Note di Matematica Vol.VII, 167-172(1987)

CONVEX HYPERSURFACES WITH TRANSNORMAL HORIZONS ARE SPHERES

F.J.CRAVEIRO DE CARVALHO-S.A.ROBERTSON (*)

Let M be a smooth ($=C^\infty$), compact, connected hypersurface of Euclidean (n+1)-space R^{n+1}, $n \geq 2$, with nowhere-zero Gaussian curvature. Thus M is diffeomorphic to the n-sphere S^n and every affine tangent hyperplane meets M in just one point.

Let λ be any (straight) line in R^{n+1} and let M_λ denote the set of points of M at which the tangent hyperplane is parallel to λ. We call M_λ the λ-*horizon* of M. If, for every λ, M_λ is a transnormal submanifold of R^{n+1} [5] we shall say that M is *horizon-transnormal*. In this paper we show that if M is horizon-transnormal then M is a round sphere. The converse is obviously true.

We show in §2 that if M is horizon-transnormal then it is transnormal. If M is transnormal then every λ-outline Ω_λ (see §1 below) is also transnormal. This is one aspect of a classical result in the theory of convex bodies of constant width [1] but we give a direct differential-geometric proof in §3. We prove in §4 that each λ-horizon M_λ is contained in a hyperplane normal to λ. It is then a consequence of a classical result that M must be an n-ellipsoid. Consequently, due to its transnormality, M is a round n-sphere.

(*) The first-named author is grateful to the Calouste Gulbenkian Foundation - LISBON, for financial support.

1. ORTHOGONAL PROJECTIONS. - From now on M will denote, as above, a smooth, compact, connected hypersurface of R^{n+1}, $n \geq 2$, with nowhere-zero Gaussian curvature. The tangent space at x to a smooth submanifold X of R^{n+1} will be denoted by T_xX and whenever convenient T_xX will be considered as a linear subspace of R^{n+1}. The affine tangent and normal spaces at x will be denoted by T_x and N_x respectively. Thus T_x and N_x are affine subspaces of R^{n+1}. Also f_{*x} will denote the linear map induced by a smooth map f between the tangent spaces at x and f(x).

Let λ be a straight line in R^{n+1} and H_λ a hyperplane orthogonal to λ. Let $\pi: R^{n+1} \to H_\lambda$ be the orthogonal projection of R^{n+1} into H_λ. The critical set M_λ of $\pi|M$ is a smooth compact (n-1)-submanifold of M. Such a restriction was studied in [2] and we recall that $x \in M$ is a critical point of $\pi|M$ if and only if T_x is parallel to λ. Since M has no non-zero asymptotic vectors it follows that if $x \in M_\lambda$ then

$$T_xM_\lambda + \ker \pi_{*x} = T_xM$$

(see [2] for details). That is, every point of M_λ is a *fold point*. Therefore $\pi|M_\lambda$ is an embedding and we denote by Ω_λ its image in H_λ. In fact Ω_λ is diffeomorphic to S^{n-1}. We shall call Ω_λ the λ-*outline* of M. Trivially, Ω_λ is independent of H_λ up to translation of H_λ along λ. If $m \in M_\lambda$ then we write $m' = \pi(m)$ and we have

$$T_{m'} = \pi(T_m), \; N_{m'} \| N_m \quad \text{and } N_{m'} = \pi(N_m),$$

where $N_{m'}$ is the normal line to Ω_λ at m' in H_λ. We remark that as far as verifying the transnormality of Ω_λ is concerned it is immaterial whether we consider $N_{m'}$ or the normal 2-plane in R^{n+1} to Ω_λ at m'.

2. **HORIZON-TRANSNORMALITY IMPLIES TRANSNORMALITY.** - We prove now that if, for every λ, M_λ is transnormal then M itself is transnormal. Let $x \in M$ and suppose that $y \in N_x$. Choose λ such that $\lambda \| T_x$ and $\lambda \| T_y$. Since x and y belong to M_λ and M_λ is transnormal then $T_x M_\lambda = T_y M_\lambda$. Also because x and y are fold points

$$T_x M_\lambda + \ker \pi_{*_x} = T_x M.$$

$$T_y M_\lambda + \ker \pi_{*_y} = T_y M.$$

But $\ker \pi_{*_x} = D(\lambda) = \ker \pi_{*_y}$, where $D(\lambda)$ stands for the direction of λ (that is to say, the line through 0 parallel to λ). Hence $T_x M = T_y M$ and, consequently, $N_x \| N_y$. Since $y \in N_x$ we have $N_x = N_y$.

