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ON THE UNIVERSAL CONNECTION OF A SYSTEM OF CONNECTIONS

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ABSTRACT. - *Given a system of connections on a fibred manifold, we study in detail the notion of universal connection and curvature. We compare different approaches and analyze the cases of linear, affine and principal bundles. We interpret the Liouville and symplectic forms in this context. Both intrinsical constructions and coordinate expressions are exhibited.*

INTRODUCTION. The universal connection and curvature were first introduced by P.L.Garcia for principal bundles [4,5]. P.L.Garcia and A.Peréz-Rendón applied extensively this technique to gauge theories [5,12]. A further application has been achieved by D.Canarutto and C.T.J.Dodson, who proved some stability properties of general relativistic singularities under variations of the connection [2].

The universal connection and curvature were generalized by L.Mangiarotti and M.Modugno to any system of connections on a fibred manifold. This general approach is based essentially on the theory of connections on fibred manifolds and on the concept of system of connections [9]. The theory of connections on fibred manifolds was introduced by C.Ehresmann [3] and pursued by P.Libermann [8], I.Kolář [7] and others. Moreover, L.Mangiarotti and M.Modugno developed a differential calculus and, in particular, an approach to curvature, by means of jet techniques and the Fröelicher-Nijenhuis bracket [9]. Furthermore, the notion of system leads to a rich geometrical theory and to interesting applications to gauge theories [10,11]. In particular, the universal connection has a strict link with the grated universal differenzial calculus [10].

Of course this approach includes principal bundles as a particular case, although it does not involve explicitely group techniques.

The purpose of this paper is to study in detail the notion of universal connection and curvature, by comparing different approaches and analysing extensively the most important cases.

In order to make the paper self contained, we begin with a recall of a few basic concepts about fibred manifolds, jet spaces, connections and systems of connections. Then, we consider a fibred manifold $p:E\to B$ and a system of connections $\xi:C\times_B E\to T^*B\otimes TE$ and we show, in three independent ways, the existence on the bundle $F\equiv C\times_B E\to C$ of the universal connection $\Lambda:F\to T^*C\otimes TF$ and its curvature $\Omega:F\to\Lambda^2T^*B\otimes VE$. Any connection of the system $c:B\to C$, or equivalently $c\equiv\xi\circ c:E\to T^*B\otimes TE$, can be obtained from Λ by a pull-back. Finally, we study the important examples related to vector, affine and principal bundles. In the particular case when E is the trivial principal bundle $E\equiv M\times\mathbb{R}\to M$ and $C\equiv T^*M$ is the space of principal connections, the universal connection Λ and curvature Ω turn out to be the Liouville and the symplectic forms.

I - SYSTEMS OF CONNECTIONS

1 - Fibred manifolds.

In this section we summarize some essential notions on fibred manifolds and on the tangent and jet functors. Further details can be found in [8].

We shall consider Hausdorff, paracompact, smooth manifolds of finite dimension and smooth maps between manifolds.

1 - A **fibred manifold** is a surjective submersion $p:E\to B$ (i.e. a surjective map of maximal rank), E is the **total space**, B is the base space and $E_x \equiv p^{-1}(x) \subset E$, with $x\in B$, are the **fibres**. We assume $\dim B=m$, $\dim E_x=l$, $\forall x\in B$, hence $\dim E=m+l$. The rank theorem yields the local trivialization of E. Namely, for each $y\in E$, there exists an open neighbourhood $V_y \subset E$, a manifold F and a diffeomorphism (called fibred chart) $\Phi:V_y\to p(V_y)\times F$, such that $p=pr_1\circ\Phi$.

We denote by Ω_B and Ω_E the sheaves of local functions $f:B\to\mathbb{R}$ and $f:E\to\mathbb{R}$, respectively.

A "fibred" coordinate system on E is a coordinate system $\{x^\lambda, y^i\}:E\to\mathbb{R}^m\times\mathbb{R}^l$, such that x^λ is factorizable through p; thus, there exists a coordinate system $\{\tilde{x}^\lambda\}$ on B such that $x^\lambda=\tilde{x}^\lambda\circ p$. When no confusion arises, we shall denote $\tilde{x}^\lambda$ simply as x^λ.

A local **section** of E, on an open subset $U\subset B$, is a map $s:U\to E$ such that $p\circ s=id_U$. Its coordinate expression is $(x^\lambda, y^i)\circ s\equiv(x^\lambda, s^i)$, with $s^i\in\Omega_B$. By abuse of notation, we shall replace some times $s:U\to E$ with $s:B\to E$.

We denote as $\mathscr{E}\equiv\mathscr{S}(E/B)$ the sheaf of local sections $s:B\to E$.

A fibred morphism between the fibred manifolds $p:E\to B$ and $p':E'\to B'$ is a map $\Phi:E\to E'$, which preserves the fibres. Thus there is a map $\phi:B\to B'$ such that $q\circ\Phi=\phi\circ p$ and we say that Φ is a fibred morphism over ϕ. The coordinate expression of a local fibred morphism Φ over ϕ is $(x'^{\lambda},y'^{i})\circ\Phi=(\phi^{\lambda},\Phi^{i})$, with $\phi^{\lambda}\in\Omega_B$ and $\Phi^{i}\in\Omega_E$. In particular, if $B=B'$ and $\phi=id_B$, then we say that Φ is a fibred morphism over B.

A fibred manifold $p':E'\to B'$ is a **fibred submanifold** of $p:E\to B$ if E' a submanifold of E, B' a submanifold of B and $p'=p_{|E'}$.

The **fibred product** over B of two fibred manifolds $p:E\to B$ and $q:F\to B$ is the submanifold $E\times_B F\hookrightarrow E\times F$ over the diagonal $B\hookrightarrow B\times B$, characterized by $p\circ pr_1=q\circ pr_2$.

A **bundle** is a fibred manifold which is locally trivializable over B, i.e. such that, for each $x\in B$, there exists a neighbourood $U_x\subset B$, a manifold F and a diffeomorphism $\Phi:p^{-1}(U_x)\to U_x\times F$, such that $p=pr_1\circ\Phi$.

A **vector bundle** is a bundle together with a smooth assignment of a vector structure on its fibres. Namely, the vector bundle structure is characterized by the fibred morphisms $+:E\times_B E\to E$, $-:E\to E$ and $\cdot:(\mathbb{R}\times B)\times_B E\to E$ over B and by the null section $0:B\to E$, which satisfy the properties of the algebraic operations.

If $p:E\to B$ and $p':E'\to B'$ are vector bundles, then a **linear fibred morphism** is a fibred morphism which is a linear map on each fibre.

A **group bundle** is a bundle together with a smooth assignment of a group structure on its fibres. Namely, the group bundle structure is characterized by the fibred morphisms $\mu: E\times_B E \to E$ (the multiplication) and $\iota: E\to E$ (the inversion) over B and by the unity section $1: B\to E$, which satisfy the properties of the algebraic operations.

If $p: E\to B$ and $p': E'\to B'$ are group bundles, then a **group fibred morphism** is a fibred morphism which is a group morphism on each fibre.

An **affine bundle** is a bundle together with a smooth assignment of an affine structure on its fibres. If $\bar{E}_x$ is the vector space associated with the affine space E_x, for each $x\in B$, then we obtain the associated vector bundle $\bar{p}: \bar{E} \equiv \cup_{x\in B}\bar{E}_x \to B$. Namely, the affine bundle structure is characterized by the fibred morphism $+: E\times_B \bar{E} \to E$ over B, which satisfies the properties of the algebraic operation.

If $p: E\to B$ and $p': E'\to B'$ are affine bundles, then an **affine fibred morphism** is a fibred morphism $f: E\to E'$ which is an affine map on each fibre, i.e. such that there exists a linear fibred morphism $\bar{f}: \bar{E}\to \bar{E}'$ satisfying $f(x+v)=f(x)+\bar{f}(v)$, $\forall (x,v)\in E\times\bar{E}$. If $\bar{E}\subset\bar{E}'$ and $\bar{f}=id_{\bar{E}}$, then the fibred morphism $f: E\to E'$ is a **translation.**

A **group affine bundle** is a bundle together with a smooth assignment of a group affine structure on its fibres. If $\bar{E}_x$ is the associated group of the group affine space E_x, for each $x\in B$, then we obtain the associated group bundle $\bar{p}: \bar{E} \equiv \cup_{x\in B}\bar{E}_x \to B$. Namely, the group affine bundle structure is characterized by the fibred morphism $\tau: E\times_B \bar{E} \to E$ over B, which is a free and transitive fibred (right) action of the fibres of the group bundle $\bar{E}$ on the fibres of E.

If $p:E\to B$ and $p':E'\to B'$ are group affine bundles, then a **group affine fibred morphism** is a fibred morphism $f:E\to E'$ which is a group affine morphism on each fibre, i.e. such that there exists a group fibred morphism $\bar{f}:\bar{E}\to\bar{E}$, satisfying $f(xg)=f(x)\bar{f}(g)$, $\forall(x,g)\in E\times_B\bar{E}$. If $\bar{E}\subset\bar{E}'$ and $\bar{f}=id_{\bar{E}}$, then the fibred morphism $f:E\to E'$ is a translation.

If G is a Lie group, then we define the category of G-principal spaces as the subcategory of G affine spaces, with the translations as morphisms.

A (right) **principal bundle**, with structure group G, is a bundle together with a smooth assignment of a G-principal structure on its fibres; in others words, a principal bundle is just a group affine bundle whose associated group bundle is the product bundle $\bar{p}:\bar{E}\equiv B\times G\to B$. In fact, the local sections $s:U\to E$ determine the local splittings $\Phi:p^{-1}(U)\to U\times G$, whose transition maps $(U\cap U')\times G\to(U\cap U')\times G$ are fibred translations over $U\cap U'$.

If $p:E\to B$ and $p':E'\to B'$ are principal bundles such that the structure group G of $p:E\to B$ is a subgroup $j:G\hookrightarrow G'$ of the structure group G' of $p':E'\to B'$, then a **principal fibred morphism** $f:E\to E'$ is a group affine fibred translation where $\bar{f}=j:G\hookrightarrow G'$.

2 - **We denote the tangent functor as T.**

So, we shall consider the vector bundles $\pi_B:TB\to B$, $\pi_E:TE\to E$. We have the induced fibred charts $(x^\lambda,\dot{x}^\lambda)$ on TB and $(x^\lambda,y^i,\dot{x}^\lambda,\dot{y}^i)$ on TE.

