Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **35** (2015) no. 1, 67–74. doi:10.1285/i15900932v35n1p67

Spectral properties of operators obtained by localization methods

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Received: 12.6.2014; accepted: 8.9.2014.

Abstract. We prove that, given two elliptic operators A_1 and A_2 in $L^p(\Omega_1)$ and $L^p(\Omega_2)$ respectively whose spectral properties are known, we can deduce those of the operator A coinciding with A_1 on Ω_1 and with A_2 on Ω_2. Conversely, if the spectral properties of the operator A are known in $L^p(\Omega)$, we deduce those of the restriction of A to a smaller subset of Ω.

Keywords: elliptic operators, resolvent estimates, analytic semigroups.

MSC 2000 classification: 35JXX, 47A10, 47D07.

1 Introduction and Notation

Given $1 \le p \le \infty$, consider the operators

$$A_1 = \sum_{i,j=1}^{N} a_{ij,1}(x)D_{ij} + \sum_{i=1}^{N} b_{i,1}(x)D_i + c_1(x),$$

$$A_2 = \sum_{i,j=1}^{N} a_{ij,2}(x)D_{ij} + \sum_{i=1}^{N} b_{i,2}(x)D_i + c_2(x)$$

in $L^p(\Omega_1)$ and $L^p(\Omega_2)$, respectively, where Ω_1 and Ω_2 are bounded or unbounded subsets of $\mathbb{R}^N$ with non empty interior set. The coefficients $(a_{ij,1})$, $(a_{ij,2})$ are uniformly elliptic matrices in Ω_1 and Ω_2 and we assume that $\Omega_1 \cup \Omega_2 = \Omega$ and $\Omega_1 \cap \Omega_2 = C$ is a bounded smooth domain of $\mathbb{R}^N$.

The operators A_1 and A_2 are endowed with the domains D_1 and D_2 contained in the respective maximal domains

$$D_{p,max}(A_j) = \{u \in L^p(\Omega_j) \cap W^{2,p}_{loc}(\Omega_j) : A_j u \in L^p(\Omega_j), \quad j = 1,2\}.$$

Our goal consists in deducing information on the operator

$$A = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij} + \sum_{i=1}^{N} b_i(x) D_i + c(x)$$

whose coefficients coincide with $a_{ij,1},\ b_{i,1}$ and c_1 on Ω_1 and with $a_{ij,2},\ b_{i,2}$ and c_2 on Ω_2 once the properties of the operators A_1 and A_2 are known.

In particular, we obtain spectral properties of the new operator A by the ones of A_1 and A_2. This allows us to deduce generation results for A. More specifically, the strategy consists in constructing an approximate resolvent $R(\lambda)$ for A,

$$R(\lambda) = \eta_1(\lambda - A_1)^{-1}\eta_1 + \eta_2(\lambda - A_2)^{-1}\eta_2,$$

where η_1, η_2 are smooth functions supported in Ω_1, Ω_2, respectively, and such that $\eta_1^2 + \eta_2^2 = 1$. If we assume the validity of gradient estimates for A_1 and A_2 in a suitable set containing $\Omega_1 \cap \Omega_2$, the operator $R(\lambda)$ satisfies $(\lambda - A)R(\lambda) = I + S(\lambda)$ and $\|S(\lambda)\| \le 1/2$ for $|\lambda|$ large. Then, for $\lambda \in \mathbb{C}_+$, $|\lambda|$ large, we have

$$(\lambda - A)^{-1} = R(\lambda)(I + S(\lambda))^{-1}$$

and hence the resolvent estimates for A follow from the ones for A_1 and A_2.

Conversely, if the spectral properties of the operator A in Ω are known, by using similar arguments we can deduce those of the restriction of the operator A in $\Omega'(\subset \Omega)$ endowed with Dirichlet boundary conditions.

We point out that the results contained in the paper have been useful to deduce generation results for some elliptic operator with singular coefficients (see [4], [3], [5], [6], [7]).

Notation. For $0 \le \theta < \pi$, $\rho > 0$, we denote by $\Sigma_{\theta,\rho}$ the closed set

$$\Sigma_{\theta,\rho} = \{\lambda \in \mathbb{C} : |\lambda| \ge \rho, |Arg\lambda| \le \theta\}.$$

2 Properties satisfied by the operator A

We shall construct a resolvent for A by gluing together the resolvents of A_1 and A_2. In some cases we can proceed in the opposite direction, that is deduce results for A_1 or A_2 from properties of A in the whole space. First we define the spectral properties of general elliptic operators L in Ω.

