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Rank 2 spanned vector bundles on $\mathbb{P}^2$ with a fixed restriction to a line or a prescribed order of stability

Edoardo Ballico[i]
Department of Mathematics, University of Trento
ballico@science.unitn.it

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Abstract. Fix a line $D \subset \mathbb{P}^2$. In this note we study rank 2 spanned vector bundles with prescribed Chern classes and either with a prescribed order of stability or whose restriction to D has a prescribed splitting type, mainly when the splitting type is either rigid or the most extremal one, $(c, 0)$. We use the description of the Chern classes of all rank 2 spanned bundles due to Ph. Ellia.

Keywords: spanned vector bundles, vector bundles on the plane, splitting type

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Introduction

Several papers are devoted to the classification of spanned vector bundles on $\mathbb{P}^n$, $n \geq 2$, with low c_1 ([1], [2], [5], [10], [11], [14], [15], [16]). For any rank 2 vector bundle $\mathcal{F}$ let $k(\mathcal{F})$ be the maximal integer k such that $h^0(\mathcal{F}(-k)) > 0$. The integer $k(\mathcal{F})$ is sometimes called the order of stability and sometimes the order of unstability or instability of $\mathcal{F}$. If $\mathcal{F}$ is spanned, then $k(\mathcal{F}) \geq 0$. $\mathcal{F}$ is stable (resp. semistable) if and only if $2k(\mathcal{F}) < c_1(\mathcal{F})$ (resp. $2k(\mathcal{F}) \leq c_1(\mathcal{F})$). Two rank 2 vector bundles $\mathcal{E}$, $\mathcal{F}$ with the same Chern numbers may have different cohomological properties. If $\mathcal{E}$ is stable, but $\mathcal{F}$ is not stable, they must have different cohomological properties (even if both are spanned), because $k(\mathcal{F}) \neq k(\mathcal{E})$. The Chern classes of all rank 2 spanned bundles on $\mathbb{P}^2$ are known ([6]). Here we use the results and proofs of [6] to consider spanned vector bundles $\mathcal{E}$ with one of the following additional conditions: we fix a line D and we prescribe in advance the splitting type of $\mathcal{E}_{|D}$ or we fix the integer $k(\mathcal{E})$ or we fix both the integer $k(\mathcal{E})$ and the splitting type of $\mathcal{E}_{|D}$.

Fix a line $D \subset \mathbb{P}^2$. Looking only at bundles whose restriction to a given line is prescribed arises in the set-up of framed sheaves ([7], [8], [4]). Fix a positive

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integer c and fix an integer t such that $0 \leq 2t \leq c$. We only look at spanned bundles $\mathcal{E}$ on $\mathbb{P}^2$ with $\mathcal{E}_{|D} \cong \mathcal{O}_D(c-t) \oplus \mathcal{O}_D(t)$ (the possible splitting types of rank 2 spanned bundles on D). It is easy to check that the answer (i.e. the possible integers $c_2(\mathcal{E})$) depends very much from t. We have a complete answer in the case $t = \lfloor c/2 \rfloor$, i.e. when $\mathcal{E}_{|D}$ is rigid (see Proposition 1.6) and partial result in the other extremal case $t = 0$ (see Propositions 4 and 5).

We recall that for all $(c, y) \in \mathbb{Z}^2$ there is a rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = c$ and $c_2(\mathcal{E}) = y$ ([17], [12, Theorem 6.2.1]). There is a stable rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = c$ and $c_2(\mathcal{E}) = y$ if and only if $4y > c^2$ and $4y - c^2 \neq -4$ ([17], [9, page 145]). However, these Chern integers (c, y) may also be realized by unstable bundles, with very different cohomological properties.

Ph. Ellia gave the complete list of all $(c, y) \in \mathbb{Z}^2$ such that there is a rank 2 spanned vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = c$ and $c_2(\mathcal{E}) = y$ ([6, Theorem 0.1]). We need $c \geq 0$ and if $c = 0$, then $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}^2$ and so $y = 0$. Hence we may assume $c > 0$. It is too long to state his full list (see [6, page 148]); suffice to say that $y \leq c^2$ and that all (c, y) with $c > 0$ and $c^2/4 \leq y \leq 3c^2/4$ are realized by some spanned $\mathcal{E}$. A minor modification of the proof of [6, Theorem 0.1] gives the following 3 results: Theorem 1 and Propositions 1.5 and 1.6.

Theorem 1. *Fix positive integers y, c such that there is a rank 2 spanned vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c$ and $c_2(\mathcal{F}) = y$.*

(i) There is a rank 2 stable and spanned vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = c$ and $c_2(\mathcal{E}) = y$ if and only if $4y > c^2$, $4y - c^2 \neq -4$.

(ii) If $y > c\lfloor c/2 \rfloor$, then any such spanned $\mathcal{F}$ is stable.

Recall again that the conditions $4y > c^2$, $4y - c^2 \neq -4$ in part (i) are the necessary and sufficient conditions for the existence of a rank 2 stable vector bundle on $\mathbb{P}^2$ with these Chern numbers ([17], [9, page 145]). Thus part (i) of Theorem 1 may be rephrased saying that some Chern numbers (c, y) are realized by a stable spanned bundle if and only if they are realized by a spanned bundle and by a stable bundle.

For odd c_1 a rank 2 semistable vector bundle on $\mathbb{P}^2$ is stable. For even c_1 we may consider properly semistable vector bundles. We get the following variation of Theorem 1.

Proposition 1. *Fix positive integers y, c such that c is even and there is a rank 2 spanned vector bundle $\mathcal{F}$ on $\mathbb{P}^2$ with $c_1(\mathcal{F}) = c$ and $c_2(\mathcal{F}) = y$.*

(i) There is a rank 2 semistable and spanned vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = c$ and $c_2(\mathcal{E}) = y$ if and only if $4y \geq c^2$.