If $\Phi:B\to B'$ is a map, then $T\Phi:TB\to TB'$ is the linear tangent morphism over Φ. Its coordinate expression is $(x'^{\lambda},\dot{x}'^{\lambda})\circ T\Phi=(\phi^{\lambda},\dot{x}^{\mu}\partial_{\mu}\Phi^{\lambda})$.

In particular, we shall consider the fibred manifold $Tp:TE\to TB$. Its coordinate expression is $(x^{\lambda},\dot{x}^{\lambda})\circ TP=(x^{\lambda},\dot{x}^{\lambda})$.

We have also the tangent prolongation $Ts:TB\to TE$ of a local section $s:B\to E$. Its coordinate expression is $(x^{\lambda},y^{i},\dot{x}^{\lambda},\dot{y}^{i})\circ Ts=(x^{\lambda},s^{i},\dot{x}^{\lambda},\dot{x}^{\mu}\partial_{\mu}s^{i})$.

We denote the vertical functor as V.

So, we have the vertical sub-bundle $VE\equiv \mathrm{Ker}Tp\subset TE$ over E, characterized locally by the condition $\dot{x}^{\lambda}=0$.

3 - The (first-order) jet space of $p:E\to B$ is the set $J(E/B)\equiv \equiv\cup_{s\in\mathscr{S},x\in B}[s]_{x}$, where $[s]_{x}$ are the equivalence classes of local sections $s:B\to E$ given by $s\underset{x}{\sim}s'$ iff $s^{i}(x)=s'^{i}(x)$ and $(\partial_{\lambda}s^{i})(x)= =(\partial_{\lambda}s'^{i})(x)$, with respect to any adapted chart. When no confusion arises, we shall write JE instead of J(E/B).

By definition, there are the natural projections

$$Jp:JE\to B \quad \text{and} \quad p_{0}:JE\to E.$$

Moreover, $Jp:JE\to B$ is a fibred manifold and $p_{0}:JE\to E$ is an affine bundle.

The induced affine fibred chart of JE is $(x^{\lambda},y^{i},y^{i}_{\lambda}):JE\to\mathbb{R}^{m}\times\mathbb{R}^{1}\times\mathbb{R}^{m1}$.

The (first) jet **prolongation** of a local section $s:B\to E$ is the local section $js:B\to JE:x\mapsto[s]_{x}$. Its coordinate expression is $(x^{\lambda},y^{i},y^{i}_{\lambda})\circ js=(x^{\lambda},s^{i},\partial_{\lambda}s^{i})$.

We have also the (first) jet **prolongation** $J\Phi:JE\to JE'$ of a fibred morphism $\Phi:E\to E'$ over B. Its coordinate expression is $(x^\lambda, y'^i, y'^i_\lambda)\circ J\Phi = (x^\lambda, \Phi^i, \partial_\lambda\Phi^i + y^j_\lambda\partial_j\Phi^i)$.

There is a canonical fibred morphism $\lambda: JE\times_B TB\to TE$ over E. We can also view λ as an affine fibred monomorphism $\lambda: JE\to T^*B\otimes TE$ over E. Its coordinate expression is $\lambda = d^\lambda\otimes\partial_\lambda + y^i_\lambda d^\lambda\otimes\partial_i$.

Hence, the bundle $p_0: JE\to E$ is the linear affine sub-bundle of the vector bundle $T^*B\otimes TE\to E$ wich is projected onto $1\subset T^*B\otimes TB$.

The associated vector bundle of $p_0: JE\to E$ is $\overline{JE}\equiv T^*B\otimes VE\to E$.

The complementary linear epimorphism of λ will be denote by $\vartheta: JE\times_E TE\to VE$. It induces naturally a vertical valued form $\vartheta: JE\to T^*JE\otimes VE$.

ϑ can also be considered as a linear morphism $\vartheta: TJE\to VE$ over $p_0: JE\to E$. Its coordinate expression is $\vartheta = d^i\otimes\partial_\lambda - y^i_\lambda d^\lambda\otimes\partial_i$.

2 - Connections on fibred manifolds.

We recall some basic notions on connections on fibred manifolds; further details can be found in [9].

1 - **A connection** on E is a (local) section

$$\gamma: E\to JE.$$

Moreover we can also interpret γ as a tangent valued 1-form

$$\gamma: E\to T^*B\otimes TE,$$

which is projectable onto $1: B\to T^*B\otimes TB$.

Its coordinate expression is

$$(x^\lambda, y^i, y^i_\lambda)\circ\gamma = (x^\lambda, y^i, \gamma^i_\lambda), \quad \text{i.e.} \quad \gamma = d^\lambda\otimes\partial_\lambda + \gamma^i_\lambda d^\lambda\otimes\partial_i,$$

with $\gamma^i_\lambda \in \Omega_E$.

2 - The connection γ can be wiewed in other equivalent ways (see [9]): in particular, a connection is equivalent to the vertical projection which is a linear fibred morphism over E

$$\omega_\gamma : TE \to VE.$$

Its coordinate expression is

$$(x^\lambda, y^i, \dot{y}^i) \circ \omega_\gamma = (x^\lambda, y^i, \dot{y}^i - \gamma^i_\mu \dot{x}^\mu) \quad \text{i.e.} \quad \omega_\gamma = d^i \otimes \partial_i - \gamma^i_\lambda d^\lambda \otimes \partial_i .$$

3 - A connection is equivalent to the affine translation over E

$$\nabla_\gamma : JE \to \overline{JE} \equiv T^*B \otimes VE : \sigma \mapsto \sigma - \gamma(p_0(\sigma)).$$

Its coordinate expression is

$$(x^\lambda, y^i, \dot{x}_\lambda \otimes \dot{y}^i) \circ \nabla_\gamma = (x^\lambda, y^i, y^i_\lambda - \gamma^i_\lambda) \quad \text{i.e.} \quad \nabla_\gamma = (y^i_\lambda - \gamma^i_\lambda) d^\lambda \otimes \partial_i .$$

We define the **covariant derivative** of $s \in \mathcal{S}$ as

$$\nabla_\gamma s \equiv \nabla_\gamma \circ js : B \to T^*B \otimes VE.$$

Its coordinate expression is

$$\nabla_\gamma s = (\partial_\lambda s^i - \gamma^i_\lambda \circ s) d^\lambda \otimes (\partial_i \circ s).$$

4 - Let $\phi : E \to \Lambda^r T^*B \otimes TE$ be a tangent valued **form**. The **covariant differential** of ϕ is the vertical valued form

$$d_\gamma \phi \equiv 1/2 [\gamma, \phi] : E \to \Lambda^{r+1} T^*B \otimes VE,$$

where $[\gamma, \phi]$ is the Frölicher-Nijenhuis bracket [9].

Its coordinate expression is

$$d_\gamma\phi = 1/2(-\partial_{\lambda 1}\phi^\mu{}_{\lambda 2\ldots\lambda r+1}\gamma^i_\mu - \partial_\mu\gamma^i_{\lambda 1}\phi^\mu{}_{\lambda 2\ldots\lambda r+1} + \partial_{\lambda 1}\phi^i_{\lambda 2\ldots\lambda r+1}$$

$$+\gamma^j_{\lambda 1}\partial_j\phi^i_{\lambda 2\ldots\lambda r+1} - \partial_j\gamma^i_{\lambda 1}\phi^j_{\lambda 2\ldots\lambda r+1})d^{\lambda 1}\wedge\ldots\wedge d^{\lambda r+1}\otimes\partial_i .$$

5 - The curvature of the connection γ is the vertical valued 2-form

$$\rho \equiv d_\gamma\gamma \equiv 1/2[\gamma,\gamma]: E \to \Lambda^2 T^*B \otimes VE .$$

Its coordinate expression is

$$\rho = (\partial_\lambda\gamma^i_\mu + \gamma^j_\lambda\partial_j\gamma^i_\mu)d^\lambda\wedge d^\mu\otimes\partial_i .$$

For the relation of this approach to more traditional ones see [9].

3 - Sistems of connections.

If $x \in B$, then the space of all connections $\gamma_x : E_x \to (JE)_x$ is not locally finite dimensional over B. So, we are led to consider finite dimensional restrictions of this space by introducing the concept of system of connections [9,10].

Namely, in a few words, we choose smoothly, for each $x \in B$, a finite dimensional sub-space C_x of the space of all pointwise connections $\gamma_x : E_x \to (JE)_x$. Then, by definition, we have, $\forall x \in B$, the evaluation map

$$\xi_x : C_x \times E_x \to (JE)_x : (\gamma_x, y) \to \gamma_x(y)$$

and the map

$$\xi_x : C_x \to \mathcal{S}((JE)_x / E_x) : c_x \to c_x \equiv \gamma_x ,$$

where $\gamma_x : E_x \to (JE)_x : y \mapsto \xi_x(c_x, y)$. Then, by gluing these objects, we obtain smooth global objects.

So, we define a system of connections of E as a pair (E,ξ) where

i) $p_C : C \to B$ is a fibred manifold, called "the space" of the system;

ii) $\xi : C \times_B E \to JE$ is a fibred morphism over E, called the "evaluation morphise" of the system.

Let $\mathcal{C}$ be the sheaf of local sections $c : B \to C$. We have the induced sheaf-morphism

$$\xi : \mathcal{C} \to \mathscr{S}(JE/E) : c \to c \equiv \xi \circ \tilde{c},$$

where $\tilde{c} : E \to C \times_B E : y \mapsto (c(p(y), y))$ is the natural extension of $c : B \to C$.

Then ξ associates with each section $c : B \to C$ the connection $c \equiv \xi \circ \tilde{c} : E \to JE$. Such sections $c \equiv \xi \circ \tilde{c}$ are the distinguished sections of the system.

The coordinate expression of ξ is

$$(x^\lambda, y^i, y^i_\lambda) \circ \xi = (x^\lambda, y^i, \xi^i_\lambda) \quad \text{i.e.} \quad \xi = d^\lambda \otimes \partial_\lambda + \xi^i_\lambda d^\lambda \otimes \partial_i, \text{ with } \xi^i_\lambda : \Omega(C \times_B E).$$

II - UNIVERSAL CONNECTION.

Let (C,ξ) be a system of connections; we shall deal with the bundle $\tilde{p} : F \equiv C \times_B E \to C$, whose base space is C, and its jet prolongation $JF \equiv J(C \times_B E/C)$.

