Note di Matematica Vol. VII, 41-45(1987)

LOCALLY DETERMINING SEQUENCES IN INFINITE DIMENSIONAL SPACES

José M. ANSEMIL (*) - Seán DINEEN (**)

SOMMARIO. *Mostriamo che ogni insieme localmente determinante in zero (per funzioni olomorfe) sopra uno spazio metrizzabile separabile e localmente convesso contiene una successione nulla che è pure localmente determinante in zero. Rispondiamo così a una domanda posta da J.Chmielowski* ([3,Question 3]).

All locally convex spaces considered are over the complex numbers. For details of the theory of holomorphic functions defined on domains in locally convex spaces we refer to [5]. For a domain U in a locally convex space H(U) and $H^{\infty}(U)$ will denote, respectively, the set of C-valued holomorphic and bounded holomorphic functions on U.

DEFINITION 1 ([1]). *A subset* L *of the locally convex space* E *is locally determining at zero for holomorphic functions if for every connected open neighbourhood* U *of zero and* $f \in H(U)$ $f|_{U \cap L}=0$ *implies* $f \equiv 0$.

(*) Partially supported by the CAICYT os Spain. Proyect 2197/83.

(**) The second author would like to thank V.B.Moscatelli and C.Sempi for their kind hospitalityat the University of Lecce where a part of this reserach was undertaken during September of 1986.

The classical identity theorem for holomorphic functions says that neighbourhoods of zero are locally determining at 0 and if E=C then every (not eventually zero) null sequence is locally determining at 0. This latter result does not, however extend to any higher dimensional spaces. In [1] Chmielowski shows that a metrizable locally convex space E contains a null sequence which is locally determining at 0 if and only if E is separable and in [3] he asks if, in such spaces, every locally determining set at 0 contains a null sequence which is also locally determining at 0. A positive answer is given in [4] for the case $E = C^n$ and in theorem 3 we give a complete answer to this question.

One of the basic results we shall use is the following proposition due to **Ng** [8]. This result is a generalisation of a result of Dixmier [6] and has recently been generalised to LB spaces by J.Mujica [7] and applied to some problems in infinite dimensional holomorphy (see also [5, p.98 and 417]. We let B_E denote the closed unit ball of a Banach space E.

PROPOSITION 2. *Let* E *denote a normed linear space and suppose there exists a Hausdorff locally convex topology* τ *on* E *such that* (B_E,τ) *is compact. If* $F = \{\phi \in E', \phi|_{B_E}$ *is* τ*-continuous}* *then* F *is a normed linear space (when normed by* $\|\phi\|=\sup\{|\phi(x)|, x\in E, \|x\|\leq 1\}$ *and* $F'_\beta = E$.

THEOREM 3. *Let* E *be a separable metrizable locally convex space and let* L *be a locally determining set at* 0 *(for holomorphic functions). Then* L *contains a null sequence which is also locally* *determining at* 0.

Proof. Let $(V_n)_{n=1}^{\infty}$ denote a decreasing neighbourhood basis at 0 consisting of convex balanced open sets. $(H^{\infty}(V_n), \|\cdot\|_{V_n})$, is a Banach space for each n. If τ_0 denotes the compact open topology then Montel's theorem implies that the closed unit ball of $H(V_n)$ is τ_0 compact. If $F_n = \{\phi \in H^{\infty}(V_n)'; \ \phi|_{B_{H^{\infty}(V_n)}}$ is τ_0 continuous$\}$ then proposition 2 implies that F_n is a predual of $H^{\infty}(V_n)$. If $x \in V_n$ then the mapping $\delta_x : f \in H^{\infty}(V_n) \mapsto f(x)$ is τ_0 continuous and hence belongs to F_n. Since E is separable we can find for each n a dense sequence of points in $V_n, (x_{n,m})_{m=1}^{\infty}$. If $f \in H^{\infty}(V_n)$ and $<f, \delta_{n,m}> = 0$ for all m then $f \equiv 0$. By the Hahn-Banach theorem this implies that $\{\delta_{x_{n,m}}\}_{m=1}^{\infty}$ spans a dense subspace of F_n and hence F_n is separable. Let $(\omega_{n,m})_{m=1}^{\infty}$ denote a dense sequence in F_n for each positive integer n. Using the identity theorem for open sets we see that $\{\delta_\xi : \xi \in L \cap V_k\}$ spans a dense subspace of F_n for each integer $k \geq n$. For each pair of positive integers (n,k), $k \geq n$, choose a finite sequence of points in $L \cap V_k$, $x_{n,k,j}$ $1 \leq j \leq \alpha(n,k)$ such that