Definition 1. Let $(L, D(L))$ be the operator L endowed with the domain $D(L) \subset D_{p,max}(L)$ on $L^p(\Omega)$. We say that $(L, D(L))$ satisfies $P(\theta, \rho, C, \gamma)$, where $C, \rho > 0$, $\gamma \geq 0$ and $0 \leq \theta < \pi$ if $\Sigma_{\theta,\rho} \subset \rho(L)$ and for every $\lambda \in \Sigma_{\theta,\rho}$ the following estimate holds

$$\|(\lambda - L)^{-1}\| \leq \frac{C}{|\lambda|^\gamma}. \tag{1}$$

Definition 2. We say that $(L, D(L))$ satisfies $P(\theta, \rho, R, C, \gamma, \delta)$, $R > 0$, $\delta \in \mathbb{R}$ if it satisfies $P(\theta, \rho, C, \gamma)$, the coefficients of the operator are bounded in $C_R := \{x \in \mathbb{R}^N \ ; \ R < |x| < 2R\}$ and moreover

$$\|(\lambda - L)^{-1}\|_{L^p(\Omega) \to W^{1,p}(C_R)} \leq \frac{C}{|\lambda|^\delta}, \tag{2}$$

where the last norm is understood as the operator norm from $L^p(\Omega)$ to $W^{1,p}(C_R)$.

Remark 1. Clearly, L generates an analytic semigroup if and only if the property $P(\theta, \rho, C, \gamma)$ holds for some $\theta > \pi/2$, $\gamma = 1$ (see e.g., [2, Section 3]).

Now we assume that (A_1, D_1), (A_2, D_2) are the previously defined operators endowed with domain D_1 and D_2 contained in $L^p(\Omega_1)$, $L^p(\Omega_2)$ respectively. We also assume that if $u_i \in D_i$ and η_i are C^∞ functions with compact support in Ω_1, Ω_2, respectively, then $\eta_i u_j \eta_i \in D_i$, for $i, j = 1, 2$. The domain of A, say D, is defined as follows.

Definition 3. For given (A_1, D_1) and (A_2, D_2), we define

$$\begin{aligned} D := &\{u \in L^p(\Omega) : \ u = u_1 + u_2, \ u_1 \in D_1, \ u_2 \in D_2, \\ &\quad u_1, u_2 \text{ with compact support contained in } \Omega_1, \ \Omega_2 \text{ respectively}\}. \end{aligned}$$

Remark 2. Observe that if $u \in D$, then $\eta_i u \in D_i$. In fact, writing $u = u_1 + u_2$ with u_i as in Definition 3, then $\eta_i u_j \in D_i$.

Proposition 1. *Under the above assumptions, suppose that the operators (A_1, D_1), (A_2, D_2) satisfy $P(\theta, \rho, R, C, \gamma, \delta)$ in $L^p(\Omega_1)$ and in $L_p(\Omega_2)$ respectively, with $\delta > 0$. If there exists $\lambda_0 > 0$ such that $(\lambda - A, D)$ is injective for every $\lambda > \lambda_0$, then (A, D) satisfies $P(\theta, \rho_1, C_1, \gamma)$ in $L^p(\mathbb{R}^N)$, where ρ_1 and C_1 depend only on $p, \theta, \rho, R, C, \gamma$. If $\gamma > \frac{1}{2}$, then (A, D) satisfies $P(\theta, \rho_1, C_1, \gamma)$ in $L^p(\mathbb{R}^N)$ without injectivity condition on A. Finally, if $P(\theta, \rho_1, R, C, \gamma, \delta)$ are satisfied both in L^p, L^q and the resolvents of A_1, A_2 are coherent in L^p, L^q, then the resolvents of A are coherent in L^p, L^q.*

Proof. Let $0 \leq \eta_1, \ \eta_2 \leq 1$ be positive C^∞-functions supported in Ω_1 and Ω_2, respectively, such that $\eta_1^2 + \eta_2^2 = 1$. For $\lambda \in \Sigma_{\theta,\rho}$ $f \in L^p(\Omega)$, set $R_i(\lambda)f = \eta_i(\lambda - A_i)^{-1}(\eta_i f) \subset D_i \cap D$ for $i = 1, 2$. Observing that $A\eta_i = A_i\eta_i$,