The aim of this section is to show that there is a canonical connection on $F \to C$, called universal connection. Any connection of the system can be obtained from the universal connection by a pull-back and an analogous result holds for the curvature. Different approaches to the concept of connection lead to different ways to introduce the universal connection.

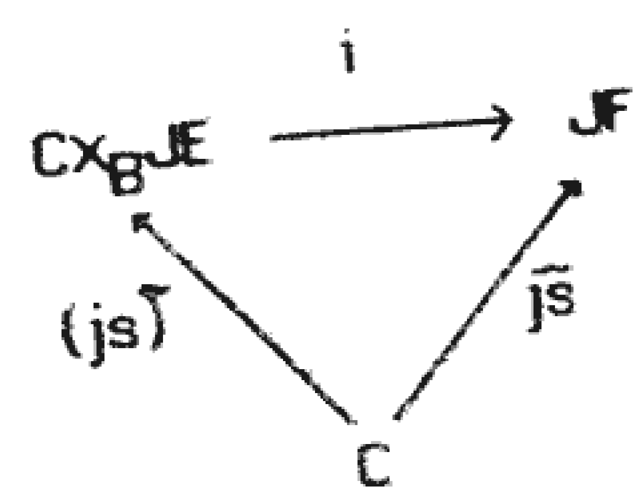
1 - Universal connection.

1 - First we need the following technical results.

LEMMA. *There is a unique fibred monomorphism over* F

$$i: C\times_B JE \to JF,$$

such that $\forall s\in\mathcal{S}(E/B)$, *the following diagram commute*



where $\tilde{s}: C\to C\times_B E$ *is the natural extension of* $s: B\to E$ *and* $(js)^\sim: C\to C\times_B JE$ *is the natural extension of* $js: B\to JE$.

We denote as (x^λ, a^α) *a fibred chart of* C, *then we have the following coordinate expression*

$$(x^i, a^\alpha, y^i_\lambda, y^i_\alpha)\circ i = (x^\lambda, a^\alpha, y^i_\lambda, 0) \qquad (*)$$

Moreover i *is affine over* $C\times_B E$.

Proof. Let $(a, y^1)\in C_x\times J_x E$, with $x\in B$. Then there is $s\in\mathcal{S}(E/B)$ such that $js(x)=y^1$. Then $(js)^\sim(a)=(a,y^1)$. So, if i exists, then we have $i(a,y^1)=(\widetilde{js})(a)$. Moreover, we have

$(x^\lambda, a^\alpha; y^i; y^i_\lambda, y^i_\alpha)\circ \widetilde{js} = (x^\lambda, a^\alpha; s^i; \partial_\lambda s^i, 0)$ and

$(x^\lambda, a^\alpha; y^i; y^i_\lambda)\circ (js)^\sim = (x^\lambda, a^\alpha; s^i; \partial_\lambda s^i)$.

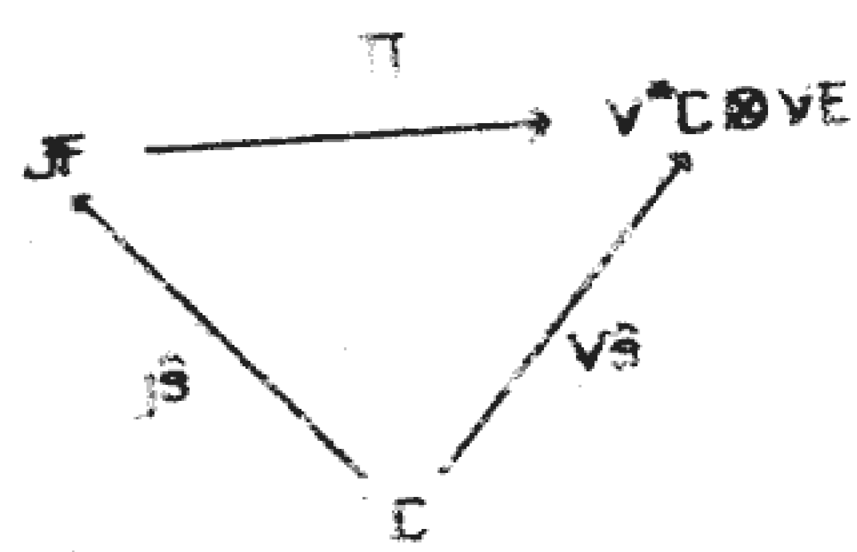
Hence, if i exists, then it has the coordinate expression (*).

Furthermore the map $i:(a,v)\mapsto (j\hat s)(a)$ exists because the coordinate expression shows that it does not depend on the choice of s.

The coordinate expression shows also that i is affine.

By the way, in order to achieve a better comprehension of this inclusion, we can state a further result.

LEMMA. *There is a unique projection* $\Pi : J\mathcal{F} \to V^*C \otimes VE$ *such that* $\forall \hat s \in \mathcal{S}(C\times_B E/C)$, *the following diagram commutes*



Moreover, we have $\Pi^i_\alpha = y^i_\alpha$.

Proof. Let $z \in J_a\mathcal{F}$, with $a\in C$. Then there is $\hat s \in \mathcal{S}(C\times_B E/C)$ such that $j\hat s(a)=z$. So, if Π exists, then we have $\Pi z = V\hat s(a)$. Moreover we have

$$(x^\lambda, a^\alpha; y^i; y^i_\lambda, y^i_\alpha)\circ j\hat s = (x^\lambda, a^\alpha; s^i; \partial_\lambda s^i, \partial_\alpha s^i);$$

$$(x^\lambda, a^\alpha; y^i; \dot\partial_\alpha \otimes \dot y^i)\circ V\hat s = (x^\lambda, a^\alpha, s^i, \partial_\alpha s^i).$$

Hence, if Π exists, then we have $\Pi^i_\alpha \equiv (\dot\partial_\alpha \otimes \dot y^i)\circ \Pi = y^i_\alpha$.

Furthermore the map $\Pi : (j\hat s(a)) \mapsto V\hat s(a)$ exists because it does not depend on the choice of $\hat s$.

COROLLARY. *We have the following fibred sequence of affine bundles over* F

$$C\times_B JE \xrightarrow{i} J(C\times_B E/C) \xrightarrow{\Pi} V^*C\otimes VE,$$

which yelds, the following exact sequence of associated vector bundles over F

$$0 \to C\times_B T^*B\otimes VE \to T^*C\otimes VE \to V^*C\otimes VE \to 0.$$

2 - Now, in view of the first approach to universal connection, we state the following proposition.

PROPOSITION. *Let* $\tilde{\xi}: C\times_B E \to C\times_B JE$ *be the natural extension of* ξ. *Then, the map*

$$\Lambda \equiv i\circ\tilde{\xi}: F \to JF: (a,y) \mapsto i(a,(\xi(a,y))$$

is a section.

The coordinate expression of $i\circ\tilde{\xi}$ *is*

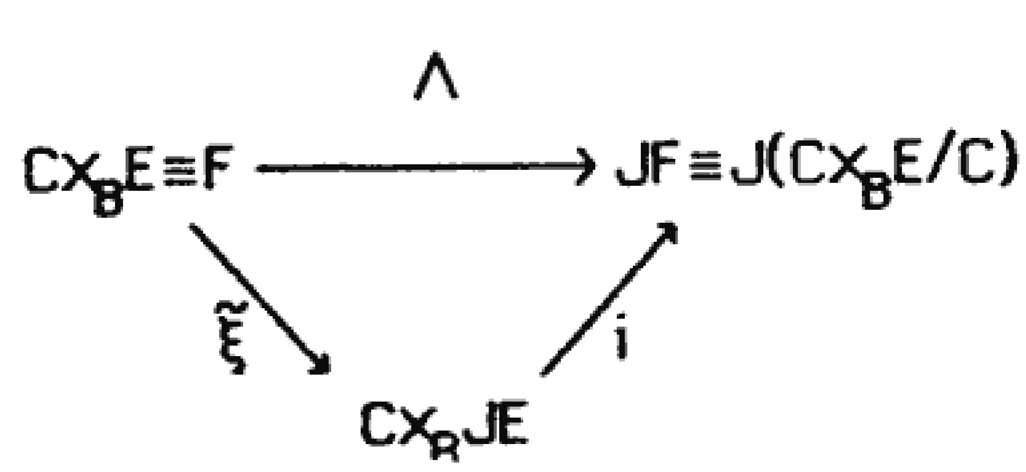
$$(x^\lambda, a^\alpha; y^i; y^i_\lambda, y^i_\alpha)\circ i\circ\tilde{\xi} = (x^\lambda, a^\alpha; y^i, \xi^i_\lambda, 0).$$

Then we can give the following definition.

DEFINITION. *Let* (C,ξ) *be a system of connections. The universal connection is the section*

$$\Lambda: C\times_B E \equiv F \to JF \equiv J(C\times_B E/C)$$

on the bundle $F\to C$ *given by the following commutative diagram*



□

3 - On the other hand, Λ can be viewed also as a tangent valued 1-form

$$\Lambda : F \to T^*C \otimes TF = T^*C \otimes TC \times_{C \times TB} T^*C \otimes TE,$$

i.e. as a linear fibred morphism over F

$$TC \times_B E \equiv TC \times_C F \to TF \equiv TC \times_{TB} TE,$$

projectable over $1 : C \to T^*C \otimes TC$.

Its coordinate expression is

$$\Lambda = d^\lambda \otimes \partial_\lambda + d^\alpha \otimes \partial_\alpha + \xi^i_\lambda d^\lambda \otimes \partial_i .$$

We can exhibit a further approach to this aspect of Λ in an indipendent way.

PROPOSITION. Λ *is the unique connection on* F *such that,* $\forall c \in C$, *we have the following commutative diagram*

$$\begin{array}{ccc} TC \times_B E & \xrightarrow{\ \Lambda\ } & TC \times_{TB} TE \\ {\scriptstyle Tc \times id_E} \uparrow & & \downarrow {\scriptstyle \pi_2} \\ TB \times_B E & \xrightarrow[\ c\ ]{} & TE \end{array}$$

Proof. It is proved analogously to the inclusion $C \times_B JE \to JF$ (II.I).

The previous diagram shows that any connection of the system can be obtained from Λ by a pull-back. This is the reason of the name "universal connection".