(ii) If $y \geq c^2/2$, then any spanned $\mathcal{F}$ is semistable.

Proposition 2. *Fix positive integers y, c. There is a rank 2 spanned vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = c$, $c_2(\mathcal{E}) = y$ and $\mathcal{E}_{|D} \cong \mathcal{O}_D(\lceil c/2 \rceil) \oplus \mathcal{O}_D(\lfloor c/2 \rfloor)$ if* *and only if either there is a spanned semistable one or c is odd and $4y = c^2 - 1$. In the latter case $\mathcal{O}_{\mathbb{P}^2}((c+1)/2) \oplus \mathcal{O}_{\mathbb{P}^2}((c-1)/2)$ is the only bundle.*

In the next results we introduce the datum $k(\mathcal{E})$. We prove the following 2 results, first without imposing the splitting type of $\mathcal{F}_{|D}$ and then imposing that it is the most unbalanced one for spanned bundles, i.e. that $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$.

Proposition 3. *Fix integers $c > k \geq 0$. There is a rank 2 spanned vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c$, $c_2(\mathcal{F}) = y$, $k(\mathcal{F}) = k$, and $h^1(\mathcal{F}) = 0$ if and only if one of the following conditions is satisfied:*

(1) $c = k + 1$ and $y = c$;

(2) $c = k + 2$ and $y = 2c$;

(3) $2k \geq c$ and $k(c-k) \leq y \leq k(c-k) + \binom{c-k+2}{2} - 3$;

(4) $2k < c$ and $k(c-k) + \binom{c-2k+1}{2} \leq k(c-k) + \binom{c-k+2}{2} - 3$.

Remark 1. Proposition 3 gives the list all triples $(c_1(\mathcal{F}), c_2(\mathcal{F}), k(\mathcal{F}))$ realized by a rank 2 spanned vector bundle $\mathcal{F}$ with $h^1(\mathcal{F}) = 0$. In particular we see that for most (c, y) several different $k(\mathcal{F})$ are possible, often with some stable bundle, some properly semistable bundle and some non semistable bundle. See Proposition 6 (resp. Proposition 7) for the list of all triples $(c_1(\mathcal{F}), c_2(\mathcal{F}), k(\mathcal{F}))$ realized by a rank 2 spanned vector bundle $\mathcal{F}$ with $h^1(\mathcal{F}(-1)) = 0$ (resp. $h^1(\mathcal{F}(-2)) = 0$. See Remark 5 for an application of Proposition 7.

Proposition 4. *Fix integer $c > k \geq 0$ and $y > 0$. There is a spanned vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c$, $c_2(\mathcal{F}) = y$, $k(\mathcal{F}) = k$, $h^1(\mathcal{F}) = 0$ and $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$ if and only if one of the following conditions is satisfied:*

(1) $c = k + 1$ and $y = c$;

(2) $c = k + 2$ and $y = 2c$;

(3) $2k \geq c$ and $(k+1)(c-k) \leq y \leq k(c-k) + \binom{c-k+2}{2} - 3$;

(4) $2k < c$ and $(k+1)(c-k) + \binom{c-2k}{2} \leq y \leq k(c-k) + \binom{c-k+2}{2} - 3$.

Any bundle $\mathcal{F}$ in Proposition 4 satisfies $h^1(\mathcal{F}(-2)) > 0$ (Lemma 3) and so it cannot have very general cohomology if c is not very small.

If we drop the condition $h^1(\mathcal{F}) = 0$, we obviously get many other cases. We point out here that for each $c_1(\mathcal{F})$ and $k(\mathcal{F})$ we realize the one with maximal c_2.

Proposition 5. *Fix integers $c > k \geq 0$.*

(a) Every spanned bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c$, $k(\mathcal{F}) = k$ and $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$ has $(k+1)(c-k) \leq c_2(\mathcal{F}) \leq c(c-k)$.

(b) There is a spanned bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c$, $k(\mathcal{F}) = k$, $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$ and $c_2(\mathcal{F}) = c(c-k)$. Any such $\mathcal{F}$ has $h^0(\mathcal{F}) = \binom{k+2}{2} + 2$ and $h^1(\mathcal{F}) = (c-k)^2 - 2 - \binom{c-k+2}{2}$.

(c) If $\mathcal{F}$ is spanned, $c_1(\mathcal{F}) = c$, $k(\mathcal{F}) = k$, $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$ and $c_2(\mathcal{F}) < c(c-k)$, then $h^0(\mathcal{F}) \geq \binom{k+2}{2} + 3$.

In the last section we briefly look at spanned bundles of rank $r > 2$ and show the informations obtained from our results on the rank 2 case.

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1 Balanced splitting type

Set $\mathcal{O} := \mathcal{O}_{\mathbb{P}^2}$.

We need the following well-known exercise (see Lemma 5 for a more difficult case).

Lemma 1. *Fix integers $a > 0$ and $s \geq 0$. Let $S \subset \mathbb{P}^2$ be a general subset with cardinality s. The sheaf $\mathcal{I}_S(a)$ is spanned if and only if either $a = 1$ and $\sharp(S) = 1$ or $a = 2$ and $\sharp(S) = 4$ or $\sharp(S) \leq \binom{a+2}{2} - 3$.*

Proof of Theorem 1 and Proposition 1.5: We first consider the stable case. A necessary and sufficient condition for the existence of a stable bundle (even a non spanned one) is $4y > c^2$ and $4y - c^2 \neq -4$. Assume that these inequalities are satisfied and that either $(c, y) \in \{(1,1), (2,4)\}$ or $c^2/4 < y \leq 2 + c(c+3)/2$. The existence of a spanned and stable bundle for these (c, y) is due to Le Potier ([6, Proposition 1.4], [9, 3.4]), who proved that in this range we may take as $\mathcal{E}$ a general stable bundle with the prescribed Chern numbers y, c. Since $2 + c(c+3)/2 \geq c^2/2$, to conclude the proof of Theorem 1 it is sufficient to prove its part (ii).