$\eta_i A = \eta_i A_i$ it follows that

$$\begin{aligned}(\lambda - A)R_i(\lambda)f &= (\lambda - A)\eta_i(\lambda - A_i)^{-1}(\eta_i f)\\ &= \eta_i(\lambda - A_i)(\lambda - A_i)^{-1}(\eta_i f) + [\eta_i, A](\lambda - A_i)^{-1}(\eta_i f)\\ &= \eta_i^2 f + [\eta_i, A](\lambda - A_i)^{-1}(\eta_i f),\end{aligned}$$

where

$$\begin{aligned}[\eta_i, A_i]g &= \eta_i(A_i g) - A_i(\eta_i g)\\ &= -\sum_{j,k=1}^{N}(a_{jk,i} + a_{kj,i})(D_j\eta_i)(D_k g) - \left(\sum_{j,k=1}^{N} a_{jk,i}D_{jk}\eta_i - \sum_{j=1}^{N} b_{j,i}D_j\eta_i\right) g\end{aligned}$$

is a first order operator supported on C_R. Therefore

$$(\lambda - A)R_i(\lambda)f = \eta_i^2 f + S_i(\lambda)f$$

where

$$\begin{aligned}S_i(\lambda)f = &-\sum_{j,k=1}^{N}(a_{jk,i} + a_{kj,i})(D_j\eta_i)(D_k(\lambda - A_i)^{-1}(\eta_i f))\\ &- \left(\sum_{j,k=1}^{N} a_{jk,i}D_{jk}\eta_i - \sum_{j=1}^{N} b_{j,i}D_j\eta_i\right)(\lambda - A_i)^{-1}(\eta_i f).\end{aligned}$$

By (2), it follows that

$$\|S_i(\lambda)\|_{L^p(\mathbb{R}^N)} \le \frac{c_1}{|\lambda|^\delta}$$

for $\lambda \in \Sigma_{\theta,\rho}$ and with c_1 depending only on C, R. Then

$$(\lambda - A)R(\lambda)f = f + S(\lambda)f,$$

where we set

$$R(\lambda) := \sum_{i=1}^{2} R_i(\lambda), \quad S(\lambda) := \sum_{i=1}^{2} S_i(\lambda).$$

Choosing $|\lambda| > \rho_1$ large enough, we find $\|S(\lambda)\|_{L^p(\mathbb{R}^N)} \le \frac{1}{2}$ and we deduce that the operator $I + S(\lambda)$ is invertible in $L^p(\mathbb{R}^N)$. Setting $V(\lambda) = (I + S(\lambda))^{-1}$ we have

$$(\lambda - A)R(\lambda)V(\lambda)f = f$$

and hence the operator $R(\lambda)V(\lambda)$, which maps $L^p(\mathbb{R}^N)$ into D, is a right inverse of $\lambda - A$ and, by (1), satisfies

$$\|R(\lambda)V(\lambda)\| \leq \frac{2C}{|\lambda|^\gamma} \tag{3}$$

for $\lambda \in \Sigma_{\theta,\rho_1}$. Clearly, $R(\lambda)V(\lambda)$ coincides with $(\lambda - A)^{-1}$ whenever this last is injective. If $\lambda - A$ is injective for $\lambda > \lambda_0$, then $]\lambda_0, \infty[\subset \rho(A)$ and the a-priori estimates (3) show that the norm of the resolvent is bounded in the set $\Sigma_{\theta,\bar{\rho}}$, hence this set is contained in $\rho(A)$, where the resolvent operator coincide with $R(\lambda)V(\lambda)$ and satisfies (3).

If $\gamma > \frac{1}{2}$, then we have to prove the injectivity of $\lambda - A$ for $|\lambda|$ large enough. Let $u \in D$, $\lambda \in \Sigma_{\theta,\rho}$. Then $\eta_i u \in D_i$, see Remark 2, $\eta_i A u = \eta_i A_i u$ and

$$\begin{aligned} R(\lambda)(\lambda - A)u &= \sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}\eta_i(\lambda - A)u \\ &= \sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}\eta_i(\lambda - A_i)u \\ &= \sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}(\lambda - A_i)\eta_i u + \sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}[A, \eta_i]u. \end{aligned} \tag{4}$$

Suppose that $(\lambda - A)u = 0$. Then (4) implies $u = -\sum_{i=1}^{2} \eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u$. It follows that

$$|\lambda|\, \|u\|_p = \|Au\|_p \leq \sum_{i=1}^{2} \left\|A\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\right\|_p . \tag{5}$$

