Note di Matematica Vol. VI, 121-137(1986)

ON REGULAR r-PACKINGS

Vikram Jha and N.L.Johnson

Abstract. *This article is concerned with the connection between regular 2-packings in* PG(2r-1,q) *and translation planes of order* q^{2r} *whose components are defined by a set of rational Desarguesian nets coordinatized by quadratic field extensions of a given field of order* q . *This work is a natural extension of the studies of Walker* [17] *and Lunardon* [14].

INTRODUCTION. Prohaska and Walker [15], Walker [17] and Lunardon [14] have, independently, shown a connection between regular 2-packings in PG(3,q) and certain translation planes of order q^4 and kernel GF(q). These planes are of particular interest as they admit a regulus $\mathcal{R}$ (of 1+q components) and the components are defined by $1+q+q^2$ derivable nets $\mathcal{D}_i$, $i=1,\dots,1+q+q^2$ such that $\mathcal{R} \subseteq \mathcal{D}_i$ and $\mathcal{R} = \mathcal{D}_i \cap \mathcal{D}_j$, $i\neq j$, $i,j = 1,\dots,1+q+q^2$.

In [8], the authors show how to connect 2-packings (or parallelisms) in PG(3,q) with general translation planes of order q^4 admitting SL(2,q) as a collineation group.

In this note, we give the natural extensions of the work of Walker [17], Lunardon [14], and the authors [8] to 2-packings in PG(2r-1,q) related to translation planes of order q^{2r} and kernel $F \simeq GF(q)$ with a regulus $\mathcal{R}$ (of 1+q components) and whose components are defined by $\frac{q^{2r-1}-1}{q-1}$ nets $\mathcal{D}_i$, $i=1,\dots,\frac{q^{2r-1}-1}{q-1}$ such that

$\mathcal{D}_i$ is a rational Desarguesian net coordinatized by a quadratic field extension of F, $\mathcal{D}_i \supseteq \mathcal{R}$ and $\mathcal{D}_i \cap \mathcal{D}_j = \mathcal{R}$ for all $i \neq j$; $i,j=1,\ldots,\frac{q^{2r-1}-1}{q-1}$.

The arguments supporting the results are quite similar or natural extensions of those of Prohaska and Walker [15] and Jha-Johnson [8]. However, we try to give direct proofs in order to make this article more or less self-contained.

We require the following results:

(1.1). **THEOREM** (Jha[5], LEMMA 2).

Let V *be an elementary abelian group of order* $p^{sr} = q^r \geq q^2$ *and suppose* U *is any non-trivial group of order* u^t *for* $t \geq 1$ *in* Aut(V,+)) *where* u *is a prime* p-*primitive divisor of* $q^{(r-1)}-1$.

Then

(a) $|\mathrm{Fix}\ U| = q$

(b) U *is semi regular on* V/Fix(U)

(c) U *is cyclic*

(d) *If* $r > 2$ *then* $V = \mathrm{Fix}U \oplus C_U$ *where* C_U *is the unique* U-*sub module of* V *which is disjoint from* Fix(U).

(e) *If* $r > 2$ *and* W *is a* U-*submodule of* V *then either* $W \subseteq \mathrm{Fix}(U)$ *or* $|W| \geq q^{r-1}$.

(1.2). **THEOREM** (Johnson [11]).

Let π *be a translation plane of order* p^{2kr} *which admits* $\mathcal{D} \simeq SL(2,p^r)$ *as a collineation group in the translation complement. Assume the* p-*elements are elations and* $\mathcal{N}$ *denotes the elation net.*

(1) *There is a rational Desarguesian net $\mathcal{D}$ containing $\mathcal{N}$ (coordinatized by a field* $\simeq GF(p^{2r})$ *which is fixed by* $\mathcal{D}$.

(2) $(\mathcal{D}-\mathcal{N})\cap \ell_\infty$ *is an orbit under* $\mathcal{D}$ *and an orbit of* $\mathcal{D}$ *of lenght* $p^{2r}-p^{r}$ *defines a rational Desarguesian net containing* $\mathcal{N}$.

(3) *If* $\mathcal{N}$ *is coordinatized by the field* $K\simeq GF(q)$ *then each such orbit net* $\mathcal{D}$ *may be coordinatized by an extension field* $K[t] \simeq GF(p^{2r})$ *(where* $K[t]$ *depends on* $\mathcal{D}$).

