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EXISTENCE OF BOUNDED SOLUTIONS FOR SET-VALUED EQUATIONS IN BANACH SPACES

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Sunto. *Si considera il problema della ricerca di soluzioni limitate, su intervalli non limitati, di problemi al contorno per equazioni differenziali multivoche in spazi di Banach. La determinazione di opportune stime a priori permetterà di risolvere il problema dell'esistenza di soluzioni mediante teoremi di punto fisso.*

INTRODUCTION. Many papers deal with the problem of existence of bounded solutions on an interval I for differential equations in Banach space (see e.g. [1],[5],[6],[9],[14] and references therein).

In this paper our aim is to prove the existence of bounded solutions for multivalued systems

$$\begin{cases} \dot{x}(t) \in Ax(t) + F(t,x(t)) \\ x \in S \end{cases}$$

where F is a convex-valued Caratheodory multivalued map, A the infinitesimal generator of a bounded semigroup and S a subset in a Banach space.

A problem of this type was solved in [5] in the case of finite-dimensional spaces.

The approach reduces the problem of proving existence of solutions for (1) to the one of finding suitable a priori bounds. Moreover, we will give an example of conditions on A and F which will ensure the existence of such a priori bounds.

NOTATIONS. Throughout this paper, X is a real or complex Banach space, J is an open interval of the real line, possibly unbounded, C(J,X) is the vector space of all continuous mappings of J into X, BC(J,X) is the vector subspace of C(J,X) of all bounded continuous mappings of J into X. If $K \subset J$ is a compact interval, the topology on C(J,X) is defined by means of a family of seminorms $N_K(x) = \max_{t \in K}(\|x(t)\|)$: thus C(J,X) is a Fréchet space, i.e. a linear, metric, locally convex, complete space. In such a space a sequence $\{x_n\}$ is said to be convergent if it is uniformly convergent on compact intervals contained in J.

In the Banach space X we shall adopt the usual meaning of strongly measurable function, strongly continuous function, strongly integrable (i.e. Bochner integrable) function (see, for instance, [10] for details).

Finally a set-valued function $F : X \to 2^X$ is called upper semicontinuous (u.s.c.) on X if for each $x \in X$ the set F(x) is nonempty, closed and for each open set $0 \subset X$, $F(x) \subset 0$, there is an open neighborhood U of x such that $F(U) \subset 0$.

1. STATEMENT OF THE PROBLEM AND SOME RESULTS

Let us consider the following multivalued boundary value problem (MBVP)

$$(1) \quad \begin{cases} \dot{x} \in G(t,x) \\ x \in S \end{cases}$$

for $t \in J$, where $G : J \times X \to 2^X$ is a multivalued mapping and S is a nonempty convex subset of $C(J,X)$.

The problem (1) can be considered, under suitable conditions on the functions and operators involved, a problem equivalent to either of the following ones:

$$(2) \quad \begin{cases} \dot{x} \in Ax + F(t,x) \\ x \in S \end{cases}$$

where $F : J \times X \to 2^X$ is a multivalued mapping obtained by "shifting" the given mapping G by means of a suitable linear bounded operator $A : D(A) \subset X \to X$,

or

$$(3) \quad \begin{cases} \dot{x} \in A(t,x)x + F(t,x) \\ x \in S \end{cases}$$

where F is obtained as above and $A(t,x)$ is a continuous operator defined on $J \times D$, where D is a locally closed subset of X, and taking values on X.

In both cases, if $y=y(t,x)$ is a selection of the multivalued mapping $F(t,x)$, there are results (see [3],[4] for problem (2),[7], [13] for problem (3)) ensuring the existence of a solution of the "corresponding" ordinary differential equation

$$(2.1) \qquad \dot{x} = Ax + y(t,x)$$

or

$$(3.1) \qquad \dot{x} = A(t,x)x + y(t,x) \quad .$$

For the sake of simplicity, we will consider only one of the previous cases in the sequel: then, by changing the hypotheses needed to obtain a solution for (2.1) (or 3.1)), we will get the other case.

