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Postulation of general unions of lines and multiplicity two points in $\mathbb{P}^r$, $r \le 5$

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Abstract. We determine the Hilbert function of general unions in $\mathbb{P}^r$, $r = 4, 5$, of prescribed numbers of lines and fat 2-points. We only have partial results in $\mathbb{P}^3$ (also if we add a few general reducible conics).

Keywords: postulation, Hilbert function, lines, zero-dimensional schemes; fat points

MSC 2000 classification: primary 14N05, secondary 14H99

Introduction

In [10], [11], [12] E. Carlini, M. V. Catalisano and A. V. Geramita started a detailed analysis of the Hilbert function of certain multiple structures on unions of linear spaces, usually prescribing the Hilbert polynomial and then requiring that the structure is general with the prescribed Hilbert polynomial. Let $X \subset \mathbb{P}^r$ be a closed subscheme. X is said to have *maximal rank* if for all integers $k \ge 0$ either $h^0(\mathcal{I}_X(k)) = 0$ or $h^1(\mathcal{I}_X(k)) = 0$, i.e. if for all integers $k \ge 0$ the restriction map $\rho_{X,k} : H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(\mathcal{O}_X(k))$ is a linear map with maximal rank, i.e. either it is injective or it is surjective, i.e. $h^0(\mathcal{I}_X(k)) = 0$ if $h^0(\mathcal{O}_X(k)) \ge \binom{r+k}{r}$ and $h^1(\mathcal{I}_X(k)) = 0$ if $h^0(\mathcal{O}_X(k)) \le \binom{r+k}{r}$.

We recall that for each $P \in \mathbb{P}^r$ the 2-point $2P$ of $\mathbb{P}^r$ is the zero-dimensional subscheme of $\mathbb{P}^r$ with $(\mathcal{I}_P)^2$ as its ideal sheaf. Hence $2P$ has degree $r+1$. For all integers $r \ge 3$, $t \ge 0$ and $a \ge 0$ let $Z(r,t,a)$ be the set of all disjoint unions $A \subset \mathbb{P}^r$ of t lines and a 2-points. For each integer $x > 0$ and any $A \in Z(r,t,a)$ we have $h^0(A, \mathcal{O}_A(x)) = t(x+1) + a(r+1)$. The *critical value* of the triple (r,t,a) or of any $A \in Z(r,t,a)$ is the minimal positive integer k such that $\binom{r+k}{r} \ge (k+1)t + a(r+1)$. Fix $A \in Z(r,t,a)$. The Castelnuovo-Mumford's lemma gives that A has maximal rank if and only if $h^1(\mathcal{I}_A(k)) = 0$ and $h^0(\mathcal{I}_A(k-1)) = 0$. The key starting point of our paper is a theorem of R. Hartshorne and A. Hirschowitz which says that for all integers $r \ge 3$ and $t > 0$

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a general union of lines $X \subset \mathbb{P}^r$ has maximal rank ([18]). A famous theorem of J. Alexander and A. Hirschowitz says that a general union of 2-points of $\mathbb{P}^r$ has maximal rank, except in a few well-understood exceptional cases ([19], [2], [3], [4], [9], [13], [14], [15]). In this paper we consider (but do not solve) the problem for a general union of a prescribed number of lines and a prescribed number of 2-points in $\mathbb{P}^r$. Since the disjoint union A of two 2-points in $\mathbb{P}^r$, $r \neq 1$, has $h^0(\mathcal{I}_A(2)) > 0$ and $h^1(\mathcal{I}_A(2)) > 0$, we do not consider the restriction map $\rho_{X,2} : H^0(\mathcal{O}_{\mathbb{P}^r}(2)) \to H^0(\mathcal{O}_X(2))$, or rather we restrict to cases in which this map is injective. It is easy to check for which $(r,t,a) \in \mathbb{N}^3$, $r \geq 3$, $\rho_{X,2}$ is injective for a general $X \in Z(r,t,a)$ and to determine for all (r,t,a) the dimension of $\ker(\rho_{X,2})$ (Lemma 1). The next case is when $k = 3$ and here there is a case in which a general $X \in Z(3,2,3)$ has not maximal rank (Example 1). For $r = 3,4,5$ we did not found any other exceptional case with respect to $\rho_{X,k}$, $k \geq 3$, (except of course the case $(r,t,a) = (4,0,7)$, which is in the exceptional list for the Alexander-Hirschowitz theorem).