2. REGULAR t-PACKINGS AND TRANSLATION PLANES.

(2.1) *Definition.* Let V be a vector space of dimension k over a field $F \simeq GF(q)$ for $q = p^{r}$, p a prime, r an integer. A *partial t-spread* $\mathcal{P}$ of V is a set of mutually disjoint t-dimensional subspaces. A *t-spread* of U is a partial t-spread which covers the vectors of V. (In this case, $t|r$).

A *Desarguesian* or *regular partial t-spread* is a partial t-spread $\mathcal{P}$ such that there is a field extension K of F and the elements of $\mathcal{P}$ are 1-dimensional subspaces over K (note that K is isomorphic to $GF(q^{t})$.

(2.2) *Definition.* Let V be a vector space of dimension 2k over a field $F \simeq GF(q)$. Let $\mathcal{N}$ be a partial k-spread and let $\mathcal{P}$ be a partial 2t-spread of V. We shall say that $\mathcal{N}$ is *t-transversal* to $\mathcal{P}$ if and only if $\mathcal{L}\in\mathcal{N}$ and $c \in\mathcal{P}$ then $\mathcal{L}\cap c$ is a t-subspace of c. We also shall say that c and $\mathcal{L}$ are t-transversal to each other.

We initially follow Prohaska and Walker [15].

(2.3). PROPOSITION. *Let* $\mathcal{P}$ *be a partial* k*-spread of a vector space of dimension* 2k *over* $F \simeq GF(q)$. *Let* $\mathcal{F}$ *denote the set of all*

2t-spaces t-transversal to $\mathcal{P}$. Let $f \in \mathcal{F}$ and let $(f)_{\mathcal{P}} = \{\mathcal{L} \cap f \mid \mathcal{L} \in \mathcal{P}\}$. If $(f)_{\mathcal{P}}$ is a t-spread of f then for every element $g \in \mathcal{F}$, $(g)_{\mathcal{P}}$ is a t-spread and $\mathcal{F}$ is a partial 2t-spread.

Proof. (We follow the argument of Prohaska and Walker.) If $(f)_{\mathcal{P}}$ is a t-spread then $(f)_{\mathcal{P}}$ is a translation plane of order q^t and $|\mathcal{P}| = 1+q^t$. Hence, $(g)_{\mathcal{P}}$ is a partial t-spread with $1+q^t$ elements. That is, $\mathcal{L} \cap g \neq \mathcal{M} \cap g$ and $(\mathcal{L} \cap g) \cap (\mathcal{M} \cap g) \subseteq \mathcal{L} \cap \mathcal{M} = \mathcal{O}$. So $(g)_{\mathcal{P}}$ is a t-spread. It remains to show that $\mathcal{F}$ is a partial 2t-spread. So, let $j,k \in \mathcal{F}$ and $j \cap k \neq \mathcal{O}$. Let $P \in j \cap k - \{\mathcal{O}\}$ (be a vector $\neq \mathcal{O}$). There exists a unique element $\mathcal{M}$ of $\mathcal{P}$ which contains P. Given $\mathcal{N}, \mathcal{L} \in \mathcal{P} - \{\mathcal{M}\}$ by projection, there is a unique 2-dimensional subspace U (line of projective space) which contains P and which intersects $\mathcal{N}$ and $\mathcal{L}$ (as $\mathcal{N} \oplus \mathcal{L} = V$). But, similarly, there is a unique 2-space $\bar{U}$ of j containing P and which intersects $j \cap \mathcal{N}$ and $j \cap \mathcal{L}$ (in a 1-space of j). That is, $U=\bar{U}$ and $U \subseteq j$ and similary, $U \subseteq k$ so $U \subseteq j \cap k$. Now suppose Q is any 2-space containing P such that $Q \subseteq j$. If $Q \not\subseteq \mathcal{M}$ then since $(f)_{\mathcal{P}}$ is a t-spread of f, it must be that Q intersects at least two elements of $\mathcal{P}$ (in $(f)_{\mathcal{P}}$ -but, this means that $Q \subseteq j \cap k$. Choose any vector in $j \cap \bar{X}$ and together with P form a 2-dimensional subspace T. As above $T \subseteq j \cap k$. Hence, $j \cap \bar{X} \subset j \cap k$ and similarly $k \cap \bar{X} \subseteq j \cap k$. And, $j \cap \bar{Y} \subseteq j$ k, $k \cap \bar{Y} \subseteq j \cap k$. So, $j=k$ so that $\mathcal{F}$ is a partial spread.