So let us assume that:

HYPOTHESIS A: $F(\cdot,x)$ is a strongly measurable function J for each $x \in X$ and $F(t,\cdot)$ is an upper semicontinuous (u.s.c.) function on X a.e. for $t \in J$. Moreover, we shall assume that each set $F(t,x)$ is nonempty, convex and compact and that there exists a function $\varphi : J \times J \to \varphi(t,u) \in \mathbb{R}$ measurable in t for each $u \in J$, continuous and non-decreasing in u for all $t \in J$ such that

$$|F(t,x)| \leq \varphi(t,|x|)$$

and

$$\int_J \varphi(t,|x|)dt < +\infty \qquad \text{for all } x \in \mathbb{R}^n.$$

HYPOTHESIS B: Assume that the following holds $A:D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous (at the origin) (see [4]) semigroup $T(t)$ of linear bounded transformations over X such that $D(A) \cap S \neq \emptyset$. In the sequel we will assume $S \subset D(A)$.

REMARK 1: If we are concerned with existence of solutions for problem (3.1), then Hypothesis B must be changed in the following way: the operator $A(t,x)$ is locally integrable w.r.t. t for each $x \in X$ and (strongly) continuous in x for $t \in J$.

Then in order to prove the existence of a solution of (1) we need the existence of integrable selections for the multivalued mapping $F(t,x)$ and this is given by the following "selection result":

PROPOSITION 1: For each $w(\cdot) \in B \subset (J,X)$ the multivalued function $F(\cdot,w(\ ))$ admits measurable selections $s_w \in L^1(J,X)$.

The above result is achieved firstly by using a general result (see Kuratowskii-Ryll-Nardewskii [11]) on measurable selectors to obtain a locally (for each compact interval K_p such that $J = \bigcup_{p=1}^{\infty} K_p$) measurable selector and then by extending it through the boundedness assumption on the set-valued function F. Now, for each $y \in BC(J,X)$ let us consider the (nonempty) set $S(y)$ of measurable selectors of the set function $F(\cdot,y)(\cdot))$ defined by

$$S(y) = \{ s_y \in L^1(J,X) : s_y(t) \in F(t,y(t)), \quad \text{a.e. in } J \}.$$

The main properties of this set are shown by the following

LEMMA 1: The subset $S(y) \subset L^1(J,X)$ is i) convex and closed, ii) weakly sequentially compact for each $y \in BC(J,X)$.

Proof. On each compact interval K_p, such that $\bigcup_{p=1}^{\infty} K_p = J$, we can analogously define a selections set $S_p(y)$: these sets are closed (see [12]). Then, as in [5] it will be easy to show that both the inclusions

$$S(y) \subset \bigcap_{p=1}^{\infty} S_p(y)$$

and

$$\bigcap_{p=1}^{\infty} S_p(y) \subset S(y)$$

hold, where $S_p(y) = \{s_y \in L^1(J,X) : s_y(t) \in F(t, y(t)), |s_y(t)| \leq \leq \varphi(t,|y|)$, a.e. in $K_p\}$. The latter set is closed for every $p \in N$: thus $S(y) = \bigcap_{p=1}^{\infty} S_p(y)$ and it is a closed set. The convexity is obvious and the weak sequential compactness follows from the boundedness of F through a standard argument in functional analysis.

A method similar to the one used in [5] will allow us to reduce the given problem to that of solving a suitable abstract equation $x \in T(x)$: then the existence of a fixed point for the operator T will be the essential tool needed to solve problem (1.1). To this aim we will need the following hypotheses:

H1) There exists a globally bounded closed convex set $\Omega \subset C(J,X)$ such that, for each $y \in \Omega$, the "linear" system

$$\text{(1L)} \qquad \begin{cases} \dot{x} \in Ax + F(t,y) \\ x \in S \end{cases}$$

has at least a solution in Ω .

H2) If T(y) denotes the (nonempty) set of solutions of the BVP (1L) we assume that $\overline{T(\Omega)} \subset S$. Then Kakutani's fixed point theorem (see [2], pg.85) will be enough for us to get the required solution of the problem (1.1).

2. MAIN RESULT

Let us consider problem (2) as a suitable form of problem (1): then we need hypotheses like A and B in order to have solutions for the "single" ordinary differential equations and hypotheses like H1) and H2) in order to apply the fixed point argument.

Thus we have:

THEOREM 1: *Assume that hypotheses* A) *and* B) *and hypotheses* H1) *and* H2) *hold. Then the* MBVP (2) *has at least one (bounded) solution.*

REMARK 1. If the set $S \subset C(J,X)$ is a closed set, then hypothesis H2) is obviously satisfied: we refer the reader to the quoted paper [6] for other details on the role played by that hypothesis and for other examples concerning the use of such an assumption.