Assume $2y \geq c^2$ and the existence of a rank 2 spanned vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c$ and $c_2(\mathcal{F}) = y$. Set $k := k(\mathcal{F})$. $\mathcal{F}$ is stable (resp. semistable) if and only if $2k < c$ (resp. $2k \leq c$). We have an exact sequence

$$0 \to \mathcal{O}(k) \to \mathcal{F} \to \mathcal{I}_Z(c-k) \to 0 \tag{1.1}$$

with Z a zero-dimensional and locally complete intersection scheme. We have $y = k(c-k) + \deg(Z)$. Since $k \geq 0$ and $h^1(\mathcal{O}(k)) = 0$, $\mathcal{F}$ is spanned if and only if $\mathcal{I}_Z(c-k)$ is spanned. If $\mathcal{I}_Z(c-k)$ is spanned, then $\deg(Z) \leq (c-k)^2$ and hence $y \leq c(c-k)$. We get part (ii) of Theorem 1 and of Proposition 1.5.

If c is odd, then stability and semistability coincide. Now assume that c is even and take any semistable, but not stable bundle $\mathcal{F}$. It fits in (1.1) with $k = c/2$ and $\mathcal{F}$ is spanned if and only if $\mathcal{I}_Z(c/2)$ is spanned. From (1.1) we get $h^0(\mathcal{F}(-1-c/2)) = 0$ and so any such $\mathcal{F}$ is semistable. We get $y = \deg(Z)+c^2/4$.

All cases with $y \ge 2 + c^2/4$ allowed by [6, Theorem 0.1] are covered by a stable spanned bundle (Theorem 1). Hence to prove part (i) of Proposition 1.5 it is sufficient to do the two cases $y \in \{c^2/4, c^2/4 + 1\}$. For any locally complete intersection scheme Z there is a locally free $\mathcal{F}$ fitting in (1.1) with $k = c/2$, because the Cayley-Bacharach condition is trivially satisfied. For the case $y = c^2/4$ use $Z = \emptyset$ (in this case $\mathcal{F} \cong \mathcal{O}(\frac{c}{2})^{\oplus 2}$). For the case $y = c^2/4 + 1$ use as Z a single point. In both cases $\mathcal{I}_Z(c/2)$ is spanned. QED

Proof of Proposition 1.6: In characteristic zero the generic splitting type of a semistable bundle $\mathcal{F}$ is rigid, i.e., $\lceil c/2\rceil, \lfloor c/2\rfloor$ is its generic splitting type, and hence for a general $g \in \mathrm{Aut}(\mathbb{P}^2)$ the bundle $g^*(\mathcal{F})$ gives a solution for Proposition 1.6.

If c is even, then every bundle $\mathcal{F}$ with $\mathcal{F}_{|D} \cong \mathcal{O}_D(\frac{c}{2})^{\oplus 2}$ is semistable.

Now take c odd and let $\mathcal{F}$ be any bundle with $\mathcal{F}_{|D} \cong \mathcal{O}_D(\frac{c+1}{2}) \oplus \mathcal{O}_D(\frac{c-1}{2})$. Either $\mathcal{F}$ is semistable or it fits in (1.1) with $k = (c+1)/2$. In the latter case we have $c_2(\mathcal{F}) = \deg(Z) + (c^2-1)/4$. Therefore $(c^2-1)/4 \le c_2(\mathcal{F}) \le c(c-1)/2$ and hence we are in the range for which there are spanned semistable bundles, unless $Z = \emptyset$, i.e. unless $\mathcal{F} \cong \mathcal{O}(\frac{c+1}{2}) \oplus \mathcal{O}(\frac{c-1}{2})$. QED

Remark 2. Take $c > 0$, $4y > c^2$, $4y - c^2 \ne 4$ and $y \le 2 + c(c+3)/2$. A general rank 2 stable bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = c$ and $c_2(\mathcal{E}) = y$ is spanned ([6, Proposition 1.4], [9, 3.4]) and it has the expected cohomology, i.e. for each $t \in \mathbb{Z}$ at most one of the integers $h^i(\mathcal{E}(t))$, $i = 0, 1, 2$, is non-zero ([3, 5.1], [9, 3.4]). In particular $h^1(\mathcal{E}(t)) = 0$ for all $t \ge 0$. In part of this range we may find $\mathcal{E}$ without the expected cohomology, but with $h^1(\mathcal{E}) = 0$. In a smaller part of this range we may find $\mathcal{E}$ with $h^1(\mathcal{E}) > 0$, i.e. with $h^0(\mathcal{E}) > \chi(\mathcal{E}) = \binom{c+2}{2} + 1 - y$.

Lemma 2. *Let $W \subset \mathbb{P}^2$ be a zero-dimensional scheme such that $\mathcal{I}_W(a)$ is spanned and $h^1(\mathcal{I}_W(a)) = 0$. Then for all $A \subsetneq W$ we have $h^1(\mathcal{I}_A(a)) = 0$ and $\mathcal{I}_A(a)$ is spanned.*

Proof. Since W is zero-dimensional, $h^1(W, \mathcal{I}_{A,W}(a)) = 0$ and hence the restriction map $H^0(\mathcal{O}_W(a)) \to H^0(\mathcal{O}_A(a))$ is surjective. Hence $h^1(\mathcal{I}_A(a)) = 0$. Hence $h^0(\mathcal{I}_A(a)) = \binom{a+2}{2} - \deg(A)$. Let B the base scheme of $|\mathcal{I}_A(a)|$. We have $h^0(\mathcal{I}_A(a)) = h^0(\mathcal{I}_B(a))$. Since $\mathcal{I}_W(a)$ is spanned, we have $B \subseteq W$ and in particular B is zero-dimensional. We saw that $h^1(\mathcal{I}_B(a)) = 0$, i.e. $h^0(\mathcal{I}_B(a)) = \binom{a+2}{2} - \deg(B)$. Since $B \supseteq A$, then $B = A$. QED