(2.4) PROPOSITION. *Let A and B be mutually disjoint k-spaces of a 2k-dimension vector space V. Let $\mathcal{A}$ be a t-spread of A, $\mathcal{B}$ a t-spread of B and f a linear bijection of V from $\mathcal{A}$ onto $\mathcal{B}$. Then $\mathcal{P} = \{\bar{x} \oplus \bar{x}^f \mid \bar{x} \in \mathcal{A}\}$ is a partial 2t-spread with A,B t-transversal to $\mathcal{P}$. Furthermore, $A_{\mathcal{P}} = \mathcal{A}$, $B_{\mathcal{P}} = B$.*

Proof. We must show that $\mathcal{P}$ is a partial 2t-spread. Let $\mathcal{R}, T \in \mathcal{P}$ and let $P \in \mathcal{R} \cap T$. Let $\mathcal{R} = \bar{x} \oplus \bar{x}^f$, $T = \bar{Y} \oplus \bar{Y}^f$ for $\bar{x}, \bar{y}$ t-spaces in $\mathcal{A}$. If $P \in A$ (or $P \in B$) then $\bar{x} = \bar{y}$ because $\mathcal{A}$ is a (partial) t-spread so that $R = T$. Assume $P \notin A$ and $P \notin B$. There is a k-space C containing P and mutually disjoint to A and B. There is a unique 2-dim space L on P which nontrivially intersects A and B. $R = \bar{x} \oplus \bar{x}^f$ is a 2t-space and $P \notin x$ or x^f so as in the previous argument there is a unique 2-space $\bar{L}$ of $x \oplus x^f$ which contains P and which intersects $R \cap A$ and $R \cap B$. That is, $\bar{L} = L$. Hence, L is in $R \cap T$. *But* T intersects A (and B). Hence R and T have a (vector) point in common on A and because $\mathcal{A}$ is a (partial) t-spread, R=T.

(2.5) PROPOSITION. *Suppose* A,B,C *are mutually disjoint* k-*subspaces (of* V *a* 2k-*dimension vector space) and let* $\mathcal{A}$ *be a* t-*spread of* A. *Then there exists precisely one partial* 2t-*spread* $\mathcal{P}$ t-*transversal to* A,B,C *and with* $(A)_{\mathcal{P}} = \mathcal{A}$. *Further, the regulus* $\mathcal{R}(A,B,C)$ *is contained in the set of all* t-*transversal to* $\mathcal{P}$.

Proof. There is a unique involution i_c of V which fixes C *pointwise* and interchanges A and B (i.e., $A = (x = \mathcal{O})$, $(y = \mathcal{O}) = B$, C is (y=x) then i_c is $\begin{bmatrix} \mathcal{O} & I \\ I & \mathcal{O} \end{bmatrix}$).

Consider $\mathcal{P} = \{\bar{x} \oplus \bar{x}^{i_c} \mid \bar{x} \in \mathcal{O}\}$. By (2.4), $\mathcal{P}$ is certainly a partial 2t-spread and $(A)_{\mathcal{P}} = \mathcal{A}$. Suppose $\bar{\mathcal{P}}$ is a partial 2t-spread t-transversal to A,B,C with $(A)_{\bar{\mathcal{P}}} = \mathcal{A}$. Then consider $\bar{x} \in \mathcal{A}$. There is a 2t-space U (of $\bar{\mathcal{P}}$) containing $\bar{x}$ and transversal to B and C. Hence, $\bar{x} \subseteq U \cap (\bar{x} \oplus \bar{x}^{i_c})$. We now again use the argument of (2.3).

We repeat part of the argument for U and $\bar{x} \oplus \bar{x}^{i_c} = T$.

Note that we may take the partial k-spread {A,B,C} and $\mathcal{F}$ the set of all 2t-spaces transversal to $\mathcal{P}$ such that $\mathcal{F}|A = \mathcal{A}$,U and T are transversal to B and C. Consider a point P$\in \bar{x}$. $\bar{x} \oplus \bar{x}^{i^c} = T$ is a 2t-space and there exists a *unique* 2-space wich contains P and intersects B and C. That is, L is also in $\bar{x} \oplus \bar{x}^{i^c}$ and in U. Thus, U and T intersect on B. Moreover, this is true for every point P of $\bar{x}$. Let $\bar{P}$,P be distinct 1-spaces on $\bar{x}$ and $\bar{L}$,L the unique 2-spaces $\bar{P}\in\bar{L}$,P$\in$L such that $\bar{L}$,L intersect B,C non-trivially. Then $L,\bar{L} \subseteq \bar{x} \in \bar{X}^{i^c}$ and $L,\bar{L} \subseteq U$. Can $L \cap \bar{L} \neq \mathcal{O}$?