PROOF OF THEOREM 1: Let us define the operator $T:y \to T(y)$. By hypothesis H1) T is properly defined. The proof will be accomplished by proving the following two steps:

Step 1: $T(y)$ is a (nonempt) convex set for each $y \in \Omega \subset C(J,X)$.

Step 2: $y \to T(y)$ is an u.s.c. operator with closed values.

Then a straightforward application of Kakutani's fixed point theorem to the multivalued operator T will finish the proof.

Step 1 is easily show by observing that, for given $y_1, y_2 \in T(y)$, by (hypothesis H1) there exist s_y^1 and $s_y^2 \in S(y)$ such that

$$\begin{cases} \dot{y}_1 = Ay_1 + s_y^1(t) \\ y_1 \in S \end{cases}$$

$$\begin{cases} \dot{y}_2 = Ay_2 + s_y^2(t) \\ y_2 \in S \end{cases}$$

The convexity of both S(y) and S completes the proof.

To prove Step 2 let us first show that, under hypothesis H1), $T(\Omega) \subset \Omega$ is a relatively compact subset of C(J,X). Thus we need to prove that, by Ascoli's theorem, the set $T(\Omega)$ is equibounded and equicontinuous in K for each compact subset K of J. Since Ω and $T(\Omega)$ are bounded, there is a continuous real function $\psi : J \to R^+$ such that $|\omega(t)| < \psi(t)$, $|y(t)| < \psi(t)$ for all $\omega \in \Omega$ and $y \in T(\Omega)$.

If $\psi(K) = \max\{\psi(t), t \in K \subset J\}$, then beside the equiboundedness of Ω (and $T(\Omega)$ we get the equicontinuity by considering

$$\|\dot{y}(t)\| \leq \|A\| \, \|y\| + |F(t,y(t))| =$$

$$= \|A\|\psi(K) + \max\{\varphi(t,\|y\|), \quad t \in K, \quad \|y\| \leq \psi(K) \ .$$

The relative compactness of $T(\Omega)$ will allow us to show that, in order to prove that T is an u.s.c. operator with closed values, T has a closed graph (see, for instance, [2]).

To that purpose assume $(y_n, x_n) \in$ graph (T) are such that $\lim_n (y_n, x_n) = (y,x) \in \Omega \times C(J,X)$: we claim that $x \in T(y)$. For such $\{x_n\}$ there is a sequence $\{s_{y_n}\} \subset S(y_n)$ such that

$$\dot{x}_n = Ax_n + s_{y_n}(t)$$

or, (see [8]), equivalently

$$x_n(t) = T(t)x_n(0) + \int_0^t T(t-\tau)S_{y_n}(\tau)d\tau \quad , \qquad (*)$$

where we assumed that $s_{y_n}(t)$ are (strongly) measurable functions, $x_n(0) \in D(A)$ for all n and the above integral is in the Bochner sense ([13]).

Since $x \in S$ by hypothesis H2), it will be enough to show the existence of $s_y \in S(y)$ such that x is a solution of the integral equation:

$$x(t) = x(0) + \int_0^t T(t-\tau)s_y(\tau)d\tau \ .$$

As usual, let $\{I_k\}$ be a sequence of nested compact intervals with the property $\bigcup_{k=0}^{\infty} I_k = J$. Put $\{z_n(t)\} = \int_0^t T(t-\tau)s_{y_n}(\tau)d\tau$: then from (*) we can say that $\{z_n(t)\}$ converges to a function $z \in C(J,X)$, uniformly on each I_k, i.e.

