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ENTROPY AND CHARACTERISTIC EXPONENTS FOR THE QUADRATIC MAP ON THE UNIT INTERVAL

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ABSTRACT. - *We point out some facts about topological and measure entropy for the quadratic map on the unit interval. We show the equality between the measure entropy and the characteristic exponent.*

INTRODUCTION. This is a continuation of a previous paper by the author [1] in which the quadratic map $S_a:[0,1]\to[0,1]$ defined by $S_a(x) = ax(1-x)$ is considered, for the countable set $\mathcal{A}$ of values of the parameter $a = a_p \in [0,4]$ satisfying the conditions of Pianigiani [5]. For such values of $a\in\mathcal{A}$, a unique absolutely continuous S_a-invariant ergodic measure exists. All this is discussed at length in [1], where topological entropy was evaluated for the set of values of $a \in\mathcal{A}$. Precisely, it was shown that the sequence of the topological entropies $\{h_t(S_{a_p})\}$ valuated for $a=a_p$, $p>2$, $p \in \mathbb{N}$ is weakly increasing with p and as $p\to\infty$ (or $a_p\to 4$) one has:

$$h_t(S_{a_p}) \to h_t(S_4) = h(S_4,\mu_4),$$

where $h(S_a, \mu_a)$ denotes the measure entropy of the map S_a with respect to the S_a-invariant measure μ_a, and for $a=4$, μ_4 is the well-known S_4-invariant measure $\mu_4(dx) = [\pi\sqrt{x(1-x)}]^{-1}dx$. Moreover, $h(S_a,\mu_a) > 0$ for $a \in\mathcal{A}$. Generally, we have $h(S_a,\mu_a) \le h_t(S_a)$ and only for a=4 the equality is known to hold.

In the present paper, we emphasize that the equality between topological and measure entropy for S_a, $a \in \mathscr{A}$ never holds for $a = a_p$, $p > 2$.

We show that $h(S_a, \mu_a) = \lambda_a$, where λ_a denotes the characteristic exponent of S_a. So, the knowledge of λ_a enables us to evaluate the measure entropy $h(S_a, \mu_a)$. We have found numerical estimates for λ_a. In particular, for p=2, $a=a_p=3.67857351...$ (this value first was estimated by Ruelle [7]) numerical evidence suggests that $\lambda_a = 0.3465.. = \log\sqrt{2} = h_t(S_a)$ (for the last equality see [1]).

Thus, the only cases in which $h(S_a, \mu_a) = h_t(S_a)$, $a \in \mathscr{A}$ are: for a=4 (this is rigorous), and for a=3.67857351...(p=2) (this is a numeric result).

1. We recall the condition of Pianigiani:

(i) $\exists\ p > 2$ such that $S_a^p(1/2) < 1/2$ and $S_a^p(1/2) = 1 - S_a^{2p}(1/2)$

(ii) $S_a^k(I) \cap (1/a, 1-1/a) = \emptyset$, $k=1,2,...p-1$ and $S_a^p(I) = 1$, where $I = (S_a^p(1/2), S_a^{2p}(1/2))$.

A countable set of values of $a \in [2,4]$ satisfying the above conditions exists ([5]). For such values we consider the measure μ_a, which is S_a-invariant, ergodic and absolutely continuous with respect to the Lebesgue measure on the interval [0,1]. The support A_I of the measure μ_a is not the interval [0,1], but a finite union of subintervals: $A_I = \bigcup_{k=0}^{p-1} A_k$, where $A_k = S_a^k(I)$; A_k is mapped onto A_{k+1} under the action of S_a, except for k=p-1 when A_k is mapped onto I.

Actually, we have for $a = a_p$:

$$(1.1) \qquad h(S_a,\mu_a) \le h_t(S_a/\operatorname{supp}\mu_a) \le p^{-1}\log 2 \le h_t(S_a)$$

and the equality never holds for $p > 2$. Infact, for $p > 2$, $p^{-1}\log 2 < h_t(S_a)$ as follows from the results of [1] (see the appendix) and so, by (1.1):

$$(1.2) \qquad h(S_a,\mu_a) < h_t(S_a), \qquad p > 2, \quad a = a_p.$$

The only case in which the equalities in (1.1) are possible is for p=2. Indeed, in this case we have $h_t(S_a) = \log\sqrt{2} = (\log 2)/2$, and one can conjecture the equality between topological and measure entropy since numerical evaluation of the characteristic exponent λ_a gives $\lambda_a = \log\sqrt{2}$.

DEFINITION 1.1. (Characteristic exponents).