Then

$$\bar{L} = <(\bar{x}_1 \ldots \bar{x}_t, \mathcal{O} \ldots \mathcal{O}), (\mathcal{O} \ldots \mathcal{O}, \bar{x}_1 \ldots \bar{x}_t)>$$

$$L = <(x^*_1 \ldots x^*_t, \mathcal{O} \ldots \mathcal{O}), (\mathcal{O} \ldots \mathcal{O}, x^*_1 \ldots x^*_t)>$$

Now

$$s = (((\bar{x}_1 \ldots \bar{x}_t)\alpha, (\bar{x}_1 \ldots \bar{x}_t)\beta))$$

$$= ((x^*_1 \ldots x^*_t)\delta, (x^*_1 \ldots x^*_t)\gamma)$$

$$\Rightarrow \bar{x}_i\alpha = x^*_i\delta \Rightarrow \bar{x}_i = x^*_i\delta\alpha^{-1} \text{ if } \alpha \neq 0$$

and

$$\Rightarrow \bar{x}_i\beta = x^*_i\gamma \Rightarrow \bar{x}_i = x^*_i\gamma\beta^{-1} \text{ if } \beta = 0.$$

Hence if α or $\beta \neq 0 \Rightarrow \bar{L} = L$. If $\alpha = 0$ then $\delta = 0$ so $\beta \neq 0$. Hence , $L \cap \bar{L} = \mathcal{O}$ or L.

So, this means $L,\bar{L} \subseteq \bar{x} \oplus \bar{x}^{i^c}$ and U, so that $\bar{x} \oplus \bar{x}^{i^c} = U$.

So, we have shown that the space $\mathcal{F}$ of all 2t-spaces which are t-trasversal to A,B,C and which when restricted to A give $\mathcal{O}$ is precisely $\mathcal{P} = \{\bar{x} \in \bar{x}^{i^c} | \bar{x} \in \mathcal{O}\}$ (for $\mathcal{F}$ is a partial 2t-spread). Now consider the regulus $\mathcal{R}$(A,B,C) and D $\in \mathcal{R}$(A,B,C). The regulus is covered by 2-spaces (little Desarguesian planes). So, if U is a 2t-spaces t-trasversal to A,B,C and considering U as a union of its 2-spaces (transversal to A,B,C) we see, by the above argument, that there exist disjoint 2-dim spaces of U which intersect D-one 2-space for each 1-space on $A \cap U$. Hence, there are $\geq \frac{q^t-1}{q-1}$ 1-spaces of U on $\mathscr{D}$.

Hence $\dim U \cap D \geq t$. But, also $\dim U \cap A = t$ and $A \cap D = \emptyset$. Hence, $\dim U \cap D = t$. Thus, every 2t-space t-transversal to A,B,C is also transversal to the elements of $\mathscr{R}(A,B,C)$.

We want to investigate the situation when there are precisely $1+q^t$ k-spaces which are t-transversal to $\mathscr{F} = \{ \bar{x} \oplus \bar{x}^{i}c \mid \bar{x} \in \mathscr{A} \}$. In this case the set $\mathscr{I}$ of t-transversal k-spaces is exactly covered by $\mathscr{F}$. Or, another way of saying this is that if $\mathscr{I}$ is the partial spread of t-transversal k-spaces to $\mathscr{F}$, i e., each element of $\mathscr{I}$ is t-transversal to $\mathscr{F}$, then (f)y is a 2t-spread of $f \in \mathscr{F}$.

By Foulser's covering theorem [12] when this happens the t-spread of $\mathscr{A}$ is Desarguesian. Conversely, consider a Desarguesian t-spread $\mathscr{A}$ of A. Then there is a field extension K of F such that $\mathscr{F} = \{ \bar{x} \oplus \bar{x}^{i}c \mid \bar{x} \in \mathscr{A} \}$ is a partial 2-spread over K. And, we may consider A,B.C as subspaces over K and V a vector space over $K \supseteq F$. Now applying the previous result the regulus $\mathscr{R}(A,B,C)$ over K is contained in the set of 1-transversals to $\mathscr{F}$ over K. But, this mens $\mathscr{R}_K(A,B,C)$ is the set of 1-transversal to $\mathscr{F}$ over K. Hence,

(2.6) THEOREM (See Prohaska and Walker [15] when t=2)

There is all 1-1 correspondence between Desarguesian t-spreads

$\mathscr{A}$ *of a* k-*space* A *of* $\mathscr{R}$(A,B,C) *(regulus over* F *generated by* A,B,C) *and partial spreads* $\mathscr{D}$ *of degree* $1+q^t$ *of* t-*transversal* k-*spaces to* A,B,C *such that surface* $\mathscr{D}$ = *surface of* $\mathscr{F}(\bar{x}\epsilon\bar{x}^{-i}C \mid \bar{x} \in \mathscr{A}\}$ *which contain the regulus* $\mathscr{R}$(A,B,C).