A bundle $\mathcal{F}$ fits in an exact sequence (1.1) with $k = k(\mathcal{F})$ and Z a locally complete zero-dimensional scheme. A bundle $\mathcal{F}$ in (1.1) has $c_1(\mathcal{F}) = c$ and $c_2(\mathcal{F}) = k(c-k) + \deg(Z) \geq k(c-k)$. A bundle $\mathcal{F}$ in (1.1) with $k \geq 0$ is spanned if and only if $\mathcal{I}_Z(c-k)$ is spanned. A bundle $\mathcal{F}$ in (1.1) has $k = k(\mathcal{F})$ if and only if $h^0(\mathcal{I}_Z(c-2k-1)) = 0$. If $k \geq -2$ we have $h^1(\mathcal{F}) = 0$ if and only if $h^1(\mathcal{I}_Z(c-k)) = 0$ (note that this is true even if $k \neq k(\mathcal{F})$).

Proof of Proposition 3: Set $s := y - k(c-k)$. Assume that $\mathcal{F}$ exists. It fits in (1.1) with $\deg(Z) = s$, $\mathcal{I}_Z(c-k)$ spanned and $h^1(\mathcal{I}_Z(c-k)) = 0$. We have $Z = \emptyset$ if and only if $s = 0$. Assume for the moment $s > 0$. We get $h^0(\mathcal{I}_Z(c-k)) \geq 2$ and that $h^0(\mathcal{I}_Z(c-k)) = 2$ if and only Z is a complete intersection of 2 plane curves of degree $c-k$. If Z is a complete intersection of 2 plane curves of degree $c-k$ we have $h^1(\mathcal{I}_Z(c-k)) = 0$ if and only if $c-k \leq 2$ and we get cases (1) and (2) in the statement of Proposition 3. Now assume $h^0(\mathcal{I}_Z(c-k)) \geq 3$. We have $h^1(\mathcal{I}_Z(c-k)) = 0$ if and only if $h^0(\mathcal{I}_Z(c-k)) = \binom{c-k+2}{2} - s$. Hence if $\mathcal{F}$ exists, then $y \leq k(c-k) + \binom{c-k+2}{3} - 2$. If $c \leq 2k$, then any sheaf $\mathcal{F}$ in (1.1) has $k(\mathcal{F}) = k$. If $c > 2k$, the condition $k = k(\mathcal{F})$ implies $\deg(Z) \geq \binom{c-2k+1}{2}$.

The existence part for cases (3) and (4) is true by Lemma 1; note that taking as Z a general union of s points in the case $c > 2k$ we have $h^0(\mathcal{I}_Z(c-2k-1)) = 0$. QED

Remark 3. Take y, c, k for which Proposition 3 gives a spanned bundle. Taking as Z a general subset with cardinality $y - k(c-k)$ gives the bundles $\mathcal{F}$ with minimal Hilbert function among all bundles with fixed $c_1(\mathcal{F})$, $c_2(\mathcal{F})$, and $k(\mathcal{F})$, i.e. $h^1(\mathcal{F}(t)) = 0$ for all t with $k - c \leq t < 0$ and $y - k(c-k) \leq \binom{c-k+t+2}{2}$. If $2k \geq c$ (i.e. if $\mathcal{F}$ is not stable) and $y \neq k(n-k)$ (i.e. $\mathcal{F} \neq \mathcal{O}(k) \oplus \mathcal{O}(c-k)$), then the maximal integer t with $h^1(\mathcal{F}(t)) > 0$ is the maximal negative integer t such that $y - k(c-k) > \binom{c-k+t+2}{2}$.

Now we prove the following two modifications of Proposition 3.

Proposition 6. *Fix integers $c > k \geq 0$. There is a rank 2 spanned vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c$, $c_2(\mathcal{F}) = y$, $k(\mathcal{F}) = k$, and $h^1(\mathcal{F}(-1)) = 0$ if and only if one of the following conditions is satisfied:*

(1) $c = k+1$ *and* $y = c$;

(2) $2k \geq c$ *and* $k(c-k) \leq y \leq k(c-k) + \binom{c-k+1}{2}$;

(3) $2k < c$ *and* $k(c-k) + \binom{c-2k+1}{2} \leq k(c-k) + \binom{c-k+1}{2}$.

Proposition 7. *Fix integers $c > k \geq 0$. There is a rank 2 spanned vector bundle $\mathcal{F}$ with $c_1(\mathcal{F}) = c$, $c_2(\mathcal{F}) = y$, $k(\mathcal{F}) = k$, and $h^1(\mathcal{F}(-2)) = 0$ if and only if one of the following conditions is satisfied:*

(1) $2k \ge c$ *and* $k(c-k) \le y \le k(c-k) + \binom{c-k}{2}$*;*

(2) $2k < c$ *and* $k(c-k) + \binom{c-2k+1}{2} \le k(c-k) + \binom{c-k}{2}$.

