spaces.*

Proof. Let (S,τ) be a T_1 MNT_2 space and consider an ultrafilter υ with two distinct limit points $x \neq y$ in S; of course x,y are the points of conditions 1),2) in proposition 4.

Let N be the complement of {x,y} in S. It is easily proved by condition 3) that $\tau(x) \cap N = \tau(y) \cap N = \upsilon \cap N$ which complete the proof.

COROLLARY 2. *Each* MNT_2 *space is a* $MNT_{2\frac{1}{2}}$ *space too.*

Proof. It follows trivially from [5] (theorem 1) and corollary 1.

COROLLARY 3. *The bicompact subsets* [1] *of a* T_1 MNT_2 *space are closed.*

Proof. Let $S = \{x,y\} \cup N$ be a T_1 MNT_2 space with the notations of example 9. If $A \subseteq S$ is an infinite subset of S, then $A \cap N$ is an infinite subset too; we can consider $U \in \phi$ such that $A-U$ is infinite; the open cover $\{\{a\} / a \in A-U\} \cup \{U \cup \{x,y\}\}$ of A has no finite subcover and consequently A is not compact. The compact subset of S are therefore the finite subsets of S which are closed of course.

Remark. The existence in (S,τ) of only two distinct non-separated points is not a sufficient condition in order to prove that all bicompact subsets of S are closed. For instance if we consider the space of example 6 we can see that $P = \{x\} \cup N$ is a bicompact non-closed subset of S.

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(1) Here bicompact means that each open cover has a finite subcover.

No T_1 MNT_2 space exists which is a bicompact space.

We are now going to verify that each T_2 non-$T_{2\frac{1}{2}}$ topology on a set S has a non-$T_{2\frac{1}{2}}$ simple extension i.e. that there exists no $MNT_{2\frac{1}{2}}$ space wich is a T_2 space too.

Let (S,τ) be a T_2 non-$T_{2\frac{1}{2}}$ space and let $x \neq y$ be two distinct points of S such that $\mathrm{cl}U \cap \mathrm{cl}V \neq \emptyset \quad \forall U \in \tau(x),\ V \in \tau(y)$. Then we have the following

LEMMA 2. *The family* $\mathscr{B} = \{P \subseteq S/\ \exists U \in \tau(x),\ V \in \tau(y) : U \cap V = \emptyset$ *and* $\mathrm{cl}U \cap \mathrm{cl}V = P\}$ *is a basis of a filter* ϕ *on* S *that has no cluster point.*

Proof. Trivially $\emptyset \notin \mathscr{B}$; if $U,U' \in \tau(x)$, $V,V' \in \tau(y)$ and $\mathrm{cl}U \cap \mathrm{cl}V = P \in \mathscr{B}$, $\mathrm{cl}U' \cap \mathrm{cl}V' = P' \in \mathscr{B}$, then $U'' = U \cap U' \in \tau(x)$, $V'' = V \cap V' \in \tau(y)$ and $P'' = \mathrm{cl}U'' \cap \mathrm{cl}V'' \subseteq P \cap P'$ is an element of $\mathscr{B}$.

Now let us call ϕ the filter wich has $\mathscr{B}$ as a basis.

If z were a cluster point of ϕ , $z \neq x$, then z would belong to the closure of every neighbourhood of x, which contradicts the assumption that (S,τ) is a T_2 space; the same argument shows that there exists no cluster point distinct from y.

Remark. Since ϕ is a closed filter, we must have $\bigcap_{F\in\phi} F = \emptyset$ from lemma 1 and consequently every element of ϕ has more than one point.

Now let us consider $M \in \tau(x)$ such that the family

$$\mathscr{V} = \{V \in \tau(y)/M \cap V = \emptyset\}$$

is non-empty.

Trivially C(clM) belongs to $\mathcal{V}$ and then it is easily seen that

$$\bigcup_{V\in\mathcal{V}} clM \cap clV = clM - int(clM) = P \in \phi$$

PROPOSITION 5. *With the notations given above and assuming that* (S,τ) *is a* T_2 *non-*$T_{2\frac{1}{2}}$ *space, then* (S,τ) *has a non-*$T_{2\frac{1}{2}}$ *simple extension.*

Proof. We shall denote again by x and y two distint points such that $clU \cap clV \neq \emptyset \quad \forall U \in \tau(x), \quad V \in \tau(y)$.

First we see that the subset $P \in \phi$ already considered above is not open in τ . Indeed if P were an open subset in τ, then we should have $int(clM) \cup P \in \tau(x)$ and, since $int(clM) \cup P = clM$, clM and N = C(clM) would be clopen neighbourhoods of x,y respectively, which is an absurd.

Now we consider a non-open subset $\{z\} \subseteq P$ and we prove that $\tau' = \tau(\{z\})$ is not a $T_{2\frac{1}{2}}$ topology.

In fact note that $z \notin \{x,y\}$, recall $\tau(x)$, $\tau(y)$ are fundamental neighbourhood systems of x,y in τ' and consider $A \in \tau(x)$, $B \in \tau(y)$; since the elements of ϕ have more than one point we have

$$cl_{\tau'}A \cap cl_{\tau'}B = clA \cap (C\{z\} \cup cl(A \cap \{z\})) \cap clB \cap (C\{z\} \cup cl(B \cap \{z\})) \supseteq$$
$$\supseteq (clA \cap clB) - \{z\} \neq \emptyset .$$

It is proved in [5] that the MNT_2 spaces are precisely the MNT_3 spaces. A similar result for MNT_2 and $MNT_{2\frac{1}{2}}$ spaces follows trivially from proposition 5 and corollary 2.

COROLLARY 4. *The* MNT_2 *spaces are precisely the* $MNT_{2\frac{1}{2}}$ *spaces.*

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