That is, there is a 1-1 *correspondence between rational Desarguesian nets of degree* $1+q^t$ *containing a regulus* $\mathscr{R}$ *and Desarguesian* t-*spreads of a component of the regulus.*

(2.7) **PROPOSITION.** *Given a regulus* $\mathscr{R}$(A,B,C) *and* Q$\notin$surf$\mathscr{R}$, *there is a unique* 4-*space containing* Q *which is* 2-*transversal to* A,B *and* C *and thus to* $\mathscr{R}$(A,B,C).

Proof. Again we argue as in Prohaska and Walker [15](3). If $\{\bar{x},\bar{y}\} \subseteq \{A,B,C\}$, then there is a unique 2-space transversal to $\bar{x}$ and $\bar{y}$ and containing Q. So there are three 2-spaces $U_{A,B}, U_{A,C}, U_{B,C}$ containing Q and transversal to (A,B),(A,C),(B,C) respectively. Then suppose two are equal. Then there is a 2-space which hits A,B,C and thus lies on the regulus. But, Q $\notin$ regulus so that these three are completely distinct spaces. $(U_{A,B}, U_{A,C}, U_{B,C})$ is a 4-dimensional space which contains Q and is the unique such 4-space which is 2-transversal.

Now suppose $\mathscr{A}_1$, $\mathscr{A}_2$ are two *distinct* regular 2-spreads. Then consider the rational Desarguesian nets $\mathscr{D}_1$, $\mathscr{D}_2$ so constructed. Then suppose $Q \in \mathscr{D}_1 \cap \mathscr{D}_2 - \mathscr{R}(A,B,C)$. Then there is a *unique* 4-space $\mathscr{L}_Q$ containing 0 and 2-transversal to $\mathscr{R}$. But, this 4-space is simultaneously then a 2-space over two fields $K_1, K_2 \simeq GF(q^2)$. So, $\mathscr{L}_Q|A$ is a 1-space over K_1 and over K_2. That is, if we obtain a partial spread we must have hat $\mathscr{A}_1$ and $\mathscr{A}_2$ do not share a 1-space and *conversely.*

We have:

(2.8) THEOREM (See Prohaska and Walker [15], Walker [17] and Lunardon [14] for order q^4.)

Let V *be a vector space of dimension* $4k$ *over* $F \simeq GF(q)$. *Let* $\mathcal{R}$ *be a regulus of* V. *Let* Γ *be a set of rational Desarguesian nets isomorphic to* $GF(q^2)$ *containing* $\mathcal{R}$. *Then* $\cup(\Gamma - \mathcal{R}) \cup \mathcal{R}$ *is a p[illegible]ial spread* $\iff$ $(A)_{\mathcal{D}} \mid \mathcal{D} \in \Gamma$ *is a partial 2-parallelism of* A *wh[illegible] is a component of* $\mathcal{R}$.

(2.9) Notes

i) Theorem (2.8) is also noted by Walker in [17]. However, our proof generally follows and extends Prohaska and Walker's unpublished notes.

ii) Stinson and Vanstone [16] have determined a great number of 2-packings in PG(5,2). It is not clear if any are regular but such regular 2-packings would correspond to translation planes of order 2^6 and kernel GF(2) which contain a regulus of 1+2=3 lines and whose components consist of $\frac{2^5-1}{2-1} = 31$ rational nets each of which may be coordinatized by a field isomorphic to GF(4) containing the same prime field.

iii) If π is a translation plane of order q^{2r} constructed as in (2.8) then $\Gamma L(2,q)$ is a collineation group of π.

Proof. The regulus $\mathcal{R}$ admits $\Gamma L(2,q)$ and since each rational Desarguesian net $\mathcal{D}_i$ is defined by an extension field of the field defining the regulus $\mathcal{R}$, $\Gamma L(2,q)$ is also a collineation group of $\mathcal{D}_i$. Hence, since $\pi = \dot{\cup}(\mathcal{D}_i - \mathcal{R}) \cup \mathcal{R}$, it follows that $\Gamma L(2,q)$ is a collineation group of the plane π.