4 - The universal connection can be viewed in this further equivalent way.

LEMMA. *The fibred morphism over* F

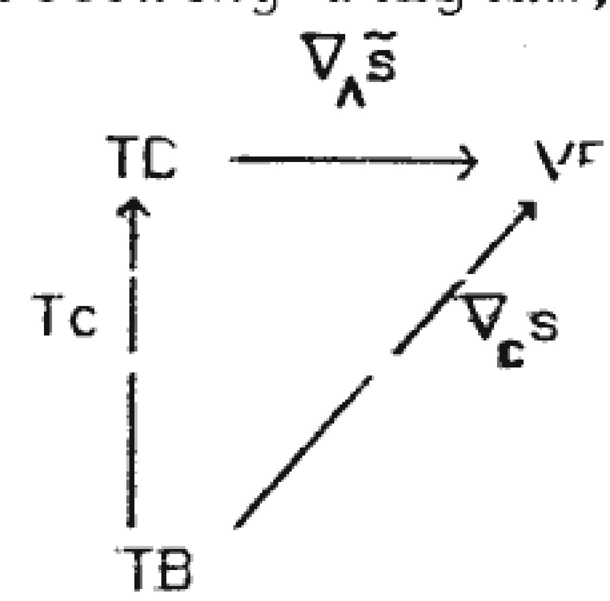
$$\omega_\Lambda : TC\times_{TB}TE \equiv TF \to VF \equiv C\times_B VE : (u_a, v_y) \to (a, \omega_a(v_y)),$$

where $\omega_a : T_y E \to V_y E$ *is the projection associated with pointwise connection* $a \in C_x$ *and* $x = p_C(a) \equiv p(y)$, *is a connection on the bundle* $F \to C$. *Moreover, the fibred morphism* ω_Λ *is just the projection associated with* $\Lambda : F \to JF$.

Proof. It is easily proved by the coordinate expression.

5 - We shall give another equivalent approach to universal connection.

PROPOSITION. *The universal connection is characterized by the commutativity of the following diagram, for any* $s \in \mathcal{S}(E/B)$ *and* $c \in \mathcal{S}(C/B)$



where $\tilde{s} \in C \to F$ *is the natural extension of* $s \in B \to E$.

Proof. It is easily proved by the coordinate expression.

2 - Universal curvature.

The universal connection yields naturally the universal curvature.

DEFINITION. *The* ***universal curvature*** *is the curvature of the universal connection.*

Hence the universal curvature is the vertical valued 2-form

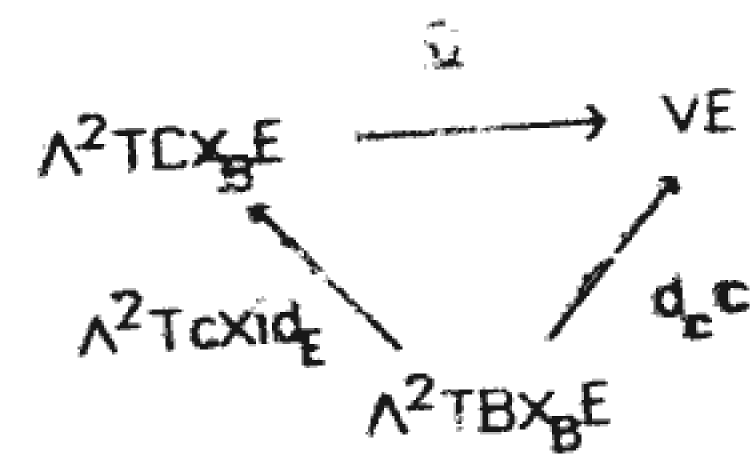
$$\Omega \equiv d_\Lambda \Lambda \equiv 1/2[\Lambda, \Lambda] : F \to \Lambda^2 T^* C \otimes VF = \Lambda^2 T^* C \otimes VE.$$

Its coordinate expression is

$$\Omega=((\partial_\lambda \xi^i_\mu+\xi^j_\lambda\partial_j \xi^i_\mu)d^\lambda\wedge d^\mu+\partial_\alpha\xi^i_\mu d^\alpha\wedge d^\mu)\otimes\partial_i .$$

The name "universal curvature" is due to its pull-back property in analogy with the universal connection.

PROPOSITION. *The universal curvature is the unique section* $\Omega:C\times_B E\to\Lambda^2T^*C\otimes VE$ *which makes the following diagram commutative*



□

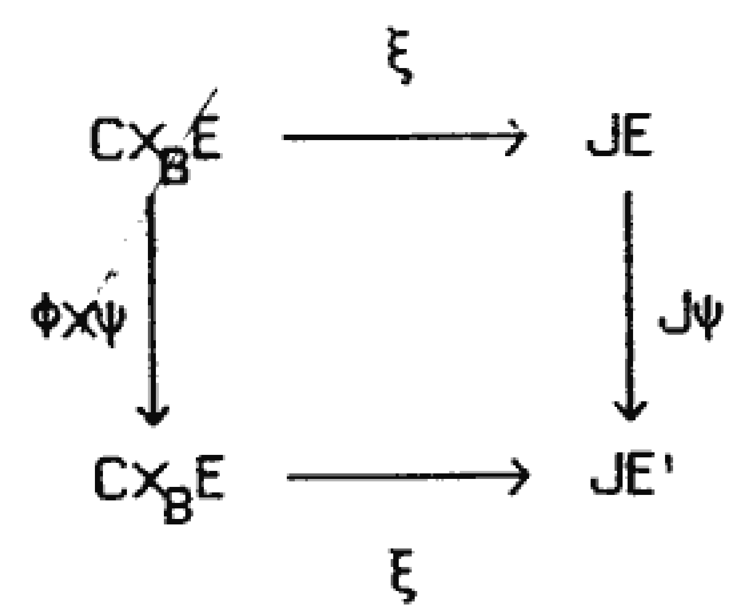
In other words, the curvature of each connection of the system is obtained from the universal curvature, by pull-back via the connection itself.

Of course, the universal curvature has all the properties of the generic curvatures. In particular, we have $d_\Lambda\Omega=0$.

3 - Invariance properties.

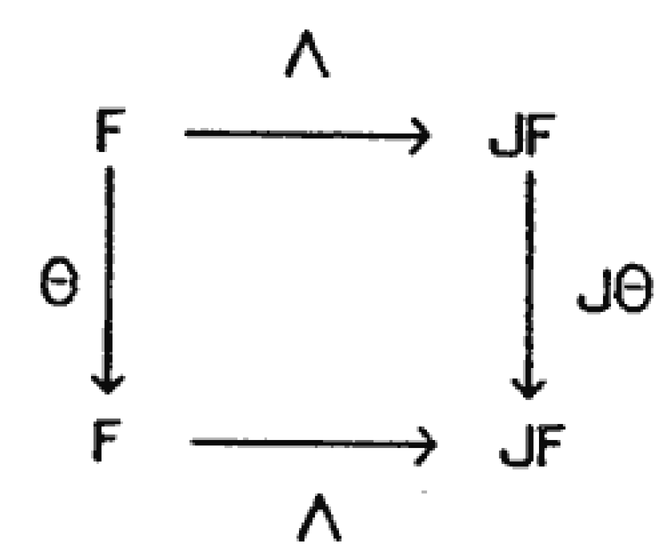
Let (C,ξ) be a given system of connections. We shall show that the universal connection and its curvature have some natural invariance properties.

DEFINITION. ***An endomorphism*** *of* (C,ξ) *is a fibred endomorphism* $\Theta\equiv(\phi\times\psi):F\equiv C\times_B E\to C\times_B E\equiv F$ *over* $\phi:C\to C$, *where* $\phi:C\to C$ *and* $\psi:E\to E$ *are, respectively, a fibred automorphim and a fibred endomorphim over* B, *such that the following diagram commutes*