Since $\nabla\eta_i, \Delta\eta_i$ have support contained in C_R and by the definition of $[A_i, \eta_i]u$, we have

$$\begin{aligned} &\left\|A\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\right\|_p \\ &\leq \left\|(\lambda - A_i)\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\right\|_p + \left\|\lambda\eta_i(\lambda - A_i)^{-1}[A, \eta_i]u\right\|_p \\ &\leq \left\|\eta_i(\lambda - A_i)(\lambda - A_i)^{-1}[A_i, \eta_i]u\right\|_p \\ &\quad + \|[A_i, \eta_i](\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p + \left\|\lambda\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\right\|_p \\ &\leq \|\eta_i[A_i, \eta_i]u\|_p + \|[A_i, \eta_i](\lambda - A_i)^{-1}[A_i, \eta_i]u\|_p + \left\|\lambda\eta_i(\lambda - A_i)^{-1}[A_i, \eta_i]u\right\|_p \\ &\leq C\left[\|u\|_{W^{1,p}(C_R)} + \frac{1}{|\lambda|^\delta}\|u\|_{W^{1,p}(C_R)} + |\lambda|^{1-\gamma}\|u\|_{W^{1,p}(C_R)}\right]. \end{aligned} \tag{6}$$

By the interpolative estimates [1, Theorem 7.28] there exists $C>0$ such that for every $\varepsilon>0$

$$\|u\|_{W^{1,p}(C_R)} \le \varepsilon\|u\|_{W^{2,p}(C_R)} + \frac{C}{\varepsilon}\|u\|_{L^p(C_R)}.$$

Using the interior estimates for elliptic operators (note that a is positive and far from the origin) as in [1, Theorem 9.11] we deduce the existence of a constant $C>0$ such that for every $\varepsilon>0$,

$$\begin{aligned}\|u\|_{W^{1,p}(C(R))} &\le C\left[\varepsilon\|Au\|_{L^p\left(B_{2R+1}\setminus B_{\frac{R}{2}}\right)} + \frac{1}{\varepsilon}\|u\|_{L^p\left(B_{2R+1}\setminus B_{\frac{R}{2}}\right)}\right] \\ &\le C\left[\varepsilon|\lambda| + \frac{1}{\varepsilon}\right]\|u\|_p, \end{aligned} \tag{7}$$

where we have used $\lambda u = Au$. Combining (6) and (7) with (5), we deduce that for every $\varepsilon, \varepsilon_1 > 0$ and some C independent of ε,

$$\|u\|_p \le C\left[|\lambda|^{-1} + |\lambda|^{-1-\delta} + |\lambda|^{-\gamma}\right]\left[\varepsilon|\lambda| + \frac{1}{\varepsilon}\right]\|u\|_p.$$

By choosing $\varepsilon = |\lambda|^{-\frac{1}{2}}$, it follows that

$$\|u\|_p \le C\left[|\lambda|^{-\frac{1}{2}} + |\lambda|^{-\frac{1}{2}-\delta} + |\lambda|^{\frac{1}{2}-\gamma}\right]\|u\|_p,$$

Since $\gamma > \frac{1}{2}$, $u=0$ for $|\lambda| > \overline{\rho}$, $\overline{\rho}$ large enough, and $\lambda - A$ is injective for every $\lambda > \overline{\rho}$. Finally, if the hypotheses hold in L^p, L^q and the resolvents of A_1, A_2 are coherent in L^p, L^q (in B_{2R}, B_R^c, respectively), we have seen in the proof that the resolvent of A is the operator $R(\lambda)V(\lambda)$ which is coherent in $L^p(\mathbb{R}^N)$, $L^q(\mathbb{R}^N)$ by construction. $\boxed{QED}$

3 Properties satisfied by A_1 and A_2 if the properties of A are known

The proof in the previous section can be adapted to deduce results both in exterior and interior domains from the whole space. In this section we consider the operator (A,D) in $L^p(\mathbb{R}^N)$, where

$$A = \sum_{i,j=1}^{N} a_{ij}(x)D_{ij} + \sum_{i=1}^{N} b_i(x)D_i + c(x)$$

and

$$D \subset D_{p,max}(A) = \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{loc}(\Omega) : Au \in L^p(\mathbb{R}^N)\}.$$

Let A_1 be the operator A restricted to B_{2R} with Dirichlet boundary conditions on ∂B_{2R}. More precisely, we define (A_1, D_1) and (A_2, D_2) as follows.