(2.10) TRANSLATION PLANES AND PARTIAL t-PACKINGS.

Let π be a translation plane of order q^{ts} and kernel GF(q) admitting a regulus $\mathcal{R}$ (of 1+q components). Suppose the components consist of $\frac{q^{ts}-q}{q^{t}-q} = \frac{q^{ts-1}-1}{q^{t-1}-1}$ (where $t-1 \mid ts-1$), rational Desarguesian nets isomorphic to $GF(q^t)$. Then on any component $\mathcal{L}$ of $\mathcal{R}$, considering $\mathcal{L}$ as PG(ts-1,q), there is an associative partial t-packing.

(sketch) If on $\mathcal{L}$ two t-space $\bar{x}_1 = \bar{x}_2$ are equal (one from two different t-spreads) then $\bar{x}_1 \oplus \bar{x}_1^{i_c} = \bar{x}_2 \oplus \bar{x}_2^{i_c}$ so that the associated nets are equal.

3. TRANSLATION PLANES OF ORDER q^{2r} ADMITTING SL(2,q).

In [8], the authors show how to obtain regular parallelisms in PG(3,q) (2-packings) directly from an associated translation plane of order q^4. In this section, it is noted that the same theorems are valid for 2-packings in PG(2r-1,q).

That is, we prove:

(3.1) THEOREM. (See (2.4)[8])

Let π be a translation plane of order q^{2r}, $q=p^s$, q a prime, s an integer which admits a collineation group $\mathcal{D}$ isomorphic to SL(2,q) *in the translation complement.*

(i) *If the* p*-elements are elations and $\mathcal{D}$ is* 1/2*-transitive on $\ell_\infty - \mathcal{N} \cap \ell_\infty$ where $\mathcal{N}$ denotes the net of elation axes then the kernel of $\mathcal{D}$ is* GF(q), *each orbit Γ union $\mathcal{N}$ is a rational Desarguesian* *net coordinatized by a field isomorphic to* $GF(q^2)$ *. If* $\mathcal{L}$ *is an elation axis then* $\mathcal{L}$ *is thought of as* $PG(2r-1,q)$ *admits a regular 2-packing.*

(ii) *Conversely, if* $\mathcal{L}$ *is a* 2r-*space over a field* $F \simeq GF(q)$ *and admits a regular 2-packing as* $PG(2r-1,q)$ *then there is a corresponding translation plane which admits a collineation group* $\mathcal{D} \simeq SL(2,q)$, $q=p^s$ *, where the p-elements are elations and such that* $\mathcal{D}$ *acts 1/2-transitively on* $\ell_\infty - \mathcal{N} \cap \ell_\infty$ *where* $\mathcal{N}$ *denotes the net of elation axes.*

Proof. (ii). By (2.8) there is a corresponding translation plane π. Since each net may be coordinatized by an extension of a field $K \simeq GF(q)$, clearly the group $\mathcal{D}$ generated by the elations of the regulus $\mathcal{R}$ is isomorphic to $SL(2,q)$ (see (2.9)(ii) and is a collineation group of π . Clearly, $\mathcal{D}$ acts 1/2-transitively on $\ell_\infty - \mathcal{R} \cap \ell_\infty$ because for each rational Desarguesian net $\mathcal{D} \supseteq \mathcal{R}$, $\mathcal{D}-\mathcal{R}$ is an orbit under $\mathcal{D}$.

(i) Suppose $\mathcal{D}$ is 1/2-transitive. By (1.2), there is at last one rational Desarguesian net $\mathcal{D}$ coordinatized by a field extension $K[t]$ of the field K defining the net $\mathcal{N}$ of elation. And, $\mathcal{D}-\mathcal{N}$ is an orbit. Hence, there exist $\frac{q^{2r}-q}{q^2-q} = \frac{q^{2r-1}-1}{q-1}$ such orbits and by (1.2)(3), each such orbit defines another rational Desarguesian net containing $\mathcal{N}$. Thus, by (2.8),(i) is proved.

We now consider translation planes of order q^{2r} that admit $SL(2,q) \times Z_{\frac{q^{2r-1}-1}{q-1}}$ as a collineation group in the translation complement. Note that the known regular 2-packings define translation planes that admit such groups (see Jha-Johnson [8]).

We prove

(3.2) THEOREM (COMPARE WITH JHA-JOHNSON [8] (2.5)).