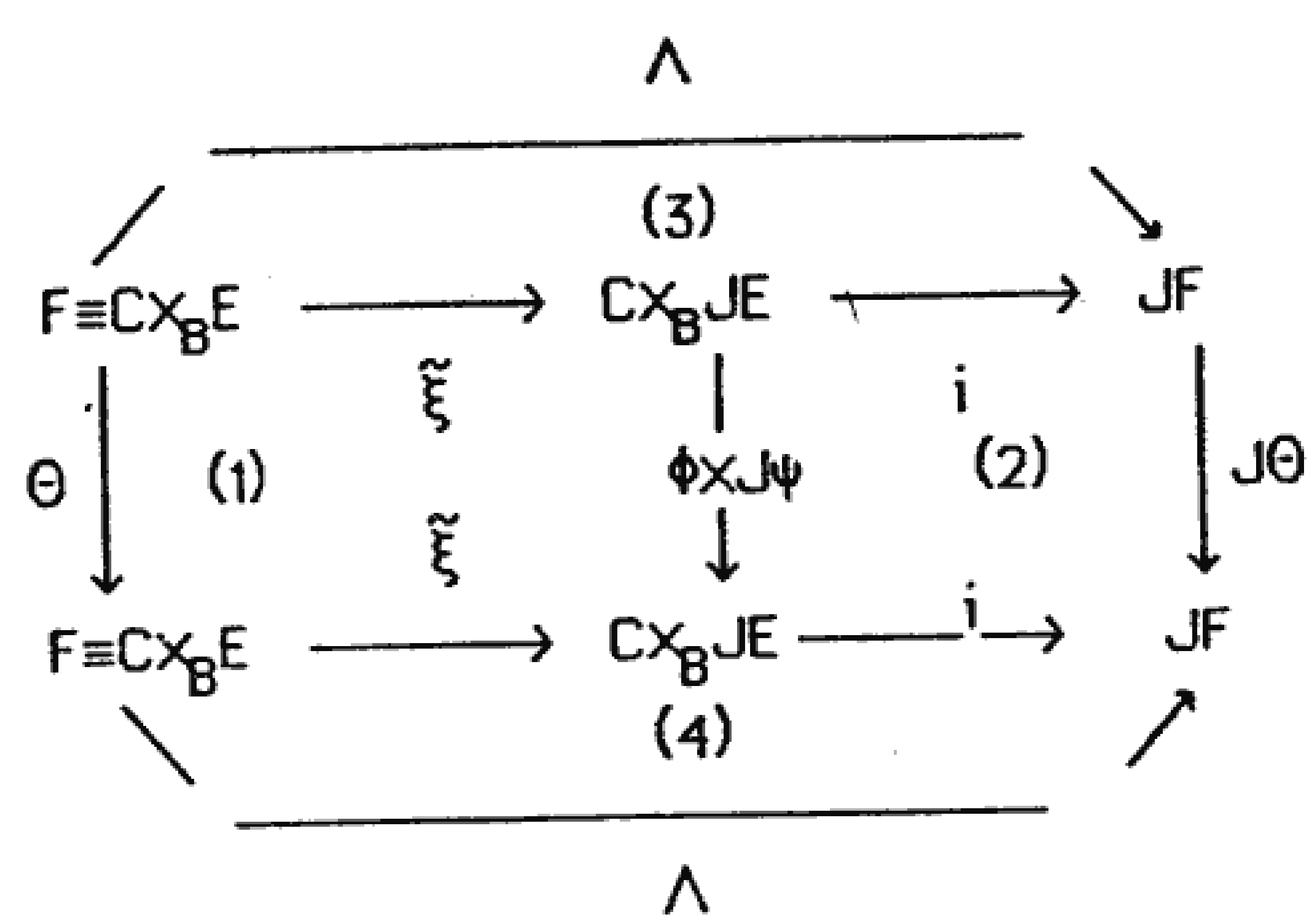


□

THEOREM. *For each endomorphism Θ, the following diagram commutes*



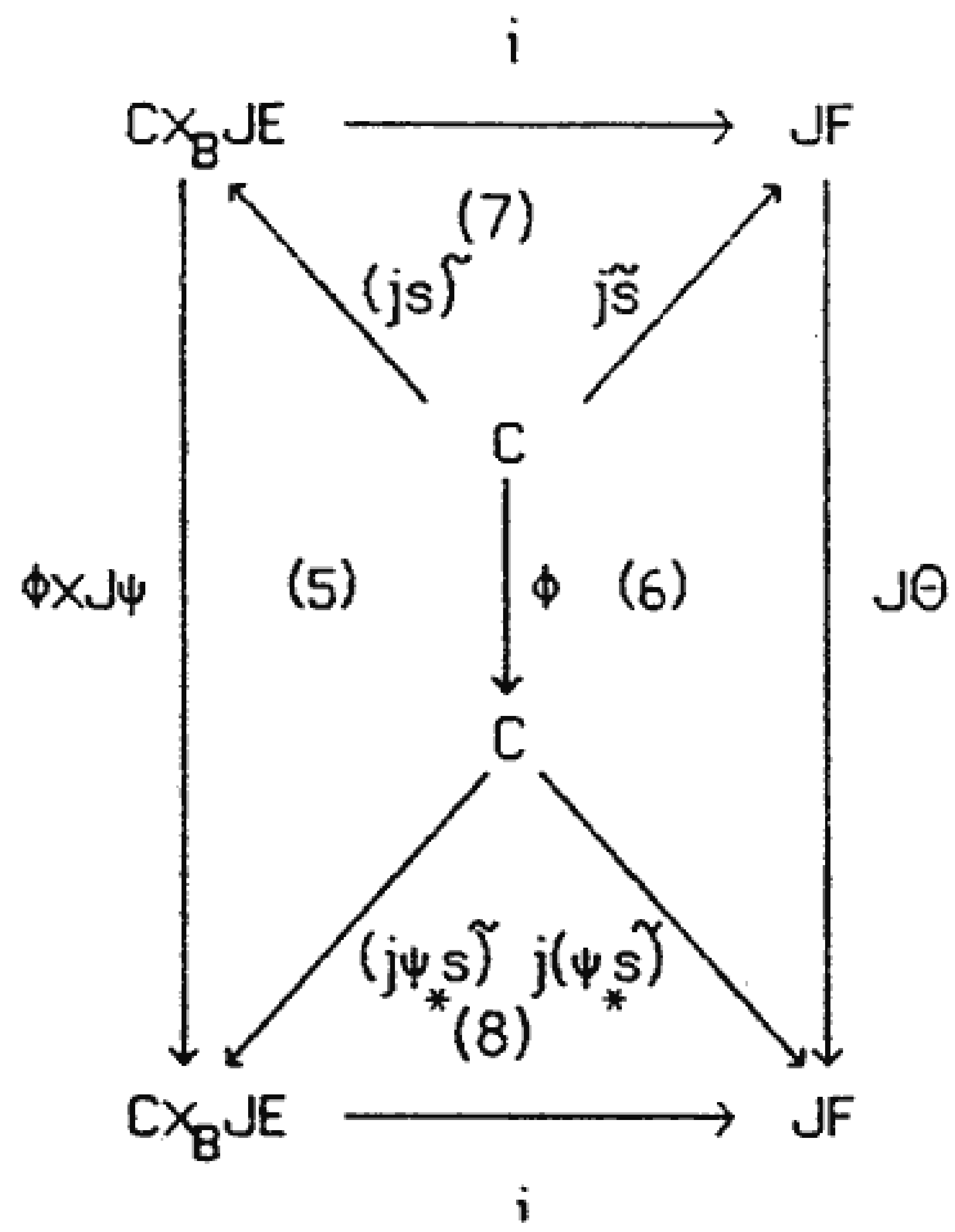
Proof. The thesis is proved if the following diagram commutes



Now, the subdiagram (3) and (4) commute by definition of Λ.

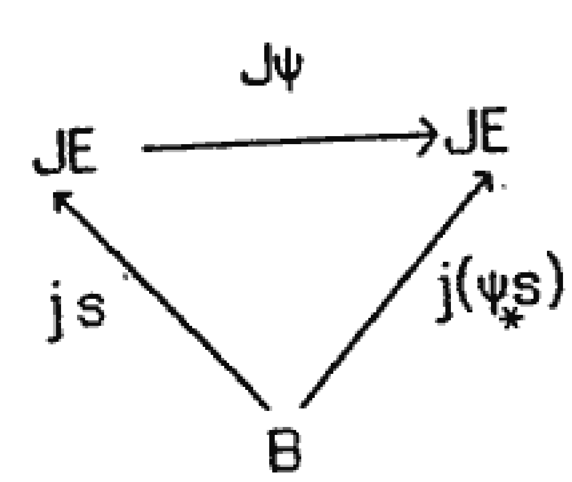
Moreover, the subdiagram (1) commutes by definition of Θ.

So, it sufficies to prove that the subdiagram (2) commutes. This fact will follow from the commutativity of the following diagram for each (local) section $s:B\to E$.

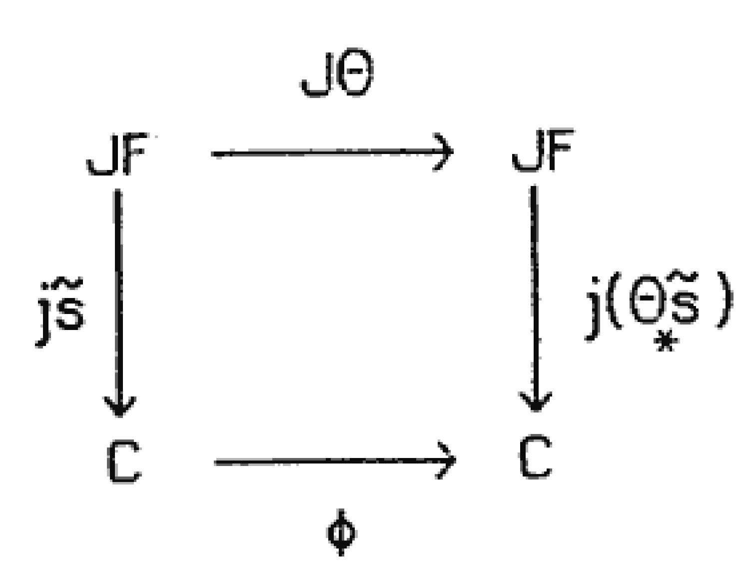


Now the subdiagram (7) and (8) commute by definition of i.

Moreover the subdiagram (5) commutes as a consequence of the commutes diagram



Furthermore, from the commutativity of



follows $\theta * \hat{s} = (\psi * s)\check{}$, and then the commutativity of the subdiagram (6).

The statement of the previous theorem can be expressed as saying that Λ is **invariant** with respect to any endomorphism θ. Next, we see that the invariance of the universal connection yields the invariance of the universal curvature.

COROLLARY. *For each endomorphism θ, the following diagram commute*

$$\begin{array}{ccc} \Lambda^2 TC & \xrightarrow{\ \Omega\ } & VE \\ {\scriptstyle \Lambda^2 T\phi}\downarrow & & \downarrow{\scriptstyle V\psi} \\ \Lambda^2 TC & \xrightarrow[\ \Omega\ ]{} & VE \end{array}$$

Proof. It follows immediately from the invariance property of the Frölicher-Nijenhuis bracket, by taking into account

$$\Omega = 1/2\,[\Lambda, \Lambda].$$

III - EXAMPLES.

The relevance of the concepts of system and universal connection is that in almost all applications one has some structure on the fibres, which induces a distinguished system. In this section we shall examine some of the most examples.

1 - The linear case.

Let $p:E\to B$ be a vector bundle. We denote by (x^λ, y^i) the linear fibred charts of E.

1 - The jet prolongations

$$J+:JE\times_B JE\to JE,\quad J-:JE\to JE,\quad J\cdot:(\mathbb{R}\times E)\times_B JE\to JE,\quad j0:B\to JE$$

of the fibred morphisms and null section that characterize the vector bundle structure, induce naturally a vector structure on the bundle $Jp:JE\to B$.

The linear fibred charts (x^λ, y^i) of E induce the linear fibred charts $(x^\lambda, y^i, y^i_\lambda)$ of JE.

Analogously, the tangent prolongations

$$T+:TE\times_{TB}TE\to TE,\quad T-:TE\to TE,\quad T\cdot:\mathbb{R}\times TE\to TE,\quad T0:TB\to TE$$

induce naturally a vector structure on the bundle $Tp:TE\to TB$; similar result holds for the vertical prolongation.