Relatively to the map S_a, we consider the following limits for almost every x with respect to μ_a:

$$(1.3) \qquad \lambda_a(x) = \lim(1/n)\log|S_a'(S_a^{n-1}x)\dots S_a'(x)|.$$

We call the number $\lambda_a(x)$ the "characteristic exponent" of S_a at the point x.

DEFINITION 1.2. We say that $x \in [0,1]$ is a "regular point" if the following conditions hold: (see [4])

(1) the sequence of measures $(1/n)\,\Sigma_{i=0,\dots n-1}\,\delta_{S_a^i(x)}$ converges vaguely toward an ergodic measure μ_x,

(2) $(1/n)\,\log|S_a^{n'}(x)| \to \lambda_a(x)$;

(3) $\lambda_a(x) = \int \log|S_a'|\,d\mu_x$.

A regular point is said to be positive regular if $\lambda_a(x) > 0$. As follows by the results in ([4]), the set of regular points has Lebesgue measure one and the set of positive regular points has positive measure. Moreover, since μ_a is ergodic, $\lambda_a(x)$ does not depend upon x choosen in the set of positive regular points, so $\lambda_a(x) = \lambda_a > 0$, as will be proved shortly.

Generally, we have $h(f,\mu) < \lambda(x)$ for a map f with characteristic exponent $\lambda(x)$. In our case the Pesin's formula holds:

$$h(S_a, \mu_a) = \lambda_a \quad \text{for } a \in \mathscr{A}.$$

This follows from the proposition:

PROPOSITION. If $a \in \mathscr{A}$, then $h(S_a, \mu_a) = \lambda_a$, where λ_a is the characteristic exponent of S_a.

Proof. We have: $\lambda_a(x) = \lim n^{-1} \log|S_a^{n'}(x)|$, and so:

$$\lambda_a(S_a(x)) = \lim n^{-1}\, \Sigma_{i=0,\dots n-1} \log|S_a'(S_a^i(S_a x))| =$$

$$= \lim n^{-1}\, \Sigma_{i=1,\dots n} \log|S_a'(S_a^i x)| =$$

$$= \lim n^{-1} \Sigma_{i=0,\dots n} \log|S'_a(S^i_a(x))| =$$

$$= \lim(n+1)^{-1} \Sigma_{i=0,\dots n} \log|S_a(S^i_a x)| =$$

$$= \lambda_a(x).$$

Thus, $\lambda_a(S_a(x)) = \lambda_a(x)$.

Since μ_a is ergodic $\lambda_a(x) = \text{constant} = \lambda_a > 0$ for μ-a.e. x.

Generally, if $\mu = \int \mu_x \; \mu(dx)$ is the ergodic decomposition of the f-invariant measure μ into ergodic measures, we have (see [4]):

$$h(f,\mu) = \int h(f,\mu_x)\,\mu(dx) = \int \max(0,\lambda(x))\,\mu(dx).$$

In our case, since μ_a is ergodic and $h(S_a, \mu_a) > 0$, we have:

$$0 < h(S_a,\mu_a) = \int h(S_a,\mu_a)\,\mu_a(dx) =$$

$$= \int \max(0,\lambda_a(x)) \;\; \mu_a(dx) = \lambda_a.$$

CONCLUSIONS. From the discussion above, it follows that, for $a \in \mathcal{A}$

$$h(S_a,\mu_a) = \lambda_a.$$

From numerical table of λ_a (see [1] and the graphic below), we see that $\lambda_a < h_t(S_a)$.

Numerical estimate of λ_a by means of a computer for a=3.67857351...

(p=2) gives $\lambda_a = 0.3465... = \log\sqrt{2}$.

So, for $a \in \mathcal{A}$, $h(S_a, \mu_a) < h_t(S_a)$, except for a = 4 when equality holds and, possibly, for a = 3.67857351... when equality is strongly suggested by numerical evidence.