$$z(t) = \lim_{n\to\infty} z_n(t) = \lim_{n\to\infty} \int_0^t T(t-\tau)s_{y_n}(\tau)d\tau$$

for $t \in I_k$. Moreover, by assuming (strongly) measurability and Bochner integrability of a suitable selector s_y (denoted s_y^k for $t \in I_k$) we can apply Proposition 1 to say that such a selector exists such that

$$z(t) = \int_0^t T(t-\tau)s_y^k(\tau)\ d\tau$$

or, equivalently,

$$x(t) = T(t)x(0) + \int_0^t T(t-\tau)s_y^k(\tau)d\tau \ .$$

Without loss of generality, we can suppose that for each $t\in I_k$

$$s_y^{k+1}(t) = s_y^k(t)$$

and define any extension $\bar{s}_y^k(t)$ of $s_y^k(t)$ to J such that $\bar{s}_y^k(t)=s_y^k(t)$ for $t\in I_k$ and $\|s_y^k(\cdot)\| \leq \varphi(t,\|y(t)\|)$ for $t\notin I_k$. This extended function belongs to the set

$$S^k(y) = \{s_y \in L^1(J,X) : \|s_y\| \leq \varphi(t,\|y(t)\|),$$

$$s_y(t) \in F(t,y(t)) \quad \text{a.e. on } I_k\}$$

and so the sequence $\{\bar{s}_y^k\}$ is a weakly compact sequence of $L^1(J,X)$: then a subsequence converging to a function $\bar{s}_y \in L^1(J,X)$ can be found such that $\bar{s}_y \in S(y) = \bigcap_{k=0}^{\infty} S^k(y)$, since the latter set is strongly closed and convex and since $\bar{s}_y$ belongs to the weak closure of the set S(y).

From the above construction, for each $p\in N$

$$\bar{s}_y^{k+p}(t) = \bar{s}_y^k(t).$$

Thus $\bar{s}_y(t) = \bar{s}_y^k(t)$ for $t\in I_k$ and the proof of both Step 2 and Theorem 1 is done.

3. SOME APPLICATIONS

As Theorem 1 shows, we need two types of a priori bounds for the existence of solutions of system (2). Hence, we want to present some cases in which the existence of such a priori bounds is given and so the hypotheses H1)-H2) are satisfied.

THEOREM 2. *Let us assume that the following conditions hold:*

i) *there exist two* (L^1*-sommable*) *functions* a, b *such that* $|F(t,x)| \leq a(t)\|x\|+b(t)$ *for* $t \in J$ *and* $x \in X$;

ii) *the linear system* (1L) *has at least one solution* x_y *for each continuous bounded* y *such that* $\|x_y(0)\| \leq \sup_{t \in J} |y(t)|$;

iii) *let* A *be such that the corresponding semigroup of bounded linear transformations over* X *has a negative exponential growth, that is*

$$\|T(t)\| \leq M \exp\,(t(\omega_0+\varepsilon)),$$

where $M = M(\varepsilon) > 0$ *and* $\omega_0 = \lim_{t\to\infty} \frac{1}{t} \lg\|T(t)\| < 0.$

Let $T(y)$ be the set as defined in hypothesis H2). Then there exists a continuous real function φ such that $\sup\{\varphi(t),\ t \in J\} < +\infty$ and $T(\Omega(\varphi)) \subset \Omega(\varphi)$ where

$$\Omega(\varphi) = \{y \in C(J,X) : \|y\| \leq \varphi(t), \quad t \in J\}.$$

Moreover, if the condition

$$(°) \qquad \overline{T(\Omega)(\varphi))} \subset S$$

holds, then the system (2) has (at least) one bounded solution.

REMARK 1: We will put in the sequel $a_0 = \exp\,(\int_J a(s)ds)$, $b_0 = \int_J b(s)ds$, $K = K(\varepsilon) = -(\omega_0+\varepsilon) > 0$. The negative exponential growth is not too restrictive a condition: there are in fact semigroups for which $\omega_0 = -\infty$ (see [3] p.166).

Proof. Let us consider the scalar differential equation

$$M\dot{\varphi}(t) = a(t)\varphi(t) + b(t) \qquad (**)$$

Then we have

$$M\varphi(t) = \exp(\int_0^t a(s)ds)\left[\varphi(0) + \int_0^t b(s)\exp(-\int_0^s a(r)dr)ds\right]$$

and so

$$\|\varphi\|_{L_1} \leq \frac{a_o}{M}[\|\varphi(0)\| + b_o] < +\infty .$$

Let now φ be such that $\varphi(0) \leq M\{\sup|y(t)|,\ t\in J\}$ and satisfying $(**)$, i.e. $\varphi(t)-\varphi(0) = \frac{1}{M}\int_0^t (a(s)\varphi(s)+b(s))ds$.

