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BI-IDEALS IN ORTHODOX SEMIGROUPS

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SOMMARIO. - *Un bi-ideale di un semigruppo S è un sottosemigruppo B di S tale che BSB ⊆ B. In questa nota si individuano delle relazioni tra i bi-ideali di un semigruppo ortodosso e i bi-ideali del suo sottosemigruppo degli idempotenti.*

Recall that a subsemigroup B (≠∅) of a semigroup S is said to be a bi-ideal of S if BSB ⊆ B. It is well-known that if S is regular then BSB = B for every bi-ideal B of S.

Let X be a non-empty subset of a semigroup S. The smallest bi-ideal of S containing X is called the bi-ideal of S generated by X and denoted by $(X)_b$. It is clear that $(X)_b = X \cup X^2 \cup XS\dot{X}$ and, if S is regular $(X)_b = XSX$.

One usually denotes by $\mathcal{B}(S)$ the semigroup of the bi-ideals of the semigroup S.

An orthodox semigroup is defined as a regular semigroup in which the idempotents form a subsemigroup. An orthodox and completely regular semigroup is called an orthogroup. A semigroup S is called intra-regular if $a = xa^2y$, where x,y are suitable elements of S, for every $a \in S$. A semigroup S is called an inverse semigroup if every a in S has a unique inverse, i.e. if there exists a unique

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element a^{-1} in S such that $a = aa^{-1}a$ $a^{-1}aa^{-1} = a^{-1}$.

For the terminology and for the definitions of algebraic theory of semigroups, we refer to J.M.Howie [1].

If S is an orthodox semigroup with the band of idempotents E, then every bi-ideal B of S has at least one idempotent. In fact, if b is an element of B and x is an element of S such that b = bxb, bxxb is an idempotent of B. Moreover, if $E(B)=E\cap B$, E(B) is a bi-ideal of E. But B is not the unique bi-ideal of S containing E(B).

Example. - Let S be the bicyclic semigroup $\mathscr{C}(p,q)$, which has the band of idempotents $E = \{q^n p^n\}_{n\in N_o}$. It is not difficult to see that $\mathscr{B}(E) = \{E_n\}_{n\in N_o}$ with $E_n = \{q^{n+h}\ p^{n+h}\}_{h\in N_o}$ and $\mathscr{B}(S) = \{B_{m,n}\}_{m,n\in N_o}$ with $B_{m,n} = \{q^a p^b : a \geq m, b \geq n\}$. For any positive integer n, E_n is the band of idempotents of the bi-ideals $B_{m,n}$ for any $m \leq n$.

THEOREM 1. *Let* S *be an orthodox semigroup with band of idempotents* E. *Then, for every bi-ideal* E' *of* E, *there exists a unique orthodox bi-ideal* B *of* S *with band of idempotents* E'; *moreover* B *is the bi-ideal of* S *generated by* E'.

Proof. Let E' be a bi-ideal of E, then B = E'SE' is a bi-ideal of S. Moreover, if e is an idempotent of B, there are elements e',e" of E' and an element s of S such that e = e'se" then e = e'se" = e'e'se"e" = e'ee" is an element of E'. Therefore E' is the band of idempotents of B. Now, if b is an element of B, there are e', e'' $\in$ E' and s, x $\in$ S such that b = e'se'' = (e'se'')x(e'se'')= = e'se''e''xe'e'se''=be''xe'b. Since e''xe'$\in$B, b is a regular element of B and therefore B is an orthodox semigroup. Finally let D be an orthodox bi-ideal of S with band of idempotents E'; then, if d is an element of D, there is x $\in$ D such that d=dxd=dxdxd with dx,xd $\in$ E'; therefore d $\in$ E'SE' and D=B.

COROLLARY 1. - Let S be an orthogroup with band of idempotents E. Then for every bi-ideal E' of E, there exists an unique bi-ideal B of S. Moreover, B is an orthogroup, with band of idempotents E'.

This corollary follows immediately from the preceding theorem and from Theorem 4 in [2].

COROLLARY 2. - Let S be an orthodox semigroup with band of idempotents E of type $\mathcal{P}$, where $\mathcal{P}$ is any one of the types of band classified by M. Petrich in [3]. Then, for every bi-ideal E' of E, there exists an unique bi-ideal B of S, B orthodox, with band of idempotents of type $\mathcal{P}$ and this is E'.

Let S be an orthodox semigroup with band of idempotents E and let E' be a bi-ideal of E. Denote by G(E') the set of bi-ideals of S whose band of idempotents is E'.

THEOREM 2. - *Let* S *be an orthodox semigroup with band of idempotents* E *and let* I *be an ideal of* S *, then there exists an ideal* E' *of* E *such that* I *is the orthodox bi-ideal in* G(E').