Let π *be a translation plane of order* $p^{2rs} = q^{2r}$ *that admits a collineation group* $\mathcal{D}$ *iomorphic to* $SL(2,q) \times Z_{\frac{q^{2r-1}-1}{q-1}}$ *in the translation complement. Then, the kernel is* $GF(q)$ *, the p-elements are elations and for any elation axis* $\mathcal{L}$ *considered as* $PG(2r-1,q)$, $\mathcal{L}$ *admits a regular 2-packing.*

Proof. We structure the proof as in Jha-Johnson [8] (2.5).

We first *assume* the p-elements are elations. By Jha-Johnson [8] (2.5), we may assume $2r > 4$ in any case.

Let the elation net be denoted by $\mathcal{N}$. By (1.2), there is at least one rational Desarguesian net $\mathcal{D} \supseteq \mathcal{N}$, $\mathcal{D}$ of degree $1+q^2$.

Suppose $g \in Z_{\frac{q^{2r-1}-1}{q-1}} = Z$ fixes $\mathcal{D}$. Let $h \in Z$ such that $|h|$ is a prime p-primitive divisor of $q^{(2r-1}-1$. Since $2r > 4$, there always exists such an element since $|h| \mid \frac{q^{2r-1}-1}{q-1}$ (note 4^3-1 is a possible exception and the argument is taken separately in [8]).

Since h fixes each elation axis and fixes points on each (as of $|h| \mid q^{2r}-1$ and $|q^{2r-1}-1$ then $|h| \mid q^{(2r,2r-1)}-1$ which cannot be the case).

By (1.1). Fix h is a subplane of order q. g acts on Fix h so that if g does not fix points of Fix h then there exists an integer j such that $g^j \neq 1$ and $|g^j|\,|q-1$. Then consider q^j with $|q^j|\,|q-1$ and fixing $\mathscr{D}$. Since $|g_j|\,|q^{2r-1}-1$ and $|g^j|q-1$ then g^j fixes affine points and since $|\mathscr{D}-\mathscr{N}| = q^2-q$, some power g^{jh} fixes infinite points of $\mathscr{D}$-$\mathscr{N}$. That is, g^{jk} fixes a subplane of order $\geq q^2$ pointwise. And, there is a subplane π_0 of order q^2 of $\mathscr{D}$ such that Fix h $\subseteq \pi_0 \subseteq$ Fix g^{jk}. However, π_0 is Desarguesian so that $|h|\,|=2$ and q must be odd. But then $|h|\,|q-1$ which cannot be the case.

Hence, if g fixes $\mathscr{D}$ then $|g^j|q-1$ and $|g^j|\,|\frac{q^{2r-1}-1}{q-1}$.

$$(q-1,1+q+q^2+\ldots+q^{2r-2})$$
$$= (q-1,(q-1)+(q^2-1)+\ldots+(q^{2r-2}-1)+(2r-1)) \ .$$

So the GCD equals (q-1,2r-1). So there are at least $\frac{q^{2r-1}-1}{(q-1)(q-1,2r-1)}=t_1$ ational Desarguesian nets $\mathscr{D}_i \supseteq \mathscr{N}$ such that $\mathscr{D}_i \cap \mathscr{D}_j = \mathscr{N}$ for i,j= $= 1,\ldots,t_1$ since SL(2,q) has $\mathscr{D}_i - \mathscr{N}$ as an orbit for all $i=1,\ldots,t_1$.

But, let $\sigma \in \mathscr{D} \simeq SL(2,q)$ be an element such that $|\sigma|\,|q^2-1$, but $|\sigma|\,\beta^k-1$ for $k \leq 2s$ $(g=p^s)$ (see e.g. Johnson [11]. σ permutes the remaining points on $\ell_\infty - \bigcup_{i=1}^{t_1} \mathscr{D}_i$.

So,

$$|\ell_\infty - \bigcup_{i=1}^{t_1} \mathscr{D}_i| = (q^{2r}-q) - \frac{(q^{2r-1}-1)(q^2-q)}{(q-1)(q-1,2r-1)} \ .$$

Let (q-1,2r-1) = s.