Proof of Propositions 6 and 7: Let $\mathcal{F}$ be any spanned rank 2 vector bundle. Fix $t \in \{1,2\}$ and let $C \subset \mathbb{P}^2$ be a smooth curve of degree t. We have an exact sequence

$$0 \to \mathcal{F}(-t) \to \mathcal{F} \to \mathcal{F}_{|C} \to 0 \qquad (1.2)$$

Since $C \cong \mathbb{P}^1$ and $\mathcal{F}_{|C}$ is a spanned vector bundle, we have $h^1(C, \mathcal{F}_{|C}) = 0$. Hence (1.2) shows that the set of all triples $(c,y,k) = (c_1(\mathcal{F}), c_2(\mathcal{F}), k(\mathcal{F}))$ which are obtained from a rank 2 spanned bundle $\mathcal{F}$ with $h^1(\mathcal{F}(-2)) = 0$ is contained in the one realized by a rank 2 spanned bundle $\mathcal{F}$ with $h^1(\mathcal{F}(-1)) = 0$ and the latter is contained in the one obtained from a rank 2 spanned bundles $\mathcal{F}$ with $h^1(\mathcal{F}) = 0$. Take a rank 2 spanned bundle $\mathcal{F}$ and set $k := k(\mathcal{F})$, $c := c_1(\mathcal{F})$ and $y := c_2(\mathcal{F})$. Hence $\mathcal{F}$ fits in (1.1) for some Z with $\deg(Z) = y - k(c-k)$. Since $k \ge 0$, we have $h^1(\mathcal{O}(k-t)) = h^2(\mathcal{O}(k-t)) = 0$. Thus $h^1(\mathcal{F}(-t)) = h^1(\mathcal{I}_Z(c-k-t))$. If we require $h^1(\mathcal{I}_Z(c-k-1)) = 0$, then we exclude case (2) of Proposition 3, while case (1) is allowed with Z a single point P and $\mathcal{F}$ any locally free extension of $\mathcal{I}_P(1)$ by $\mathcal{O}(c-1)$. If we require $h^1(\mathcal{I}_Z(c-k-2)) = 0$, then we exclude cases (1) and (2) of Proposition 3. Now we look at cases (3) and (4) of Proposition 3. If $h^1(\mathcal{I}_Z(c-k-t)) = 0$, $t \in \{1,2\}$, then $y - k(c-k) \le \binom{c-k-t+2}{2}$. Recall that to get the existence part for Proposition 3 we took as Z a general subset of $\mathbb{P}^2$ with cardinality $y - k(c-k)$. Such a set Z has $h^1(\mathcal{I}_Z(c-k-t)) = 0$ if and only if $y - k(c-k) \le \binom{c-k-t+2}{2}$. We have $\binom{c-k+2}{2} - 3 \le \binom{c-k+1}{2}$ for all $c \ge k+2$. Hence for our general Z in cases (2) and (3) of Proposition 6 we may apply Lemma 1 with $a = c-k$. If $c = k+1$ we only get case (1) of Proposition 6, because if $Z = \emptyset$, then $\mathcal{F} \cong \mathcal{O}(c) \oplus \mathcal{O}$ and $k(\mathcal{O}(c) \oplus \mathcal{O}) = c$. Since $c > k$, we have $\binom{c-k+2}{2} - 3 \ge \binom{c-k}{2}$ and so we may apply Lemma 1 with $a = c-k$ to prove Proposition 7. QED

In Propositions 3, 4, 5, 6 and 7 we assumed $c > k \ge 0$, because if $\mathcal{F}$ is spanned, then $k(\mathcal{F}) \ge 0$ and $c_1(\mathcal{F}) = k(\mathcal{F})$ if and only if $\mathcal{F} \cong \mathcal{O}(c_1(\mathcal{F})) \oplus \mathcal{O}$.

Proposition 8. *Fix integers* $c > k > 0$*. There is a rank* 2 *spanned vector bundle* $\mathcal{F}$ *with* $c_1(\mathcal{F}) = c$, $c_2(\mathcal{F}) = y$, $k(\mathcal{F}) = k$, *and* $h^1(\mathcal{F}(-3)) = 0$ *if and only if one of the following conditions is satisfied:*

(1) $2k \ge c$ *and* $k(c-k) \le y \le k(c-k) + \binom{c-k-1}{2} - 3$*;*

(2) $2k < c$ *and* $k(c-k) + \binom{c-2k+1}{2} \le k(c-k) + \binom{c-k-1}{2}$.

Proof. Take a spanned rank 2 vector bundle $\mathcal{F}$ fitting in (1.1) with $c = c_1(\mathcal{F})$, $k = k(\mathcal{F})$ and $\deg(Z) = c_2(\mathcal{F}) - k(c-k)$. Since $k > 0$, we have $h^1(\mathcal{O}(k-3)) =$

$h^2(\mathcal{O}(k-3)) = 0$ and so $h^1(\mathcal{F}(-3)) = h^1(\mathcal{I}_Z(c-k-3))$. Thus $c_2(\mathcal{F}) \le k(c-k) + \binom{c-k-1}{2}$ if $h^1(\mathcal{F}(-3)) = 0$. Assume $c_2(\mathcal{F}) \le k(c-k) + \binom{c-k-1}{2}$ and take as Z a general subset of $\mathbb{P}^2$ with cardinality $c_2(\mathcal{F}) - \binom{c-k-1}{2}$. Any sheaf $\mathcal{F}$ in (1.1) with this scheme Z satisfies $h^1(\mathcal{F}(-3)) = 0$. The proof of Proposition 3 gives that Z gives a spanned vector bundle $\mathcal{F}$ with $k(\mathcal{F}) = k$. QED

2 Splitting type $(c, 0)$

In this section we consider necessary or sufficient conditions for the existence of spanned bundles $\mathcal{F}$ with $c_1(\mathcal{F}) = c$, $c_2(\mathcal{F}) = y$ and $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$.

Lemma 3. *Let $\mathcal{F}$ be a rank $r \ge 2$ spanned vector bundle with no trivial factor and with $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D^{\oplus(r-1)}$. Then $h^1(\mathcal{F}(-2)) \ge r-1$.*

Proof. Since $\mathcal{F}$ has no trivial factor and it is spanned, we have $h^0(\mathcal{F}^\vee) = 0$ and $c > 0$. From the exact sequence

$$0 \to \mathcal{F}^\vee(-1) \to \mathcal{F}^\vee \to \mathcal{F}^\vee_{|D} \to 0 \tag{2.1}$$

we get $h^1(\mathcal{F}^\vee(-1)) \ge r-1$. Duality gives $h^1(\mathcal{F}^\vee(-1)) = h^1(\mathcal{F}(-2))$. QED

The next lemma settles the case $c = 1$.