2 - We define a linear connection as a local section $\gamma:E\to JE$ which is a linear fibred morphism over B. Its coordinate expression is

$$\gamma = d^\lambda\otimes\partial_\lambda + \gamma^i{}_{\lambda j}\, y^j d^\lambda\otimes\partial_i, \qquad \text{with} \quad \gamma^i{}_{\lambda j}\in\Omega_B.$$

The curvature ρ of a linear connection γ turns out to be a linear fibred morphism $\rho:E\to\Lambda^2T^*B\otimes VE$ over B. Since $VE=E\times_B E$, we can see the curvature also as a section $\rho:B\to\Lambda^2T^*B\otimes E^*\otimes E$. Its coordinate expression is

$$\rho=(\partial_\lambda\gamma^i_{\mu j}+\gamma^h_{\lambda j}\gamma^i_{\mu h})y^j d^\lambda\wedge d^\mu\otimes\partial_i.$$

3 - Now we can show that the linear connections constitute a system (C,ξ) of connections.

For this purpose, we define, $\forall x\in B$, C_x as the subspace $C_x\subset L(E_x,J_xE)$ constituted by all linear maps $c_x:E_x\to J_xE$ which are sections, i.e. such that $p_{0x}\circ c_x=id:E_x\to E_x$. Hence $C_x\subset L(E_x,J_xE)$ turns out to be an affine subspace, whose vector space is $T^*_xB\otimes E^*_x\otimes E_x$, (i.e. the vector subspace of $L(E_x,J_xE)$ which is projected onto $0:E_x\to E_x$).

Obviously, we have, $\forall x\in B$, the evalutation map

$$\xi_x:C_x\times E_x\to J_xE:(c_x,y)\to c_x(y).$$

By gluing these objects for all $x\in B$, we obtain smooth objects. Namely, we have the affine bundle

$$p_C:C\equiv\cup_{x\in B}C_x\to B,$$

whose vector bundle is $T^*B\otimes E^*\otimes E\to B$, and the fibred morphism over E

$$\xi\equiv\cup_{x\in B}\xi_x:C\times_B E\to JE.$$

The linear fibred charts (x^λ,y^i) of E induce naturally affine fibred charts of C, which will be denoted by $(x^\lambda,a^i_{\lambda j})$.

Then, the coordinate expression of ξ is

$$(x^\lambda, y^i, y^i_\lambda)\circ\xi = (x^\lambda, y^i, a^i_{\lambda j} y^j) \quad \text{i.e.} \quad \xi = d^\lambda \otimes \partial_\lambda + a^i_{\lambda j} y^j d^\lambda \otimes \partial_i .$$

By construction, (C,ξ) is the system of linear connections.

Namely, with each local section $c:B\to C$ is associated a linear connection $c\equiv \xi\circ\tilde{c}:E\to JE$. Conversely, each linear connection $\gamma:E\to JE$ comes from a local section $c:B\to C:x\to c(x)\cong\gamma_x$.

The coordinate expression of c is

$$c=\xi\circ\tilde{c}=d^\lambda\otimes\partial_\lambda + c^i_{\lambda j} y^j d^\lambda\otimes\partial_i, \quad \text{with } c^i_{\lambda j}\in\Omega_B .$$

4 - Finally, we can consider the vector bundle $F\equiv C\times_B E\to C$. The induced linear fibred charts are $(x^\lambda, a^i_{\lambda j}; y^i)$.

The universal connection $\Lambda:F\to JF$ turns out to be a linear connection and the universal curvature $\Omega:F\to\Lambda^2T^*C\otimes VE$ a linear fibred morphism over C.

Their coordinate expressions are

$$\Lambda = d^\lambda\otimes\partial_\lambda + d^i_{\lambda j}\otimes\partial^{\lambda j}_i + a^i_{\lambda j} y^j d^\lambda\otimes\partial_i$$

$$\Omega = (a^h_{\lambda j} a^i_{\mu h} d^\lambda\wedge d^\mu + d^i_{\lambda j}\wedge d^\mu) y^j\otimes\partial_i .$$

2 - The affine case.

Let $p:E\to B$ be an affine bundle and $\bar{p}:\bar{E}\to B$ its vector bundle. We denote by (x^λ, y^i) the affine fibred chart of E and by $(x^\lambda, \bar{y}^i)$ the induced linear fibred chart of $\bar{E}$.

1 - The jet prolongation $J+:JE\times_B J\bar{E}\to JE$, of the fibred morphism that characterizes the affine bundle structure, induces naturally

an affine structure on the bundle $Jp:JE\to B$ with associated vector bundle $J\bar{p}:J\bar{E}\to B$.

The fibred charts (x^λ,y^i) of E and $(x^\lambda,\bar{y}^i)$ of $\bar{E}$ induce the affine fibred chart $(x^\lambda,y^i,y^i_\lambda)$ of JE and the linear fibred chart $(x^\lambda,\bar{y}^i,\bar{y}^i_\lambda)$ of $J\bar{E}$.

Analogously, the tangent prolongation $T+:TE\times_{TB}T\bar{E}\to TE$ induces naturally an affine structure on the bundle $Tp:TE\to TB$ with associated vector bundle $T\bar{p}:T\bar{E}\to TB$; similar results hold for the vertical prolongation.

2 - We define a linear affine connection as a local section $\gamma:E\to JE$ which is a linear affine morphism over B. Its coordinate expression is

$$\gamma = d^\lambda\otimes\partial_\lambda+(\gamma^i_{\lambda j}y^i+\hat{\gamma}^i_\lambda)d^\lambda\otimes\partial_i \quad \text{with} \quad \gamma^i_{\lambda j},\hat{\gamma}^i_\lambda\in\mathcal{F}_B$$

The curvature ρ of a linear affine connection γ turns out to be an affine fibred morphism $\rho:E\to\Lambda^2T^*B\otimes VE$ over B. Since $VE=E\times_B\bar{E}$, we can see the curvature also as section $\rho:B\to A(E,\Lambda^2T^*B\otimes\bar{E})$, where A indicates the set of all affine morphism. Its coordinate expression is

$$\rho=((\partial_\lambda\gamma^i_{\mu j}+\gamma^h_{\lambda j}\gamma^i_{\mu h})y^j+\partial_\lambda\hat{\gamma}^i_\mu+\hat{\gamma}^h_\lambda\gamma^i_{\mu h})d^\lambda\wedge d^\mu\otimes\partial_i$$

By deriving γ and ρ with respect to the fibres, we obtain the linear connection $\bar{\gamma}:\bar{E}\to J\bar{E}$ and its linear curvature $\bar{\rho}:\bar{E}\to\Lambda^2T^*B\otimes V\bar{E}$.

In other words, a connection $\gamma:E\to JE$ is linear affine if there exists a linear connection $\bar{\gamma}:\bar{E}\to J\bar{E}$ such that $\gamma(y+v)=\gamma(y)+\bar{\gamma}(v)$, $\forall(y,v)\in E\times_B\bar{E}$, i.e. such that the following diagram commutes

$$\begin{array}{ccc} E\times_B E & \xrightarrow{(\gamma,\bar{\gamma})} & JE\times_B J\bar{E} \\ \tau\downarrow & & \downarrow J\tau \\ E & \xrightarrow[\gamma]{} & JE \end{array}$$

3 - Next we show that linear affine connections constitute a system (C,ξ) of connections.

For this purpose we define C_x, $\forall x\in B$, as the subspace $C_x\subset A(E_x,J_xE)$ constituted by all affine maps $c_x:E_x\to J_xE$ which are sections, i.e. such that $p_{0x}\circ c_x=id:E_x\to E_x$. Hence, C_x turns out to be an affine subspace of $A(E_x,J_xE)$ and its vector space is the vector subspace of $A(E_x,J_x\bar{E})$ which is projected onto $0:E_x\to\bar{E}_x$.

Obviously we have, $\forall x\in B$, the evaluation map

$$\xi_x;C_x \times E_x\to J_xE:(c_x,y)\mapsto c_x(y).$$

By gluing these objects for all $x\in B$, we obtain smooth objects. Namely, we obtain the linear affine bundle

$$p_C:C\equiv\cup_{x\in B}C_x\to B,$$

whose vector bundle is $A(E,T^*B\otimes\bar{E})$, and the linear affine fibred morphism over E

$$\xi \equiv \cup_{x\in B}\xi_x \;:C\times_B E\to JE.$$

The linear fibred charts (x^λ,y^i) of E induce naturally affine fibred charts of C, which will be denoted by $(x^\lambda,a^i_{\lambda j},\hat{a}^i_\lambda)$.

Then, the coordinate expression of ξ is

$(x^\lambda, y^i, y^i_\lambda)\circ\xi \; (x^\lambda, y^i, a^i_{\lambda j}y^j+\hat{a}^i_\lambda)$ i.e. $\xi = d^\lambda \otimes \partial_\lambda + (a^i_{\lambda j}y^j+\hat{a}^i_\lambda)d^\lambda\otimes\partial_i$.

With respect to the vector bundle $\bar{p}:\bar{E}\to B$, by following the same procedure as in the previous example, we obtain the linear affine bundle

$$p_{\bar{C}}:\bar{C}\equiv \cup_{x\in B}\bar{C}_x \to B,$$

whose vector bundle is $T^*B\otimes\bar{E}^*\otimes\bar{E}\to B$, the affine fibred morphism over $\bar{E}$

$$\bar{\xi}\equiv \cup_{x\in B}\bar{\xi}_x : \bar{C}\times_B\bar{E}\to J\bar{E}$$

which is a linear morphism over $\bar{C}\to B$, the induced linear fibred charts $(x^\lambda, \bar{a}^i_{\lambda j})$ of $\bar{C}$ and the coordinate expression of $\bar{\xi}$

$$(x^\lambda, \bar{y}^i, \bar{y}^i_\lambda)\circ\bar{\xi} = (x^\lambda, \bar{y}^i, \bar{a}^i_{\lambda j}\bar{y}^j) \quad \text{i.e.} \quad \bar{\xi} = d^\lambda\otimes\partial_\lambda + \bar{a}^i_{\lambda j}\bar{y}^j d^\lambda\otimes\partial_i.$$

The "derivative" map, $\forall x\in B$, $D_x : A(E_x, J_xE)\to L(\bar{E}_x, J_x\bar{E}) : c_x\mapsto\bar{c}_x$, induces a fibred morphism $D:C\to\bar{C}$ over B. Its coordinate expression is $(x^\lambda, \bar{a}^i_{\lambda j})\circ D = (x^\lambda, a^i_{\lambda j})$.

By construction (C,ξ) is the system of affine connections.

Namely, with each local section $c:B\to C$ is associated an affine connection $c\equiv\xi\circ\tilde{c}:E\to JE$. Conversely, each local affine connection $\gamma:E\to JE$ comes from the local section $c:B\to C:x\mapsto c(x)\cong\gamma_x$.