More generally, suppose there are $\frac{T}{s}(\frac{q^{2r-1}-1}{q-1}) = t_2$ rational Desarsian nets. Then

$$|\ell_\infty - \bigcup_{i=1}^{t_2}\mathcal{D}_i| = (q^{2r}-q) - \frac{T}{s}(\frac{q^{2r-1}-1}{q-1}(q^2-q)$$

$$q(q^{2r-1}-1)(1-\frac{T}{s})$$

$$= q(q^{2r-1}-1)(\frac{s-T}{s}).$$

Then if

$$|\sigma|\,|q(q^{2r-1}-1)(\frac{s-T}{s})$$

then

$$|\sigma|\,|(q^{2r-1}-1,q^2-1) = q^{(2r-1,2)}-1 = q-1.$$

Hence, σ fixes *additional* points on ℓ_∞. Now apply the previous argument inductively. That is, remove another set of at least $(\frac{q^{2r-1}-1)}{(q-1)s}$ rational Desarguesian nets. Obtain another set of cardinality $q(q^{2r-1}-1)(\frac{s-2}{s})$. By (1.2) and induction, there are $\frac{q^{2r-1}-1}{q-1}$ rational Desarguesian nets $\mathcal{D}_i \supseteq \mathcal{N}$ such that $\mathcal{D}_i \cap \mathcal{D}_j = \mathcal{N}$. Now apply (2.8).

Now assume the p-elements in SL(2,q) are planar. Note the proof in [8] extends directly.) Let π_0 be a subplane of order p^k=Fix,$\sigma\,|\sigma|=$ $= p$, $\sigma\in\mathcal{D}\simeq SL(2,q)$. $Z_{\frac{q^{2r-1}-1}{q-1}} \equiv Z$ must leave π_0 invariant and if $g\in Z$ has order a prime p-primitive divisor of $q^{(2r-1}-1)$ (recall $2r>4$) then we assert that g must fix a component of π_0. That is, by Foulser's Dimension Theorem (e.g., see Jha [7]) k must divide

$2rs$ if $q=p^s$. However, if $|g|\,|1+p^k$ then $|g|\,|(p^{2k}-1,p^{(2r-1)s}-1)=$ $=(p^{(2k,(2r-1)s)}-1)$. Hence we have a contradiction unless $(2r-1)s|2k|4rs$ so $(2r-1|s|4rs$ or $2r-1|4r$. But since $2r-1$ is odd, $2r-1|r$ so that $r=1$. But then the order is q^2 and the group is $SL(2,q)$ and the planes are determined in Foulser-Johnson ([5],[6]).

Now let g fix p^t points on a fixed component $\mathcal{L}$ of π_0. As g is completely reducible on $\mathcal{L}\cap\pi_0$, $\mathcal{L}\cap\pi_0$ = (Fix g on $\mathcal{L}\cap\pi_0)\oplus W$ where $|W| = \frac{p^k}{p^t}$. Hence $|g|\,|(p^{(k-t,(2r-1)s)}-1)$. However, this cannot be the case unless $(2r-1)s|(k-t)$. But $k|2rs$ so $k-2rs-t$ and $k \le rs$. Then $(2r-1)s|2rs-t$ so $s|t$ and $k-t\le rs-s$. Hence, $(2r-1)s|2rs-\ell s$ so that $\ell = 1$ and $s=t$. But, if $(2r-1)s|k-s$ then $(2r-1)s \le rs-s$ which obviously cannot be.

Thus, it must be that $k=t$ so that g fixes π_0 pointwise. By (1.1)(a) Fix g on each fixed component has order q. So $\pi_0 \subseteq$ Fix g and Fix g is a subplane of order q.

Let $\mathcal{L}$ be a component of π_0 then $\mathcal{L}=((\text{Fix } g)|\mathcal{L})\oplus C_{g,\mathcal{L}}$ where $C_{g,\mathcal{L}}$ is the unique g-submodule on $\mathcal{L}$ which is disjoint from $((\text{Fix } g)|\mathcal{L})$. But, σ fixes $\mathcal{L}$ and therefore must fix the module $C_{g,\mathcal{L}}$ since σ permutes the g-submodules on $\mathcal{L}$. However, this implies that fixes additional points on $\mathcal{L}$.

Hence, the p-elements cannot be planar.

Now let σ be a p-element and assume Fix σ lies in a component $\mathcal{L}$. The previous argument shows that Fix $\subseteq$ Fix g, Fix g has order q on $\mathcal{L}$ and σ must fix the complement $C_{g,\mathcal{L}}$ of g on $\mathcal{L}$. That is, again σ must fix additional points of $\mathcal{L}$ (since $|\sigma| = p$).
This proves (3.2).

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Ricevuto il 31/1/1986

V. JHA
Mathematics Department
Glasgow College of Technology
Cowcaddens Road
Glasgow GA OBA
SCOTLAND

N.L.JOHNSON
Mathematics Department
University of Iowa
Iowa City, IA 52242
U.S.A.