Lemma 4. *Let $\mathcal{E}$ be a rank r spanned vector bundle such that $c_1(\mathcal{E}) = 1$. Then either $\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{O}^{\oplus(r-1)}$ or $\mathcal{E} \cong T\mathbb{P}^2(-1) \oplus \mathcal{O}^{\oplus(r-2)}$.*

Proof. First assume $r = 2$. In this case $\mathcal{E}$ is uniform of splitting type $(1,0)$ and hence either $\mathcal{E} \cong \mathcal{O}(1) \oplus \mathcal{O}$ or $\mathcal{E} \cong T\mathbb{P}^2(-1)$ ([18]). Now assume $r > 2$ and that the lemma is true for bundles of rank $r-1$. Since $r > \dim(\mathbb{P}^2)$ a general section of $\mathcal{E}$ induces an exact sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{G} \to 0$$

with $\mathcal{G}$ a spanned vector bundle with $c_1(\mathcal{G}) = 1$. Use the inductive assumption and that $h^1(\Omega_{\mathbb{P}^2}(1)) = 0$. QED

From now on we assume $c \ge 2$.

Remark 4. Let $\mathcal{F}$ be a vector bundle fitting in (1.1). If $\mathcal{F}$ is spanned, then $c \ge k$ and $\deg(Z \cap T) \le c-k$ for each line $T \subset \mathbb{P}^2$. If $k = c$, then $Z = \emptyset$ and so $\mathcal{F} \cong \mathcal{O}(c) \oplus \mathcal{O}$. If $0 < k < c$, then $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$ if and only if $\deg(Z \cap D) = c-k$.

Lemma 5. *Fix an integer $a > 0$ and a line $D \subset \mathbb{P}^2$. Fix an integer z such that $a \le z \le \binom{a+2}{2} - 3$. Let $A \subset D$ be any degree a zero-dimensional scheme. Let $B \subset \mathbb{P}^2 \setminus D$ be a general subset with $\sharp(B) = z - a$. Then $h^1(\mathcal{I}_{A\cup B}(a)) = 0$ and $\mathcal{I}_{A\cup B}(a)$ is spanned.*

Proof. By Lemma 2 it is sufficient to do the case $z = \binom{a+2}{2} - 3$. Since $B \cap D = \emptyset$, there is a residual exact sequence

$$0 \to \mathcal{I}_B(a-1) \to \mathcal{I}_{A\cup B}(a) \to \mathcal{I}_{A,D}(a) \to 0 \tag{2.2}$$

Since B is general, we have $h^0(\mathcal{I}_B(a-1)) = 2$ and $h^1(\mathcal{I}_B(a-1)) = 0$. Hence (2.2) gives $h^1(\mathcal{I}_{A\cup B}(a)) = 0$ and $h^0(\mathcal{I}_{A\cup B}(a)) = 3$. Fix a general $(C, C') \in |\mathcal{I}_B(a-1)|^2$. For a general B, the curves C, C' are general plane curves of degree $a-1$ and hence $C \cap C' = B \sqcup E$ with E a finite set with cardinality $(a-1)^2 - \sharp(B)$ and $C \cap C' \cap E = \emptyset$. Using $T \cup D$ with $T \in |\mathcal{I}_B(a-1)|$ we see that the scheme-theoretic base locus of $|\mathcal{I}_{A\cup B}(a)|$ is contained in $A \cup E \cup D$. Let $Z \subset D$ be any zero-dimensional scheme such that $\deg(Z) = a+1$ and $Z \supset A$. Using Z instead of A in (2.2) we get $h^0(\mathcal{I}_{B\cup Z}(a)) = 2$. Hence $W \cap D = A$ (as schemes). Therefore $W \subseteq A \cup B \cup E$. Hence to prove the lemma it is sufficient to prove that $E \cap W = \emptyset$. Assume the existence of $o \in E \cap W$. We fixed the scheme A, but we are allowed to move B. Recall that C is a smooth plane curve of degree $a-1$. Hence $|\mathcal{O}_C(a-1)|$ is induced by $|\mathcal{O}_{\mathbb{P}^2}(a-1)|$. Since the lemma is easy if $a \le 3$, we may assume $a \ge 4$. In this case $h^0(\mathcal{O}_C(a-1)) \ge 8$. By [13, Theorem 2.4] the monodromy group of the set of divisors $|\mathcal{O}_C(a-1)|$ contains the alternating group and hence it is $(a-1)^2 - 1$-transitive. For a general C' we get that the union with A of any two subset of $B \cup E$ with cardinality $\sharp(B) + 1$ have the same Hilbert function. Since $o \in W$, we get $E \subset W$, i.e. $h^0(\mathcal{I}_{B\cup E\cup A}(a)) = 2$. Since $B \cup E = C \cap C'$, the equations of C and C' generate the homogeneous ideal of $B \cup E$ and so we $h^0(\mathcal{I}_{B\cup E}(a-1)) = 2$ and $h^0(\mathcal{I}_{B\cup E}(a) = 6$. This is true for any A, D and hence we may first assume that D is a general line and then that A is a general subset of D with cardinality $a \ge 4$. For a general $A \subset D$ with cardinality $a \ge 4$, we get $h^0(\mathcal{I}_{B\cup E\cup A}(a)) = 2$, contradicting the inclusion $E \subset W$, which gives $h^0(\mathcal{I}_{B\cup E\cup A}(a)) = 3$. QED