Definition 4. For given (A, D), we respectively define (A_1, D_1) and (A_2, D_2) as

$$\begin{aligned} D_1 =& \{u \in L^p(B_{2R} \cap W^{2,p}(B_{2R} \setminus B_\varepsilon) \ \forall \varepsilon > 0 : u_{|\partial B_{2R}} = 0, \\ & \eta u \in D \ \forall \eta \in C_c^\infty(B_{2R}), \eta \equiv 1 \text{ near } 0\}. \end{aligned} \tag{8}$$

and

$$\begin{aligned} D_2 =& \{u \in L^p(B_R^c \cap W^{2,p}(B_\rho \setminus B_R) \ \forall \rho > R : u_{|\partial B_R} = 0, \\ & \eta u \in D \ \forall \eta \in C^\infty(B_R^c), \eta \equiv 1 \text{ near } \infty, \ \eta \equiv 0 \text{ near } \partial B_R\}. \end{aligned} \tag{9}$$

Then we have

Proposition 2. *Let (A, D) satisfy $P(\theta, \rho, R, C, \gamma, \delta)$ in $L^p(\mathbb{R}^N)$. Let (A_2, D_2) in B_{R^c} as defined in (9). If there exists $\lambda_0 > 0$ such that $\lambda - A_2$ is injective for every $\lambda > \lambda_0$, then A_2 satisfies $P(\theta, \rho_2, C_2, \gamma)$ in L^p, where ρ_2, C_2 depend only on $p, \theta, \rho, R, C, \gamma$. If $\gamma > \frac{1}{2}$, then A_2 satisfies $P(\theta, \rho_1, C_1, \gamma)$ in L^p without injectivity condition on A_2. Finally, if $P(\theta, \rho_1, R, C, \gamma, \delta)$ is satisfied both in L^p, L^q and the resolvent of A are coherent in L^p, L^q, then the resolvents of A_2 are coherent in L^p, L^q.*

Proof. The proof is similar to that of Proposition 1 and we only outline the main steps. Let $0 \le \eta_1, \ \eta_2 \le 1$ be positive C^∞-functions supported in B_{2R} and B_R^c, respectively, such that $\eta_1^2 + \eta_2^2 = 1$. Let A_R be the operator A in the annulus C_R with Dirichlet boundary conditions, that is with domain

$$D_p(A_R) = \{u \in W^{2,p}(C_R) \cap W_0^{1,p}(C_R)\}.$$

Since $a > 0$ in C_R, A_R is uniformly elliptic and generates an analytic semigroup in C_R, see [2]. In particular, A_R satisfies $P(\theta, \rho, R, C, \gamma, \delta)$.

For $\lambda \in \Sigma_{\theta,\rho}$ $f \in L^p(B_R^c)$ (extended to zero outside B_R^c) we set

$$R(\lambda)f = \eta_1(\lambda - A_R)^{-1}(\eta_1 f) + \eta_2(\lambda - A)^{-1}(\eta_2 f) \in D_2.$$

Then we argue as in Proposition 1. $\boxed{QED}$

Proposition 3. *Let (A, D) satisfy $P(\theta, \rho, R, C, \gamma, \delta)$ in $L^p(\mathbb{R}^N)$. Let (A_1, D_1) in B_{2R} as defined in (8). If there exists $\lambda_0 > 0$ such that $\lambda - A_1$ is injective for every $\lambda > \lambda_0$, then A_1 satisfies $P(\theta, \rho_1, C_1, \gamma)$ in L^p, where ρ_1, C_1 depend only* *on $p, \theta, \rho, R, C, \gamma$. If $\gamma > \frac{1}{2}$, then A_1 satisfies $P(\theta, \rho_1, C_1, \gamma)$ in L^p without injectivity condition on A_1. Finally, if $P(\theta, \rho_1, R, C, \gamma, \delta)$ is satisfied both in L^p, L^q and the resolvents of A are coherent in L^p, L^q, then the resolvents of A_1 are coherent in L^p, L^q.*

Proof. Keeping the notation of the proof of Proposition 2, for $f \in L_p(B_{2R})$ (extended to zero outside B_R) we set

$$R(\lambda)f = \eta_1(\lambda - A)^{-1}(\eta_1 f) + \eta_2(\lambda - A_R)^{-1}(\eta_2 f) \in D_1$$

and we argue as in Proposition 1. $\boxed{QED}$

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