Proof. It is known ([4]) that the ideals of a regular semigroup are regular. Therefore, if S is an orthodox semigroup and I is an ideal in S, then I is an orthodox bi-ideal. Moreover, if E' is the band of idempotents of I, then E' is an ideal of E and I is the orthodox bi-ideal in G(E').

Observe that if E' is an ideal of E, the orthodox bi-ideal in G(E') generally is not an ideal of S. In fact in the bicyclic semigroup every bi-ideal of E is an ideal, because E is a semilattice, but the semigroup is simple.

THEOREM 3. - *Let* S *be an inverse semigroup with semilattice of idempotents* E *and let* E' *be a bi-ideal of* E. *If* B *is a bi-ideal of* S *and* B *in* G(E'), *then* $B' = \{y \in S : y^{-1} \in B\}$ *is a bi-ideal of* S *and* $B' \in G(E')$. *Moreover* $B = \{y \in S : y^{-1} \in B'\}$.

Proof. Let E' be a bi-ideal of E and let B be a bi-ideal of S such that $B \in G(E')$. If $y_1, y_2 \in B'$ and $x \in S$, then $(y_1 x y_2)^{-1} = y_2^{-1} x^{-1} y_1^{-1} \in B$, i.e. $y_1 x y_2 \in B'$ and B' is a bi-ideal of S. Moreover, if e is an idempotent of B', then $e = e^{-1}$ is an idempotent of B. Analogously, if e is an idempotent of B, $e = e^{-1}$ is an idempotent of B'. Therefore B' is in G(E'). Finally, from the definition of B' it follows that $B = \{y \in S : y^{-1} \in B'\}$.

Let S be an orthodox semigroup with band of idempotents E; then $Z : \mathcal{B}(E) \to \mathcal{B}(S)$ defined by $Z(E') = E'SE'$ for every bi-ideal E' of E is a mapping.

THEOREM 4. - *If* S *is an orthodox semigroup with band of idempo-*

tents E, *the mapping* Z *is one - to - one.*

Proof. Let E' and E'' be two bi-ideals of E such that E'SE' = E''SE''. Then E' = E ∩ (E'SE') = E ∩ (E''SE'') = E''. Thus Z is one - to - one.

THEOREM 5 (Theorem 6 of [2]). *If* S *is an orthogroup with band of idempotents* E, *then the mapping* Z *is a homomorphism, i.e.*

$$(E'SE')(E''SE'') = E'E''SE'E'' \quad \text{for all } E', E'' \text{ in } \mathscr{B}(E).$$

Example. - Let S be the bicyclic semigroup $\mathscr{C}(p,q)$ and let E_n, E_m $(n,m \in N_o)$ be two bi-ideals of E. Then $Z(E_nE_m) = B_{t,t}$, where $t = \max(m,n)$, whereas $Z(E_n)Z(E_m) = B_{n,m}$. Therefore $Z(E_nE_m) \neq Z(E_n)Z(E_m)$.

THEOREM 6. - *Let* S *be an orthodox and intra-regular semigroup. Then the following are equivalent:*

1) S *is an orthogroup;*
2) *the mapping* Z *is a homomorphism;*
3) *the mapping* Z *is onto.*

Proof. 1) ⟹ 2). It is Theorems 5.

2) ⟹ 3) Let Z be a homomorphism, let B be a bi-ideal of S and let $\bar{E}$ be its band of idempotents. Then $\bar{E}S\bar{E} \subseteq B$. Moreover, if b is an element of B and x is one of its inverses and if E' = bxEbx and E'' = xbExb, then by the hypothesis and because $\mathscr{B}(E)$ is a normal band ([5])

$$E'E''SE'E'' = E'SE'E''SE'' = E'SEE'E''ESE'' = E'SEE''E'ESE'' = E'SE''E'SE''.$$

Moreover $(E'SE'')^2 = E'SE''$ because $E'SE'' \in \mathscr{B}(S)$ and $\mathscr{B}(S)$ is a band ([4]). Hence $E'E''SE'E'' = E'SE''$ and $b \in E'E''SE'E''$. Moreover $E'E'' = bxEbx \quad xbExb \subseteq B \cap E = \bar{E}$ and so $b \in \bar{E}S\bar{E}$. Finally $B = \bar{E}S\bar{E}$ and Z is onto.

3) $\Longrightarrow$ 1) Let Z be onto; then for every $B \in \mathcal{B}(S)$ there is $E' \in \mathcal{B}(E)$ such that $B = E'SE'$. Moreover, by Theorem 1, B is an orthodox bi-ideal of S. Then, by Theorem 4 of [2], S is an orthogroup.

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