Proof of Proposition 4: Any $\mathcal{F}$ with $c_1(\mathcal{F}) = c$ and $k(\mathcal{F}) = k$ fits in (1.1) with $h^0(\mathcal{I}_Z(c-2k-1)) = 0$ and $\deg(Z) = c_2(\mathcal{F}) - k(c-k)$. We have $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$ if and only if $\deg(Z \cap D) = c-k$. In particular we have $c_2(\mathcal{F}) \ge (k+1)(c-k)$. Cases (1) and (2) corresponds to the case in which $1 \le c-k \le 2$ and $h^0(\mathcal{I}_Z(c-k)) = 2$, i.e. Z a complete intersection of two plane curves C_1, C_2 of degree $c-k$; this case is realized taking $C_1 \supseteq D$ and then taking C_2 a general curve of degree $c-k$. Therefore it is sufficient to test which (c, y) of the cases (3) and (4) of Proposition 3 give a solution for Proposition 4. Let $\mathrm{Res}_D(Z)$ be the residual scheme of Z with respect to D, i.e. the closed subscheme of $\mathbb{P}^2$ with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$.

First assume $2k \geq c$, so that any $\mathcal{F}$ fitting in (1.1) has $k(\mathcal{F}) = k$. Use Lemma 5.

Now assume $2k < c$. In this case we also have the condition $h^0(\mathcal{I}_Z(c-2k-1)) = 0$. Since $\deg(Z \cap D) = c - k > c - 2k - 1$, we have $h^0(\mathcal{I}_Z(c-2k-1)) = h^0(\mathcal{I}_{\mathrm{Res}_D(Z)}(c-2k-2))$. The h^1-part of Lemma 5 with $a = c - k$ shows that we may satisfy it taking $Z = A \sqcup B$ with $\deg(A) = c - k$, $A \subset D$, and B general in $\mathbb{P}^2 \setminus D$ as soon as $\sharp(B) \geq \binom{c-2k}{2}$. Hence we need $\deg(Z) \geq c - k + \binom{c-2k}{2}$ and hence $y \geq (k+1)(c-k) + \binom{c-2k}{2}$. QED

Proof of Proposition 5: Any $\mathcal{F}$ with $c_1(\mathcal{F}) = c$, $k(\mathcal{F}) = k$ and $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$ fits in (1.1) with $\deg(Z \cap D) = c - k$. Without the condition $\deg(Z \cap D) = c - k$, the maximal integer $\deg(Z)$ is obtained if and only Z is a complete intersection of 2 plane curves C, C' of degree $c - k$ and in this case we have $c_2(\mathcal{F}) = c(c-k)$ and $h^0(\mathcal{F}) = 2 + \binom{k+2}{2}$. We satisfy the condition $\deg(D \cap Z) = c - k$ taking as C a reducible curve with D as a component. QED

3 Rank $r > 2$

In this section we consider rank $r > 2$ spanned vector bundles $\mathcal{E}$ on $\mathbb{P}^2$ without trivial factors with $c_1(\mathcal{E}) = c$ and $c_2(\mathcal{E}) = y$. The situation is different for certain sectors of triples (c, y, r) of c_1, c_2 and rank r. First of all $r \leq \binom{c+2}{2} - 1$. If $r = \binom{c+2}{2} - 1$, then the spanned bundle $\mathcal{E}$ exists, it is unique, it is homogeneous and hence its splitting type,$c_2(\mathcal{E}) = c^2$, $\mathcal{E}$ is homogeneous and for each line D the bundle $\mathcal{E}_{|D}$ has splitting type $(1, \dots, 1, 0, \cdots, 0)$ with c 1's. So we cannot achieve all splitting types. For the more unbalanced splitting type $(c, 0, \dots, 0)$ we may use the statements of Propositions 4 and 5 and give some existence results, summarized in Remark 6.

Fix an integer $r > 2$. Let $\mathcal{F}$ be a rank 2 vector bundle on $\mathbb{P}^2$ with no trivial factors. There is a rank r vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ fitting in an exact sequence

$$0 \to \mathcal{O}^{\oplus(r-2)} \to \mathcal{E} \to \mathcal{F} \to 0 \tag{3.1}$$

and with no trivial factor if and only if $r \leq h^1(\mathcal{F}^\vee) + 2$. If $\mathcal{E}$ exists, then it is spanned if and only if $\mathcal{F}$ is spanned.

Remark 5. Take a spanned rank 2 vector bundle $\mathcal{F}$. Duality gives $h^1(\mathcal{F}^\vee) = h^1(\mathcal{F}(-3))$. Hence Proposition 8 gives the list of all $(c_1(\mathcal{F}), c_2(\mathcal{F}), k(\mathcal{F}))$ with $\mathcal{F}$ a rank 2 spanned vector bundle such that any extension of $\mathcal{F}$ by a trivial vector bundle is the trivial extension.

Fix a line $D \subset \mathbb{P}^2$. Look at the exact sequence (2.1). From (2.1) we get a linear map $u : H^1(\mathcal{F}^\vee(-1)) \to H^1(\mathcal{F}^\vee)$. Set $\alpha := \mathrm{rank}(u)$. We have $r \le 2 + \alpha$ if and only if there is an extension (3.1) whose restriction to D is the trivial extension. Note that the restriction of (3.1) to D is the trivial extension if and only if $\mathcal{E}_{|D} \cong \mathcal{F}_{|D} \oplus \mathcal{O}_D^{\oplus(r-2)}$. Now assume that $\mathcal{F}$ is spanned and set $c := c_1(\mathcal{F})$. We assume $c > 0$ i.e. $\mathcal{F} \ne \mathcal{O}^2$. Since $\mathcal{F}$ has no trivial factor, then $h^0(\mathcal{F}^\vee) = 0$. Hence the map u is injective if and only if $\mathcal{F}_{|D}$ has no trivial factor (in this case $\alpha = h^1(\mathcal{F}^\vee(-1))$), while if $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$, then u has a one-dimensional kernel (in this case $\alpha = h^1(\mathcal{F}^\vee(-1)) - 1$). Duality gives $h^1(\mathcal{F}^\vee(-1)) = h^1(\mathcal{F}(-2))$. So to know the integer α it is sufficient to compute the integer $h^1(\mathcal{F}(-2))$.