In this paper we prove the following result.

Theorem 1. *Fix $r \in \{4,5\}$, $(t,a) \in \mathbb{N}^2 \setminus \{0,0\}$ and an integer $k \geq 3$. Assume $(r,t,a,k) \notin \{(4,0,7,3),(4,0,14,3)\}$. Fix a general $X \in Z(r,t,a)$. Then $\rho_{X,k}$ has maximal rank.*

In the case $r = 3$ we only have partial results (Propositions 4 and 5). The latter one look at general disjoint unions of lines, 2-points and reducible conics. In [18] both reducible conics and sundials are used as a tool to get their main theorem related to disjoint unions of lines. Sundials may simplify the proofs in [18] ([10]). We use sundials, but we found easier to avoid reducible conics and use instead the +lines introduced in [6]. As usual for fat points we use the very powerful Differential Horace Lemma ([5, Lemma 2.3]), although only for 2-points.

Remark 1. After this paper was submitted we solved the general case for $r = 3$ ([7]) (it does not cover Proposition 5, but it could be used to shorten the proof of the case $r = 4$ done here in section 4).

We work over an algebraically closed field with $\mathrm{char}(\mathbb{K}) = 0$. To apply the Differential Horace Lemma for double points it would be sufficient to assume $\mathrm{char}(\mathbb{K}) \neq 2$ or the characteristic free [14, Lemma 5], but for a smooth quadric surface we also use [20, Propositions 4.1 and 5.2 and Theorem 7.2].

1 Preliminaries

For each integer $x > 0$ and any $A \in Z(r,t,a)$ we have $h^0(A, \mathcal{O}_A(x)) = t(x+1) + a(r+1)$. The *critical value* of the triple (r,t,a) or of any $A \in Z(r,t,a)$

is the minimal positive integer k such that $\binom{r+k}{r} \ge (k+1)t + a(r+1)$. Fix $A \in Z(r,t,a)$. The Castelnuovo-Mumford lemma gives that A has maximal rank if and only if $h^1(\mathcal{I}_A(k)) = 0$ and $h^0(\mathcal{I}_A(k-1)) = 0$. Let $Z(r,t,a)'$ the closure of $Z(r,t,a)$ in the Hilbert scheme of $\mathbb{P}^r$. Fix natural numbers r,t,a,k with $r \ge 3$. By the semicontinuity theorem for cohomology ([17, III.12.8]) to prove that $h^1(\mathcal{I}_X(k)) = 0$ (resp. $h^0(\mathcal{I}_X(k)) = 0$) it is sufficient to prove the existence of at least one $W \in Z(r,a,t)'$ such that $h^1(\mathcal{I}_W(k)) = 0$ (resp. $h^0(\mathcal{I}_W(k)) = 0$). The set $Z(r,t,a)$ is irreducible and hence $Z(r,t,a)'$ is irreducible. Therefore to prove that a general element of $Z(r,t,a)$ has maximal rank it is sufficient to find $A, B \in Z(r,t,a)'$ such that $h^1(\mathcal{I}_A(k)) = 0$ and $h^0(\mathcal{I}_B(k-1)) = 0$, where k is the critical value of the triple (r,t,a). A sundial $B \subset \mathbb{P}^r$ is an element of $Z(r,2,0)'$ ([10]).

Lemma 1. *Fix a general $X \in Z(r,t,a)$, $r \ge 3$.*

(a) We have $h^0(\mathcal{I}_X(2)) = 0$ if and only if either $a \le r-3$ and $3t \ge \binom{r+2-a}{2}$ or $a = r-2$ and $t \ge 3$ or $a = r-1, r$ and $t > 0$ or $a \ge r+1$.