Moreover, if $\gamma:E\to JE$ is the affine connection, which comes from the section $c:B\to C$, then its derived linear connection $\bar{\gamma}\equiv D\gamma:\bar{E}\to J\bar{E}$ comes from the derived section $\bar{c}\equiv Dc:B\to\bar{C}$.

Consequently the coordinate expressions of c and $\bar{c}\equiv Dc$ are

$$c=d^\lambda\otimes\partial_\lambda+(c^i_{\lambda j}y^j+\hat{c}^i_\lambda)d^\lambda x\partial_i \quad \text{and} \quad \bar{c}=d^\lambda\otimes\partial_\lambda+c^i_{\lambda j}\bar{y}^j d^\lambda\otimes\partial_i,$$

with $c^i_{\lambda j},\hat{c}^i_\lambda\in\Omega_B$.

4 - Finally, we can consider the affine bundle $F\equiv C\times_B E\to C$. The induced linear affine fibred charts are $(x^\lambda,a^i_{\lambda j},\hat{a}^i_\lambda;y^i)$.

The universal connection $\Lambda:F\to JF$ turns out to be an affine connection and the universal curvature $\Omega:F\to\Lambda^2T^*C\otimes VE$ an affine fibred morphism over C. Their coordinate expressions are

$$\Lambda=d^\lambda\otimes\partial_\lambda+d^i_{\lambda j}\otimes\partial^{\lambda j}_i+(a^i_{\lambda j}y^i+\hat{a}^i_\lambda)d^\lambda\otimes\partial_i,$$

$$\Omega=((a^h_{\lambda j}y^j a^i_{\mu h}+\hat{a}^h_\lambda\ a^i_{\mu h})d^\lambda\wedge d^\mu+y^j d^i_{\lambda j}\wedge d^\mu)\otimes\partial_i.$$

3 - The principal case.

1 - First let us recall some results about group bundles.
Let $\bar{p}:\bar{E}\to B$ be a group bundle.
The jet prolongations

$$J\mu:J\bar{E}\times_B J\bar{E}\to J\bar{E},\quad J\iota:J\bar{E}\to J\bar{E},\quad j1:B\to J\bar{E}$$

of the fibred morphisms and unity section, that characterize the group bundle structure, induce naturally a group structure on the bundle $J\bar{p}:J\bar{E}\to B$.

Analogously, the tangent prolongations

$$T\mu:T\bar{E}\times_{TB}T\bar{E}\to T\bar{E},\quad T\iota:T\bar{E}\to T\bar{E},\quad T1:TB\to T\bar{E}$$

induce naturally a group structure on the bundle $T\bar{p}:T\bar{E}\to TB$; similar results hold for the vertical prolongation.

In particular, if G is a Lie group and $\bar{E}\to B\times G$, then we have the canonical fibred isomorphism $J\bar{E}=T^*B\otimes TG\simeq T^*B\otimes(G\times\mathcal{G})=G\times(T^*B\otimes\mathcal{G})$, where $\mathcal{G}$ is the Lie algebra of G. Observe that $J\bar{E}\to B$ may be not trivial, even if $\bar{E}\equiv B\times G\to B$ is trivial. Moreover, as $\bar{E}\to B$ is trivial, we have the fibred morphism $j:\bar{E}\to J\bar{E}:(x,g)\mapsto(jg)(x)$ over B, where we identify $g\in G$ with the constant section $g:B\to\bar{E}$. Actually j turns out to be induced by the canonical group monomorphism $G\hookrightarrow TG=G\times\mathcal{G}$.

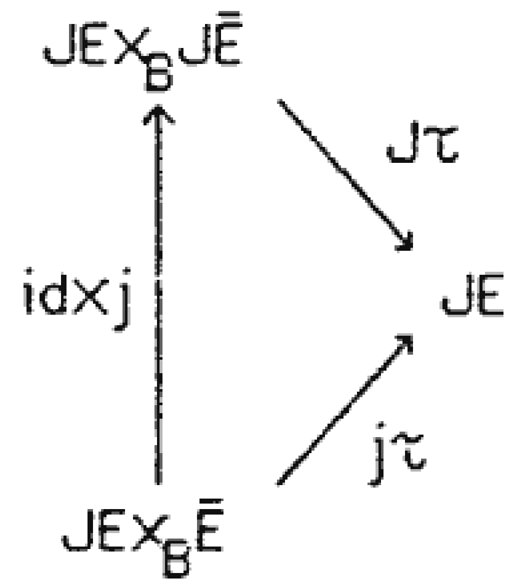
2 - Now, let $p:E\to B$ be a principal bundle with structure group G and right action $\tau:E\times_B\bar{E}\equiv E\times G\to E$ over B.

The jet prolongation $J\tau:JE\times_B J\bar{E}\to JE$ of the (right) action $\tau:E\times\bar{E}\to E$ is a fibred (right) free and transitive action over B, with respect to the group bundle $J\bar{E}=G\times(T^*B\otimes\mathcal{G})\to B$. Thus, $JE\to B$ turns out to be a group affine bundle, but not a principal bundle, just because $J\bar{E}$ is not a product bundle over B.

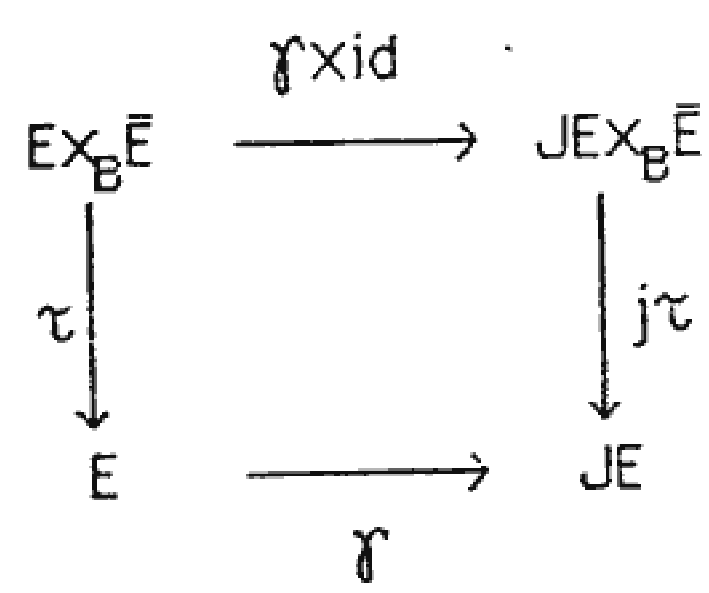
Analogously, the tangent prolongation $T\tau:TE\times_{TB}T\bar{E}\to TE$ is a fibred (right) free and transitive action over TB, with respect to the group bundle $T\bar{E}$.

However, by taking into account $j:\bar{E}\to J\bar{E}$, we obtain the restricted (right) fibred action $j\tau:JE\times\bar{E}\to JE$ over B, which is free, but no more transitive.

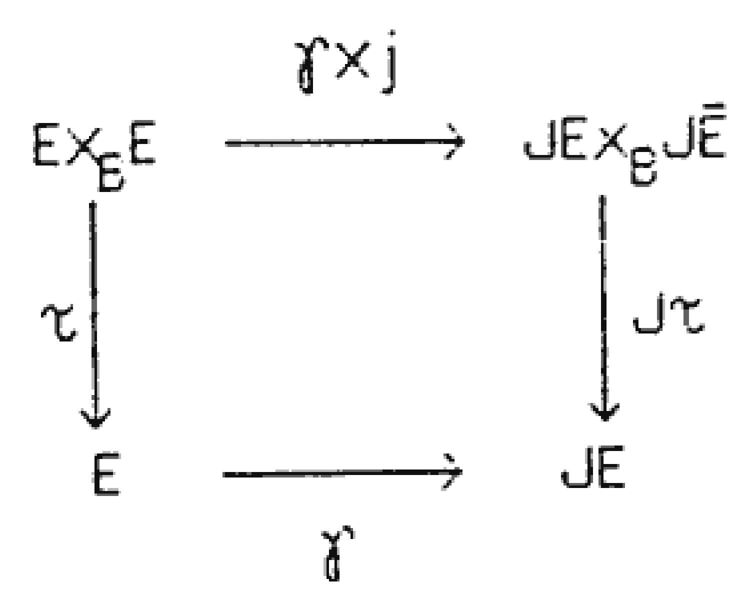
By construction, the following diagram commutes



3 - We define a principal connection as a local section $\gamma: E \to JE$, which is invariant with respect to the action of the group, i.e. such that the following diagram commutes



Equivalently, a connection $\gamma: E \to JE$ is a principal connection if $\gamma(xg) = \gamma(x)g$, $\forall (x,g) \in E \times \bar{E}$, i.e. if the following diagram commutes



Thus a principal connection is just a group affine connection, whose "group fibre derivative" is j.

A fibred chart (x^λ) on $X \subset B$ and a chart (u^i) on $U \subset G$ together with a section $s_o: X \to p^{-1}(X) \subset E$ induce a fibred chart, called "principal", on E, as follows. We denote as

$$\bar{\tau}: E \times_B E \to G$$

the map characterized by

$$\tau(a,\bar{\tau}(a,b))\equiv b.$$

For any $g\in G$, we put

$$\tau_g : E\to E : a \mapsto \tau(a,g).$$

We consider the map

$$\tau_o \equiv \bar{\tau}\circ(s_o\circ p, id_E) : p^{-1}(X)\to G : a\mapsto \bar{\tau}(s_o(p(a)),a).$$

Then, we consider the G-principal bundle $E_X\equiv p^{-1}(X)\subset E$ and the local principal trivialization $(p,\tau_o):p^{-1}(X)\to X\times G$ of the principal bundle $p:E\to B$ and its inverse local principal fibred isomorphism

$$\sigma_o : X\times G\to p^{-1}(X).$$

The principal fibred chart (x^λ, y^i) on $Y\equiv E_X\cap\sigma_o(X\times U)\subset E$ is defined by

$$x^\lambda \equiv x^\lambda\circ p : Y\to \mathbb{R}^m$$

$$y^i\equiv u^i\circ\tau_o : Y\to\mathbb{R}^n .$$

We remark that the coordinate expression of τ does not depend on the coordinates on the base space, i.e. we have $\partial_\lambda\tau^i=0$.