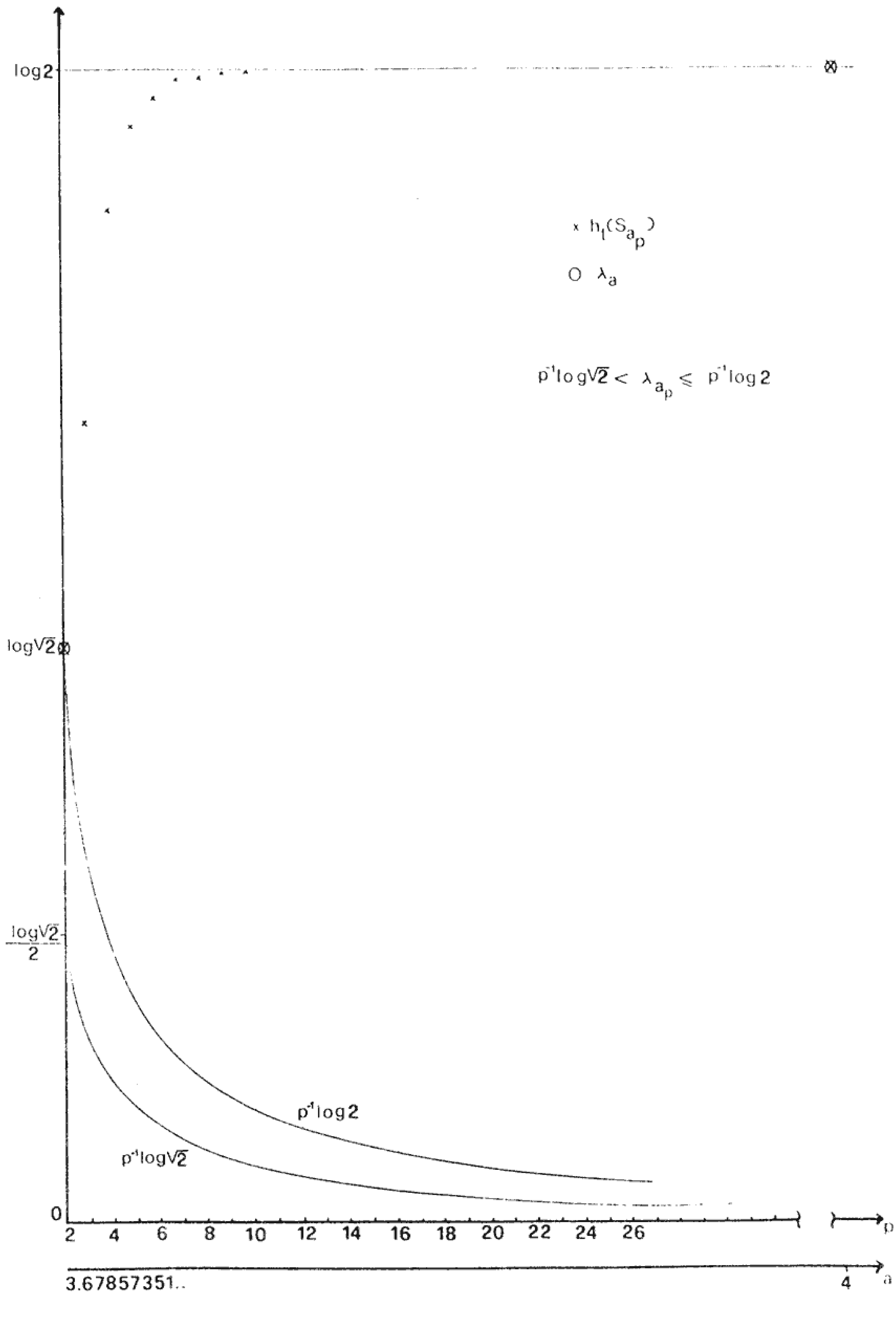
REMARK. Really, our set $\mathcal{A}$ is a proper subset of the set $\mathcal{A}^\wedge$ of values of $a \in [0,4]$ such that the a.c. S_a-invariant ergodic measure μ_a exists; indeed $\mathcal{A}^\wedge$ has positive measure ([2],[3]). It was conjectured that $\{a \in [0,4]: \lambda_a > 0\}$ has positive measure, too (for instance, see [6]).

The proposition above is also true for $a \in \mathcal{A}^\wedge$, so $h(S_a, \mu_a) = \lambda_a$, for $a \in \mathcal{A}^\wedge$.

Estimates are known of λ_a, $a \in [0,4]$ and $h_t(S_a)$ computed by means of a purely numerical approach (see [6]). From these estimates it appears that $\lambda_a < h_t(S_a)$ for $a \in [0,4]$ except near the value of a for which $h_t(S_a) = \log\sqrt{2}$, and for a=4. Moreover, it seems that $\{a \in [0,4] : \lambda_a > 0\}$ may be a set of positive measure, and λ_a is zero or near zero for a lot of values of a. This agrees with our bounds to λ_{a_p} and the fact that $\lambda_{a_p} \to 0$, as $a_p \to 4$ (see the graphic below).

We observe that our values of $h_t(S_{a_p})$, $a_p \in \mathcal{A}$ are rigorous estimates, since they derive from numerical solutions of algebraic equations of degree p (see [1]).

The numerical computations have been executed by means of a Micro Vax II computer.


log2
log√2
log√2 / 2
0
× h_t(S_{a_p})
O λ_a
p⁻¹log√2 < λ_{a_p} ≤ p⁻¹log2
p⁻¹log2
p⁻¹log√2
2 4 6 8 10 12 14 16 18 20 22 24 26
p
3.67857351..
4
a

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