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The Theorem of Klee on The Density of Support Points

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Abstract. A new proof is given of the theorem of Victor Klee which states that the support points of a closed, convex, and locally weakly compact set in a real Hausdorff locally convex topological vector space are dense in the boundary of the set.

Keywords: Convex, Klee, Support Point

MSC 2000 classification: 32F27

Some years ago Klee asked in [5] if a non-empty bounded closed convex subset K of a real Banach space B must have any support points. Bishop and Phelps in [2] answered affirmatively. The proof of the latter used Zorn's Lemma on a set of cones where the cone construction was suggested by the ingenious technique they used to show every Banach space is subreflexive [1] and both of these results have since become part of the foundation of Functional Analysis. Phelps [7] gave a new proof of Klee's theorem (Theorem 2 of [7]) stated in the abstract using a generalization the support cone construction used in [2]. The proof given here avoids the cone contruction and the explicit use of Zorn's Lemma employed by Phelps (Lemma 1 and Theorem 2 of [7]) and is a natural extension of the an R^n proof "translate a convex body in the complement of K and intersect K " [10, p.84]. We use the known basic results referred to as Propositions where S is a real Hausdorff locally convex topological vector space.

Proposition 0.1 ([8]). Let $A \subset S$ be a convex set with non-empty interior and let B be a non-empty convex set disjoint from the interior of A. Then A and B can be separated by a closed hyperplane.

Proposition 0.2 ([8]). Let $A \subset S$ be a convex set with non-empty interior. Then every boundary point of A is a support point of A.

Proposition 0.3 ([8]). Let $A \subset S$ be a closed set. Then every convergent net in A has its limit in A.

Proposition 0.4 ([4]). A topological space X is compact if and only if every net in X has a subnet convergent to a point in X.

Theorem 0.5. (Klee) Suppose K is a non-empty convex, closed and locally weakly compact set of a real Hausdorff locally convex topological vector space S. Then the support points of K are dense in the boundary of K.

Proof. We may assume S is the closed linear hull of K so K is not contained in a closed hyperplane of S. Also if $\operatorname{int}(K) \neq \emptyset$ then every boundary point of K is a support point of K by Proposition 2 and we are done so we may assume $\operatorname{int}(K) = \emptyset$. Let $k \in K$ and there exists a closed convex neighborhood P_k of k with $P_k \cap K$ weakly compact. If M_k is a closed convex neighborhood of k let $N_k = M_k \cap P_k$ and then $N_k \cap K$ is weakly compact with $k \in N_k \cap K$. As $\operatorname{int}(K) = \emptyset$ and K is closed choose $y \in int\,(N_k \sim K)$ and without loss of generality we may assume $y = 0_v$, the origin. Since K is closed there exists a closed convex neighborhood $N_{0_v} \subset N_k$ with $0_v \in int N_{0_v}$and $N_{0_v} \cap K = \emptyset$.

For each real α, $0 \leq \alpha \leq 1$, let $N(\alpha, k) = (\alpha k + N_{0_v}) \cap N_k$ and then either $N(\alpha, k) \cap K \neq \emptyset$ and $N(\alpha, k)\ \cap K$ is weakly compact or $N(\alpha, k) \cap K = \emptyset$. Let $\theta = \inf\{\alpha, 0 \leq \alpha \leq 1 | N(\alpha, k) \cap K \neq \emptyset\}$ and θ exists as $N(1, k) \cap K \neq \emptyset$. We assert $\theta < 1$. If φ_n is a sequence of reals with $0 < \varphi_n < 1$ and $\varphi_n \to 1$ then as $(k - \varphi_n k) \to 0_v$ there exists a positive integer M such for all $n \geq M$, $(k - \varphi_n k) \in N_{0_v}$ for otherwise 0_v is in the boundary of N_{0_v}, contradiction. Thus $(k - \varphi_M k) \in N_{0_v}$ and so $k \in (\varphi_M k + N_{0_v})$ which gives $k \in N(\varphi_M, k) \cap K$ and as $\varphi_M < 1$ the assertion follows and $\theta < 1$.

We assert that if $N = int\,(N(\theta, k)) \cap K$ then $N = \emptyset$. If $\theta = 0$ this is true since $N(0, k) = N_{0_v}$ and $N_{0_v} \cap K = \emptyset$. Suppose $\theta > 0$ and the assertion is false. Then there exists $k_1 \in N$ with $k_1 \in int\,(N(\theta, k))$. Since $k_1 \in int\,(N(\theta, k))$ there exists δ with $0 < \delta < \theta$ such that if $\lambda \in [\theta - \delta, \theta + \delta]$ then $\lambda k_1 \in int\,(N(\theta, k))$. Let $k_2 = (\theta + \delta)k_1$ and as $k_2 \in int\,(N(\theta, k))$ we may write $k_2 = \theta k + n$ for some $n \in N_{0_v}$. Then $k_1 \in (0_v, k_2) \subset (0_v, \theta k + n]$ and so there exists a positive $\beta < 1$ with $k_1 = \beta k_2$. Then $k_1 = \beta k_2 = \beta(\theta k + n) = \beta\theta k + \beta n$ and as $\beta < 1$ we have $\beta n \in N_{0_v}$. Thus $k_1 \in N(\beta\theta, k)$ contradicting the definition of θ because $\beta\theta < \theta$ and the assertion is established and so $int\,(N(\theta, k)) \cap K = \emptyset$. Since $int\,(N(\theta, k)) \cap K = \emptyset$, Proposition 1 implies the existence of a continuous linear functional f and a real β with $f(N(\theta, k)) \leq \beta$ and $f(K) \geq \beta$. The definition of θ implies for every $\delta \in (0, 1 - \theta)$ that $N(\theta + \delta, k) \cap K \neq \emptyset$. For each $\delta \in (0, 1 - \theta)$ choose $x_\delta \in N(\theta + \delta, k) \cap K$. Define a binary relation $\succeq$ on $\{x_\delta\}$ by $x_{\delta_1} \succeq x_{\delta_2}$ if $\delta_1 \leq \delta_2$. Then $(\{x_\delta\}, \succeq)$ is a net and since $\{x_\delta\} \subset N_k \cap K$ the net $(\{x_\delta\}, \succeq)$ has a convergent subnet by Proposition 4 as $N_k \cap K$ is weakly compact. Without loss of generality we may suppose $x_\delta \to x \in K$ and the net of numbers $\delta \to 0$. Since $x_\delta \in N(\theta + \delta, k) \cap K$ we may write $x_\delta = (\theta + \delta)\,k + n_{0_v}^\delta$ for each δ where $n_{0_v}^\delta \in N_{0_v}$. Since $x_\delta \to x$ and $\delta \to 0$ then the net $\left\{n_{0_v}^\delta\right\} = \{x_\delta - (\theta + \delta)\,k\} \to (x - \theta k)$ and so $(x - \theta k) \in N_{0_v}$ by Proposition 3. Let $n = (x - \theta k)$ and then $x = (\theta k + n) \in (N(\theta, k) \cap K)$. Then $f(x) = \beta$, $f(K) \geq \beta$ and $K \not\subset \{z | f(z) = \beta\}$

so x is a support point of K. For any $k \in K$, as M_k is any closed convex neighborhood of k, and $N_k \subset M_k$ we conclude the support points of K are dense in K and this completes the proof. The approach here was suggested the work of Cel [3] and by the author's proof [9] of Krasnoselsky's Lemma for weak compacta in a real Hausdorff locally convex topological vector space which answered a question of Valentine [10, p. 84] of extending this Lemma for weak compacta to locally convex spaces. $\boxed{QED}$

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