Let us now consider $x_y \in T(y)$, with $y\in\Omega(\varphi)$: then, from the corresponding integral equation $(*)$, we have

$$\|x_y(t)\| \leq \|T(t)\|\|x_y(0)\| +$$

$$+ \int_0^t \|T(t-s)\|(a(s)\varphi(s)+b(s))ds \leq$$

$$\leq M\exp(-kt)\ \|x_y(0)\| + \int_0^t (a(s)\varphi(s)+b(s))\ ds \leq$$

$$\leq M\|x_y(0)\| + (\varphi(t)-\varphi(0)) =$$

$$= \varphi(t) + M\|x_y(0)\| - \varphi(0) \leq \varphi(t).$$

Then $T(\Omega(\varphi)) \subset \Omega(\varphi)$. Finally the assertion follows from Theorem 1 by recalling condition (°).

REMARK 2: Condition ii) can be improved by putting

$$\|x_y(0)\| \leq \alpha \{\sup|y(t)|, \quad t\in J\} + \beta$$

where $\alpha \geq 0$ and $\beta > 0$. If $\alpha = 0$, then the assumption is always satisfied: this occurs, for instance, if the boundary condition implies a bound on the initial condition or, equivalently, if $S \subset S_\gamma$, where

$$S_\gamma = \{y\in C(J,X) : \|y(0)\| \leq \gamma \quad , \quad \gamma > 0\} .$$

REMARK 3. Whether condition H2) is satisfied or not depends essentially upon the special kind of boundary conditions associated with the differential system. If the set $\Omega(\varphi)$, introduced in Theorem 2, is such that $\Omega(\varphi) \subset S$, then the assumption $T(\Omega(\varphi)) \subset \Omega(\varphi)$ is verified. This is the case, for instance, when we are looking for bounded solutions or when the boundary conditions are of Cauchy's or Nicoletti's type (in these latter cases both the assumptions $T(\Omega) \subset \Omega$ and $\overline{T(\Omega)} \subset S$ are satisfied.

REFERENCES

[1] G. ANICHINI-P.ZECCA: *Multivalued Differential Equations in Banach Space. An application to control theory"*, J.O.T.A. 21, 4(1977).

[2] J.P.AUBIN-A.CELLINA: *Differential Inclusions*, Springer,New York, (1984).

[3] A.V.BALAKRISHNAN: *Applied Functional Analysis*, Springer, New York (1976).

[4] A.BELLENI MORANTE: *Applied Semigroups and Evolution Equations*, Clarendon Press, Oxford (1979).

[5] M.CECCHI-M.MARINI-P.ZECCA: Existence of bounded solutions for multivalued differential systems, *Nonlinear Analysis* T.M.A. 9, 8(1985) 775-786.

[6] M.CECCHI-M.FURI-M.MARINI: On continuity and compactness of some nonlinear operator associated with differential equations in non-compact intervals, *Nonlinear Analysis* T.M.A.9(1985) 171-180.

[7] A.CELLINA: Multivalued functions and multivalued flows,*Tech.Note*, BN 615, Univ. of Maryland 1969.

[8] J.L.DALEKII-M.KREIN: Stability of solutions of differential equations in Banach spaces, *Trans. of Math.Monograph*,A.M.S., 1974.

[9] M.DAVIDOWSKI: On existence of bounded solutions of differential equations in Banach spaces, *Commentationes Math.Univ.Carolinae*, 26, 3(1985) 611-616.

[10] R.E.EDWARDS: *Functional Analysis. Theory and Applications,* Holt-Rinehart-Winston, New York 1965.

[11] K.KURATOWSKI-C.RYLL NARDEWSKII: A general theorem on selectors, *Bull. Ac. Polon.Sci.Ser.Sci.Math.Astron-Phys* 13(1965) 397-403.

[12] A.LASOTA-Z.OPIAL: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull.Ac.Polon. Sci. Ser. Sci-Math-Astron-Phys* 13(1965) 781-786.

[13] R.MARTIN: *Nonlinear Operator and Differential Equations in Banach Spaces,* John Wiley & Sons, New York (1976).

[14] P.ZECCA-P.ZEZZA: Nonlinear Boundary Value Problems in Banach Space for Multivalued Differential Equations on a non compact interval, *Nonlinear Analysis* T.M.A., 3(1979) 347-352.

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