3. **TRANSNORMALITY IS EQUIVALENT TO TRANSNORMAL OUTLINES.** - We show next that M is transnormal if and only if, for every λ, Ω_λ is transnormal.

Assume that, for every λ, Ω_λ is transnormal. Let $m \in M$ and suppose that $y \in N_m$. Choose λ such that N_m and N_y are parallel to H_λ. Then $m, y \in M_\lambda$ and $m' = \pi(m)$, $y' = \pi(y)$ belong to Ω_λ. Since Ω_λ is transnormal and $y' \in N_{m'}$ we have $N_{m'} = N_{y'}$. Consequently $N_m \| N_y$ and as they have non-empty intersection they coincide.

Suppose now that M is transnormal and take a fixed λ. Let $m' = \pi(m)$ be a point in Ω_λ. Since every affine tangent (n-1)-plane to Ω_λ

only intersects it in one point it follows that $N_{m'}$ intersects Ω_λ in just one more point, say y'. Also N_m intersects M in just another point z and $N_m = N_z$ which implies that $T_m \| T_z$ and, consequently, that $z \in M_\lambda$. Moreover $T_{m'} \| T_{z'}$ and $N_{m'} \| N_{z'}$, with $z' = \pi(z)$. Since $z' \neq m'$ and $N_{m'} = \pi(N_m)$ we conclude that $z' = y'$. Therefore $N_{m'} = N_{y'}$.

4. HORIZONS ARE CONTAINED IN HYPERPLANES. - We may assume without any loss of generality that λ is the straight line generated by $(0,\dots,0,1)$. Then M_λ can be regarded as the graph of a smooth map $h : \Omega_\lambda \to R$. That M_λ is contained in a hyperplane normal to λ is implied by the following generalization of corollary 1 in [3].

PROPOSITION 4.1. Let X be a smooth, compact, connected, transnormal hypersuface of R^n and $h : X \to R^k$, $k \leq n-1$, a smooth map. If the graph of h is transnormal then h is constant.

Proof. Since X is transnormal there is an antipodal involution [5] $\delta : X \to X$. Then if $N_{(x,h(x))} = N_{(y,h(y))}$ either $x=y$ or $\delta(x)=y$ and it follows that the graph G_h of h is also 2-transnormal. Therefore the diameter of G_h is $\|(x,h(x))-(\delta(x),h(\delta(x)))\|$, for every $x \in X$, and, consequently, $\|h(x)-h(\delta(x))\|$ is constant. By a result from [4] there exists $x_0 \in X$ such that $h(x_0)=h(\delta(x_0))$. Hence $h = h \circ \delta$.

Next we establish that, for every $x \in X$, h_{*x} is the zero linear map. Regarding δ_{*x} as an automorphism of a linear subspace of R^n all its eigenvalues $\lambda_1,\dots,\lambda_{n-1}$ are real and negative. Choose a basis (e_i) for T_xX formed by eigenvectors of δ_{*x}.

Then $T_{(x,h(x))}G_h = \ll(e_i, h_{*x}(e_i)), i=1,\ldots,n-1\gg$ and $\Gamma_{(\delta(x),h(\delta(x)))}G_h =$ $= \ll(\lambda_i e_i, (h\circ\delta)_{*x}(e_i)), i=1,\ldots,n-1\gg$, where $\ll \gg$ means generated by. Since $h=h\circ\delta$ it follows that $h_{*x}(e_i) = (h\circ\delta)_{*x}(e_i)$, $i=1,\ldots,n-1$ and therefore there are negative real numbers μ_i such that $h_{*x}(e_i) = \mu_i h_{*x}(e_i)$. Consequently, for every i, $h_{*x}(e_i) = 0$ and h_{*x} is the zero map.

5. HORIZON-TRANSNORMAL HYPERSURFACES ARE ROUND SPHERES. - Having proved that the λ-horizons are contained in hyperplanes we can use a classical result [6] to conclude that M is an n-ellipsoid. Since the only transnormal n-ellipsoids are round n-spheres we obtain our main result, namely:

THEOREM 5.1. *Let* M *be a* C^∞, *compact, connected hypersurface of Euclidean* (n+1)*-space* R^{n+1}, $n\geq 2$, *with nowhere-zero Gaussian curvature. If* M *is horizon-transnormal then it is a round* n*-sphere.*

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Ricevuto il 18/12/1987

Departamento de Matematica
Universidade de Coimbra
3000 COIMBRA - PORTUGAL

Faculty of Mathematical Studies
University of Southampton
Southampton S09 5NH - U.K.