(b) Assume $0 < a \le r-3$. Then $h^0(\mathcal{I}_X(2)) = \max\{\binom{r-a+2}{2} - 3t, 0\}$.

(c) Assume $a = r-2$. We have $h^0(\mathcal{I}_X(2)) = 6-2t$ if $t \le 1$, $h^0(\mathcal{I}_X(2)) = 1$ if $t = 2$ and $h^0(\mathcal{I}_X(2)) = 0$ if $t \ge 3$.

Proof. Write $X = Y \sqcup A$ with $Y \in Z(r,t,0)$ and $A = \cup_{P \in S} 2O$, $\sharp(S) = a$, and S general in $\mathbb{P}^r$. The case $a = 0$ is covered by [18]. Assume $a > 0$. Let $\langle S \rangle$ be the linear span of S. Since S is general, then $\dim(\langle S \rangle) = \min\{r, a-1\}$. The linear system $|\mathcal{I}_X(2)|$ is the linear system of all quadric cones containing Y and with vertex containing $\langle S \rangle$. In particular we get $h^0(\mathcal{I}_X(2)) = 0$ if $a \ge r+1$ or $a = r$ and $t > 0$. Now assume $1 \le a < r$. Let $\ell : \mathbb{P}^r \setminus \langle S \rangle \to \mathbb{P}^{r-a}$ be the linear projection from $\langle S \rangle$. If $a \le r-2$, then $Y \cap \langle S \rangle = \emptyset$ and $\ell(Y)$ is a general union of t lines of $\mathbb{P}^{r-a}$ (it is $\mathbb{P}^1$ if $a = r-1$). We get the lemma in the case $a = r-2$, because 3 distinct lines of $\mathbb{P}^2$ are not contained in a conic. If $1 \le a \le r-3$, then $\ell(Y)$ is a general element of $Z(r-a,t,0)$ and we apply [18] to $\ell(Y)$, because $h^0(\mathcal{I}_X(2)) = h^0(\mathbb{P}^{r-a}, \mathcal{I}_{\ell(Y)}(2))$. $\boxed{QED}$

Fix an integral variety T, an effective divisor D of T and any closed subscheme $X \subset T$. The residual scheme $\mathrm{Res}_D(X)$ of X with respect to D is the closed subscheme of T with $\mathcal{I}_X : \mathcal{I}_D$ as its ideal sheaf. For each $\mathcal{L} \in \mathrm{Pic}(T)$ we have an exact sequence of coherent $\mathcal{O}_T$-sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_D(X)} \otimes \mathcal{L}(-D) \to \mathcal{I}_X \otimes \mathcal{L} \to \mathcal{L}|D \otimes \mathcal{I}_{X \cap D, D} \to 0. \tag{1}$$

From (1) we get the following inequalities:

(1) $h^0(T, \mathcal{I}_X \otimes \mathcal{L}) \le h^0(T, \mathcal{I}_{\mathrm{Res}_D(X)} \otimes \mathcal{L}(-D)) + h^0(D, \mathcal{L}|D \otimes \mathcal{I}_{X \cap D, D})$;

(2) $h^1(T, \mathcal{I}_X \otimes \mathcal{L}) \le h^1(T, \mathcal{I}_{\mathrm{Res}_D(X)} \otimes \mathcal{L}(-D)) + h^1(D, \mathcal{L}|D \otimes \mathcal{I}_{X\cap D,D})$.

As in [11], [12], [10] we say that something is true by " *the Castelnuovo's sequence* " if it is true by one of the two inequalities above for suitable T, D, X, $\mathcal{L}$.

We use the following form of the Differential Horace Lemma for double points ([2, (1.3), (1.4)(5)], [14, Lemma 5] (although stated only in $\mathbb{P}^n$); [5, Lemma 2.3] contains the case with arbitrary multiplicities).