Remark 6. Take the set-up of Propositions 5, i.e. the set-up of Proposition 4 without the assumption $h^1(\mathcal{F}) = 0$. By duality we have $h^1(\mathcal{F}^\vee) = h^1(\mathcal{F}(-3))$. Since $k > 0$, (1.1) gives $h^1(\mathcal{F}(-2)) = h^1(\mathcal{I}_Z(c-k-3))$. Since $\deg(Z) = y - k(c-k)$, we have $h^1(\mathcal{I}_Z(c-k-3)) \ge \max\{0, \binom{c-k-1}{2} - y + k(c-k)\}$, but the condition $\mathcal{F}_{|D} \cong \mathcal{O}_D(c) \oplus \mathcal{O}_D$ gives $h^1(\mathcal{F}(-2)) > 0$ (Lemma 3). In case (1) (resp. (2)) of Proposition 4 Z is a point (resp. the complete intersection of 2 conics) and hence $h^1(\mathcal{F}^\vee) = 1$ and $h^1(\mathcal{F}(-2)) = 1$ (resp. $h^1(\mathcal{F}^\vee) = 4$ and $h^1(\mathcal{F}(-2)) = 3$). Hence in case (1) $\mathcal{F}$ extends as a spanned bundle with no trivial factor, up to rank 3, but the associated bundle has not $(c, 0, 0)$ as its splitting type over D. In case (2) $\mathcal{F}$ extends up to rank 6 as a spanned bundle with no trivial factor, but only up to rank 4 if we add the condition that $(c, 0, \dots, 0)$ is the splitting type over D.

Now look at cases (3) and (4) of Propositions 3 and 4. For very large y in cases (3) and (4) we have $h^1(\mathcal{F}(-2)) \ge 2$, but for many y there are different schemes Z with $h^1(\mathcal{I}_Z(c-k)) = 0$, but with different values for $h^1(\mathcal{F}(-2))$. Since $h^0(D, \mathcal{I}_{A,D}(c-k-2)) = 0$, $h^1(D, \mathcal{I}_{A,D}(c-k-2) = 1$ and $h^2(\mathcal{I}_Z(c-k-3)) = h^2(\mathcal{O}(c-k-3)) = 0$, the one used to solve the existence part for Proposition 4 has $h^1(\mathcal{I}_Z(c-k-3)) = \max\{1, \binom{c-k-1}{2} - y + (k+1)(c-k)\}$. Any spanned bundle $\mathcal{F}$ has $h^1(\mathcal{F}(-2)) \ge \binom{c-k-1}{2} - y + (k+1)(c-k)$.

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References

[1] C. Anghel, I. Coandă, N. Manolache: *Globally generated vector bundles on $\mathbb{P}^n$ with $c_1 = 4$*, arXiv:1305.3464v2.

[2] C. Anghel, N. Manolache: *Globally generated vector bundles on* $\mathbb{P}^n$ *with* $c_1 = 3$, Math. Nachr. **286** (2013), no. 13–15, 1407–1423.

[3] J. Brun: *Les fibrés de rang deux sur* $\mathbb{P}_2$ *et leurs sections*, Bull. Soc. Math. France **107** (1979), 457–473.

[4] U. Bruzzo, D. Markushevich: *Moduli of framed sheaves on projective surfaces*, Documenta Math. **16** (2011), 399–410.

[5] L. Chiodera, P. Ellia: *Rank two globally generated vector bundles with* $c_1 \leq 5$, Rend. Istit. Mat. Univ. Trieste **44** (2012), 413–422

[6] Ph. Ellia: *Chern classes of rank two globally generated vector bundles on* $\mathbb{P}^2$, Rend. Lincei, Mat. Appl. **24** (2013), no. 2, 147–163.

[7] D. Huybrechts, M. Lehn: *Framed modules and their moduli*, Internat. J. Math. **6** (1995), 297–324

[8] D. Huybrechts, M. Lehn: The geometry of moduli spaces of sheaves, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[9] J. Le Potier: *Stabilitè et amplitude sur* $\mathbb{P}^2(\mathbb{C})$, in: Vector Bundles and Differential Equations, A. Hirschowitz (ed.), 145–182, Progress in Math. 7, Birkhäuser, Boston, 1980.

[10] L. Manivel: *Des fibrés globalment engendre sur l'espace projectif*, Math. Ann. **301** (1995), 469–484.

[11] N. Manolache: *Globally generated vector bundles on* $\mathbb{P}^3$ *with* $c_1 = 3$, Preprint, arXiv:1202.5988 [math.AG], 2012.

[12] Ch. Okonek, M. Schneider, H. Spindler: Vector bundles on complex projective spaces, Progress in Mathematics, 3. Birkhäuser, Boston, Mass., 1980.

[13] J. Rathmann: *The uniform position principle for curves in characteristic* p, Math. Ann. **276** (1987), no. 4, 565–579.

[14] J. C. Sierra: *A degree bound for globally generated vector bundles*, Math. Z. **262** (2009), no. 3, 517–525.

[15] J.C. Sierra, L. Ugaglia: *On globally generated vector bundles on projective spaces*, J. Pure Appl. Algebra **213** (2009), 2141–2146.

[16] J.C. Sierra, L. Ugaglia: *On globally generated vector bundles on projective spaces II*, J. Pure Appl. Algebra **218** (2014), 174–180.

[17] R. L. E. Schwarzenberger: *Vector bundles on the projective plane*, Proc. London Math. Soc. **11** (1961), 623–640.

[18] A. Van de Ven: *On uniform vector bundles*, Math. Ann. **195** (1972), 245–248.