$$\sup_{1 \leq m \leq k} d(\omega_{n,m}, F_{n,k}) \leq \frac{1}{k}$$

where $F_{n,k}$ is the subspace of F_n spanned by $\{\delta_{x_{n,k,j}}, \ 1 \leq j \leq \alpha(n,k)\}$ and d is the distance defined by the norm of F_n.

Let

$$L^* = \bigcup_{\substack{(n,k),k\geq n,n=1,2,\ldots \\ 1\leq j\leq \alpha(n,k)}} x_{n,k,j}$$

By construction L^* is a subset of L and $|\{\xi \in L^*;\ \xi \notin V_n\}| < \infty$ for each n. Hence L^* can be rearranged to form a null sequence in E. If f is a holomorphic function defined on a connected domain $\mathscr{D}$ containing the origin then there exists a positive integer n such that $g := f|_{V_n} \in H^\infty(V_n)$. If $f|_{L^*\cap V_n}=0$ then for each positive integer $k \geq n$, $<F_{n,k},g>= 0$. Since $\bigcup_{k\geq n} F_{n,k}$ spans a dense subspace of F_n it follows that $<F_n,g> = 0$ and hence $g\equiv 0$. Since $\mathscr{D}$ is connected the identity theorem for open sets implies that $f\equiv 0$. This completes the proof.

An obvious modification of the proof of the above theorem yields the following.

COROLLARY 4. *If* U *is a connected domain in a separable Banach space then* $(H^\infty(U),\|\cdot\|_U)$ *has a separable predual.*

So far we have only considered locally determining sets for holomorphic functions but one can also define locally determining sets for continuous polynomials and continuous homogeneous polynomials. In all these situations the proof of Theorem 3 extends to show that locally determining sets at 0 for such collections on separable metrizable locally convex spaces contain null sequences which are also locally determining at 0. This answers Question 1 of Chmielowski [3].

REFERENCES

[1] J.CHMIELOWSKI: "Ensemble déterminants pour les fonctions analytiques", *C.R.A.Sc., Paris*, t279, Serie A (1974), 639-641.

[2] J.CHMIELOWSKI: "Théoremes d'identité en dimension infinite", *Ann.Pol.Math.*, 33(1976), 35-37.

[3] J.CHMIELOWSKI: "Constructions des ensembles déterminants pour les fonctions analytiques", *Studia Math.*, LVII (1976),141-146.

[4] J.CHMIELOWSKI and G.LUBCZONOK: "A property of determining sets for Analytic Functions", *Studia Math.*,LX(1977), 285-288.

[5] S.DINEEN: "Complex analysis on locally convex spaces", *North Holland Math. Studies*, 57(1981).

[6] J.DIXMIER: "Sur un théorème de Banach", *Duke Math. J.*, 15(1948), 1057-71.

[7] J.MUJICA: "A completeness criterion for inductive limits of Banach spaces", in Functional Analysis, Holomorphy and Approximation Theory II, Ed. G.Zapata, *North Holland Math. Studies* (1984), 319-329.

[8] K - F Ng: "On a theorem of Dixmier", *Math. Scand.*, 29(1971) 279-280.

(*) Departamento de Analisis Mathematico,
Facultad de Matématicas,
Universidad Complutense,
28040 MADRID,
SPAIN

(**) Department of Mathematics,
University College Dublin
Belfied,
DUBLIN
IRELAND