We shall use the following notation

$$\sigma^i_j : a\mapsto \partial\tau^i/\partial y^j(s_o(p(a)),\tau_o(a)) : Y\to\mathbb{R}.$$

Then, the coordinate expression of the principal connection turns out to be

$$\gamma = d^\lambda\otimes\partial_\lambda + \gamma^j_{o\,\lambda}\sigma^i_j d^\lambda\otimes\partial_i ,$$

where $\gamma^j_{o\lambda}(a) = \gamma^j_\lambda(s_o \circ p(a))$. In others words, we have

$$\gamma^i_\lambda = \gamma^j_{o\lambda}\sigma^i_j \quad ,$$

where $\gamma^j_{o\lambda}$ dependes only on the x^λ,s and σ^i_j depends only on the y^i,s.

The curvature ρ of a principal connection γ turns out to be a G-invariant (because of the G-invariance of the connection and the naturality of the F.N.bracket)fibred morphism $\rho:E\to\Lambda^2T^*B\otimes VE$ over B.

Its coordinate expression is

$$\rho=(\sigma^i_h\partial_\lambda\gamma^h_{o\mu}+\gamma^h_{o\lambda}\gamma^k_{o\mu}\sigma^j_h\partial_j\sigma^i_k)d^\lambda\wedge d^\mu\otimes\partial_i .$$

4 - Now we can show that principal connections constitute a system (C,ξ) of connections.

For this purpose, we consider the quotient space JE/G.

Let us recall that the bundle $JE/G\to B$ is the affine subbundle of the bundle $(T^*B\otimes TE)/G=T^*B\otimes(TE/G)\to B$, which is projected onto $1\subset T^*B\otimes TB$, whose vector bundle is $(T^*B\otimes VE)/G=T^*B\otimes(VE/G)\to B$.

We define, $\forall x\in B$, C_x as the subspace $C_x\subset P(E_x,J_xE)$ constituted by the principal (i.e. G-invariant cf.(1.1)), maps $c_x:E_x\to J_xE$ which are sections, i.e. such that $p_{ox}\circ c_x=id:E_x\to E_x$.

Let us show that there is a canonical isomorphism $C_x\cong J_xE/G$. For each $x\in B$, the map

$$C_x\to J_xE/G:c_x\mapsto[c_x(a)]$$

is well defined (i.e. does not depend on the choice of $a\in E_x$, because c_x is G-invariant). Similarly, the map

$$q_x : J_xE/G \to C_x : bG \to c_x$$

where $$c_x : E_x \to J_xE : a \mapsto b\bar{\tau}(p_o(b), a),$$

is well defined (i.e. does not depend on the choice of $b \in J_xE$, because of the definitions of J_xE/G and C_x).

Then, these maps determine a diffeomorphism $C_x \to J_xE$, by which we shall identify the two spaces.

Obviously, we have, $\forall x \in B$, the evaluation map

$$\xi_x : C_x x E_x \to J_xE : (c_x, y) \mapsto c_x(y).$$

We can easily prove that $\xi_x : C_x x E_x \to J_xE$ is a diffeomorphism, whose inverse is $(q_x, p_{ox}) : J_xE \to C_x x E_x$.

By gluing these objects for all $x \in B$, we obtain smooth objects. Namely, we have the linear affine bundle

$$p_C : C \equiv \cup_{x \in B} C_x \equiv \cup_{x \in B} J_xE/G \to B,$$

whose vector bundle is $T^*B \otimes VE/G \to B$, and the fibred isomorphism over B

$$\xi \equiv \cup_{x \in B} \xi_x : C x_B E \to JE.$$

We denote by (x^λ, a^i_λ) the fibred chart of $C \equiv JE/G$, where

$$a^i_\lambda : JE/G \to \mathbb{R}^n : [v] \to ((\sigma^{-1})^i_j y^j_\lambda)(v).$$

Then, the coordinate expression of ξ is

$$(x^\lambda, y^i, y^i_\lambda) \circ \xi = (x^\lambda, y^i, \sigma^i_j a^j_\lambda) \quad \text{i.e.} \quad \xi = d^\lambda \otimes \partial_\lambda + (\sigma^i_j a^j_\lambda) d^\lambda \otimes \partial_i.$$

By construction, (C,ξ) is the system of principal connections. Namely, with each local section $c:B\to C$ is associated a principal connection $c\equiv\xi\circ\tilde{c}:E\to JE$. Conversely, each local principal connection $\gamma:E\to JE$ comes from the local section $c:B\to C:x\mapsto c(x)\tilde{=}\gamma_x$.

The coordinate expression of c is

$$c=\xi\circ\tilde{c}=d^\lambda\otimes\partial_\lambda+\sigma^i_j c^j_\lambda d^\lambda\otimes\partial_i, \quad \text{with } c^j_\lambda\in\Omega_B.$$

5 - Finally, we can consider the bundle $F\equiv C\times_B E\to C$. The induced linear fibred charts are $(x^\lambda,a^i_\lambda;y^i)$.

The universal connection $\Lambda:F\to JF$ turns out to be a principal connection and the universal curvature $\Omega:F\to\Lambda^2T^*C\otimes VE$ a principal fibred morphism over C.

Their coordinate expressions are

$$\Lambda=d^\lambda\otimes\partial_\lambda+d^i_\lambda\otimes\partial^\lambda_i+(\sigma^i_j a^j_\lambda)d^\lambda\otimes\partial_i,$$
$$\Omega=[(a^h_\lambda a^k_\mu\sigma^j_h\partial_j\sigma^i_k)d^\lambda\wedge d^\mu+\sigma^i_h d^h_\mu\wedge d^\mu]\otimes\partial_i.$$

4 - Liouville and symplectic forms.

Let $p:E\equiv M\times\mathbb{R}\to B\equiv M$ be the trivial principal bundle with structure group $\mathbb{R}$.

1 - Let us recall the fibred monomorphism $\lambda:JE\hookrightarrow T^*B\otimes TE$ over E and the complementary linear epimorphism $\tilde{\vartheta}:TJE\to VE$. In our particular case, we have

$$J(M\times\mathbb{R}/M)\hookrightarrow T^*M\otimes T(M\times\mathbb{R})=T^*M\otimes TM\times(T^*M\times\mathbb{R})$$

and, more precisely, we have

$$J(M\times\mathbb{R}/M) = id_{TM}\times(T^*M\times\mathbb{R}).$$

Then, we have the canonical fibred isomorphism over M

$$JE \simeq \mathbb{R} \times T^*M$$

which given, also,

$$T^*JE \simeq \mathbb{R} \times(\mathbb{R}\times T^*T^*M) \quad \text{and} \quad VE \simeq (M\times\mathbb{R})\times_M(M\times\mathbb{R}).$$

Hence

$$T^*JE\otimes VE \simeq \mathbb{R}\times\mathbb{R}\times T^*T^*M.$$

So, we have the 1-form

$$\tilde{\vartheta}:\mathbb{R}\times T^*M\to\mathbb{R}\times\mathbb{R}\times T^*T^*M.$$

We can see that ϑ is the pull-back of ϑ with respect to $T^*M\times\mathbb{R}\to T^*M$ of

$$\vartheta:T^*M\to T^*T^*M$$

which turns out to be the Liouville form.

2 - The principal connections $c:E\equiv M\times\mathbb{R}\to JE\equiv\mathbb{R}\times T^*M$ are the sections projectable on the usual forms $\varphi:M\to T^*M$, which characterize them.

Their curvatures, $\rho:E\equiv M\times\mathbb{R}\to \Lambda^2T^*M\otimes VE\equiv\Lambda^2T^*M\otimes\mathbb{R}$, are projectable on the usual differentials $d\varphi:M\to\Lambda^2T^*M$, which characterize them.

3 - The principal connections constitute a system (C,ξ) where $p_C:C\equiv T^*M\to M$ is the "space",

$\xi=id:C\times_M E\equiv T^*M\times\mathbb{R}\to JE\equiv\mathbb{R}\times T^*M$ is the "evaluation morphism".

We denote by $(\dot{x}_\alpha, y^o)$ the fibred chart of T*MxR, where $y^o \equiv id:\mathbb{R}\to\mathbb{R}$. The coordinate expression of ξ is

$$(y^o,\dot{x}_\alpha)\circ\xi=(y^o,\dot{x}_\alpha) \quad \text{i.e.} \quad \xi=d^\lambda\otimes\partial_\lambda+\dot{x}_\alpha d^\lambda\otimes\partial_o.$$

With each local section $c:M\to T^*M$ is associated a principal connection $c:\xi\circ\tilde{c}:M\times\mathbb{R}\to\mathbb{R}\times T^*M$.

4 - Finally, we can consider the principal bundle $F:C\times_M E\equiv T^*M\times\mathbb{R}\to$ $\to C\equiv T^*M$.

We have

$$J(T^*M\times\mathbb{R}/T^*M) = id_{TT^*M}\times(T^*T^*M\times\mathbb{R})$$

Then, we have the canonical fibred isomorphism over T*M

$$J(C\times_B E/C) \simeq \mathbb{R}\times T^*T^*M.$$

We obtain the universal connection

$$\Lambda:\ C\times_B E\equiv T^*M\times\mathbb{R}\to J(C\times_B E/C)\equiv\mathbb{R}\times T^*T^*M$$

and the universal curvature

$$\Omega:C\times_B E\equiv T^*M\times\mathbb{R}\to\Lambda^2 T^*C\otimes VE\equiv\Lambda^2 T^*T^*M\times\mathbb{R}.$$

Their coordinate expression are

$$\Lambda=d^\lambda\otimes\partial_\lambda+d^\alpha\otimes\partial_\alpha+\dot{x}_\lambda d^\lambda\otimes\partial_o$$
$$\Omega=d\dot{x}_\alpha\wedge dx^\lambda+y^o\,\partial_o.$$

We note that $\Lambda=1\times\tilde{\vartheta}$ and $\Omega=\tilde{\omega}$ where $\tilde{\omega}$ is the pull-back with respect to $T^*M\times\mathbb{R}\to T^*M$ of the symplectic forms $\omega:T^*M\to\Lambda^2 T^*T^*M$.

5 - The university of the universal connection and curvature, in this case, turn out to be the following well-known classical properties:

if $\phi:M\to T^*M$ is any 1-form, then we have

$$\phi^*\vartheta=\phi \quad ; \quad \phi^*d\vartheta= d\phi .$$

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