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ON DUALS OF $L^1(\mu)$

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Abstract. *In questa rota si dà una rappresentazione del duale di uno spazio* $L^1(\mu)$ *quando la misura "μ" non è σ-finita e il carattere di densità dello spazio è il continuo.*

The aim of this note is to give an isomorphic representation of the duals of some spaces $L^1(\mu)$ when the measure μ is not σ-finite.

Of course, $L^1(\mu)$ stands for the classical Banach space $L^1(X,\Sigma,\mu)$ where (X,Σ,μ) is a measure space with positive measure μ. We also abbreviate $L^\infty(X,\Sigma,\mu) = L^1(X,\Sigma,\mu)'$ to $L^\infty(\mu)$. When μ is the counting measure on X, $L^1(\mu)$ and $L^\infty(\mu)$ will be denoted by ℓ^1_d and ℓ^∞_d respectively, where $d = |X|$ = the cardinaly of X (dropping the d, of course, when $d = \aleph_0$). The algebra Σ is said to be an *m-algebra*, where m is a cardinal number $\geq \aleph_0$, if the union of m members of Σ also belongs to Σ. The measure μ is called *m-finite* if X is the disjoint union of m members of Σ each having a finite μ-measure. For a Banach space E, we denote by $\aleph(E)$ the *density character of* E, i.e. the smallest cardinality of a dense subset of E. Also we recall that, for any μ, $L^\infty(\mu)$ is complemented in every Banach space containing it. Finally we shall use the notation $E = F$ and $E < F$ to indicate that E is isomorphic to F or to a complemented subspace of F, respectively.

To avoid trivialities, we always assume that if $A \subset X$ is such that $\mu(A) = \infty$, then there exists $B \subset A$ for which $0<\mu(B)<\infty$. We also assume that $L^1(\mu)$ is not separable and that μ is not σ-finite, since otherwise the following results hold.

(a) *If* $E = L^1(\mu)$ *is separable, then* $E = \ell^1$ *or* $E = L^1(0,1)$ *(Lebesgue measure) and hence* $E' = \ell^\infty$ (cf. [1])

(b) *If* μ *is* σ*-finite, then* $L^\infty(\mu) = L^\infty(\nu)$ *if and only if* $\chi[L^1(\mu)] = \chi[L^1(\nu)]$, (cf. [3], Th. 3.5 p.221).

We need the following

LEMMA. *Let* $m \geq \chi_0$ *and let* $E = L^1(X,\Sigma,\mu)$ *where* Σ *is an m-algebra and* μ *is not m-finite. Then* E *contains a complemented subspace* F *isometric to* $\ell^1_{2^m}$ *with a norm-one projection.*

Proof. Let $\mathcal{P}$ be the collection of all families $L=\{A_i\}$, with $A_i \in \Sigma$, $0 < \mu(A_i) < \infty$ and $A_i \cap A_j = \emptyset$ $(i \neq j)$, ordered by inclusion. By Zorn's Lemma $\mathcal{P}$ has a maximal element $M = \{A_i : i \in I\}$. Put $d = |I|$ and suppose that $d \leq m$. Then $A = \cup\{A_i : i \in I\} \in \Sigma$, since Σ is an m-algebra and $\mu(X \setminus A) = \infty$ because μ is not m-finite. If $B \subset X \setminus A$ and $0 < \mu(B) < \infty$, then $M \cup \{B\} \in \mathcal{P}$ and we have reached a contradiction. Therefore, $d \geq 2^m$. It is easy to see that the closed linear span of the characteristic functions χ_{A_i} of the sets A_i is isometric to ℓ^1_d. Now let $f \in L^1(\mu)$ and note that the set $\{x \in X : |f(x)| > 0\}$ has a σ-finite measure and hence $\int_X f \chi_{A_i} \, d\mu \neq 0$ only for a countable set of indices. It follows that

$$\sum_{i \in I} \left| \int_X f \chi_{A_i} \, d\mu \right| \leq \sum_{i \in I} \int_{A_i} |f| \, d\mu \leq \int_X |f| \, d\mu ,$$

showing that $P : L^1(\mu) \to \ell^1_d$, defined by $Pf = (\int_X f\chi_{A_i}\, d\mu)$, is a norm-one projection of $L^1(\mu)$ onto ℓ^1_d from which the lemma follows.

We now have the

THEOREM. *Under the hypotheses of the lemma, if* $\chi(E) = 2^m$ *then* $E' = \ell^\infty_{2^m}$.

Proof. $\ell^\infty_{2^m} < E'$, since $\ell^1_{2^m} < E$ by the lemma. But also $E' < \ell^\infty_{2^m}$, because $E' = L^\infty(\mu)$ and E is a quotient of $\ell^1_{2^m}$, since $\chi(E) = 2^m$. The desired isomorphism $E' = \ell^\infty_{2^m}$ then follows by noting that $\ell^\infty_{2^m} = (\ell^\infty_{2^m} \oplus \ell^\infty_{2^m} \oplus \ldots)_{\ell^\infty}$ and applying Pełczynski's decomposition method (cf. [2]).

COROLLARY. *Let* $E = L^1(\mu)$ *with* $\chi(E) = c$ *and* μ *not* σ*-finite. Then* $E' = \ell^\infty_c$.

Remark. If $E = L^1(\mu)$ and $\chi(E) = c$, then the situation is as follows:

(i) if μ is σ-finite, then $E' = L^\infty([0,1]^c, \nu)$ where ν is the Haar measure on $[0,1]^c$ (cf. [3], Th.3.5 p.221);

(ii) if μ is not σ-finite, then $E' = \ell^\infty_c$.

It is interesting to note that ℓ^∞_c and $L^\infty([0,1]^c, \nu)$ are not isomorphic: indeed, $\chi(\ell^\infty_c) = 2^c$, while $\chi\{L^\infty([0,1]^c, \nu)\} = c$ (cf. [3], p.222).

REFERENCES

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