Lemma 2. *Let $T \subset \mathbb{P}^r$ be a degree t irreducible hypersurface. Fix a closed subscheme $X \subset \mathbb{P}^r$ not containing T and integers $\alpha > 0$, $k \ge t$. Fix a general $S \subset T$ such that $\sharp(S) = \alpha$ and set $B := \cup_{O\in S} 2O$, $B' := B \cap T$. Let W be the union of X and α general 2-points of $\mathbb{P}^r$. Then $h^i(\mathcal{I}_W(k)) \le h^i(\mathcal{I}_{Res_T(X)\cup B'}(k-t)) + h^i(T, \mathcal{I}_{(X\cap T)\cup S}(k))$, $i = 0, 1$.*

Take the set-up of Lemma 2. Since S is general in T, then

(1) $h^0(T, \mathcal{I}_{(X\cap T)\cup S}(k)) = \max\{0, h^0(T, \mathcal{I}_{X\cap T}(k)) - \alpha\}$;

(2) $h^1(T, \mathcal{I}_{(X\cap T)\cup S}(k)) = h^1(T, \mathcal{I}_{X\cap T}(k)) + \max\{0, \alpha - h^0(T, \mathcal{I}_{X\cap T}(k))\}$.

The following result is an elementary consequence of [10]. We will not use it here. Sometimes it may be used in a situation in which we would like to apply Lemma 4 below.

Lemma 3. *Fix integers $n \ge 3$, $k > 0$, $e \ge 0$ and $t \ge 0$. Let $X \subset \mathbb{P}^n$ be a general union of t lines and e reducible conics.*

(a) If $e(2k+1) + t(k+1) \le \binom{n+k}{n} - e$, then $h^1(\mathcal{I}_X(k)) = 0$.

(b) If $\binom{n+k}{n} - e < e(2k+1) + t(k+1) \le \binom{n+k}{n}$, then $h^1(\mathcal{I}_X(k)) \le e(2k+1) + t(k+1) - \binom{n+k}{n} + e$ and $h^0(\mathcal{I}_X(k)) = h^1(\mathcal{I}_X(k)) + \binom{n+k}{n} - e(2k+1) - t(k+1)$.

(c) If $e(2k+1) + t(k+1) \ge \binom{n+k}{n} - e$, then $h^0(\mathcal{I}_X(k)) \le e$.

Proof. For each $P \in \mathrm{Sing}(X)$ let $C_P \subset X$ the connected component of X containing P. Write $X = Y \sqcup \bigcup_{P\in \mathrm{Sing}(X)} C_P$. For each $P \in \mathrm{Sing}(X)$ let $N_P \subseteq \mathbb{P}^n$ be a general 3-dimensional linear space containing C_P. Let $E_P \subset N_P$ be the sundial with C_P as its support. Set $W := Y \sqcup \bigcup_{P\in\mathrm{Sing}(X)} E_P$. Since X and each N_P is general, W is a general union of e lines and f sundials. Hence either $h^0(\mathcal{I}_W(k)) = 0$ (case $(t+2e)(k+1) \ge \binom{n+k}{n}$) or $h^1(\mathcal{I}_W(k)) = 0$ (case $(t+2e)(k+1) \le \binom{n+k}{n}$). Let η be the nilradical of $\mathcal{O}_W$, i.e. let η the $\mathcal{O}_W$-ideal sheaf of the subscheme X of W. We have $\eta \equiv \mathcal{I}_X/\mathcal{I}_W$. Hence we have the exact sequence

$$0 \to \mathcal{I}_W(k) \to \mathcal{I}_X(k) \to \eta(k) \to 0. \qquad (2)$$

Since η is supported by finitely many points, we have $h^1(\eta(k)) = 0$ and $h^0(\eta(k)) = e$. Hence (2) gives $h^1(\mathcal{I}_X(k)) \le h^1(\mathcal{I}_W(k)) \le h^1(\mathcal{I}_X(k)) + e$ and $h^0(\mathcal{I}_X(k)) \le$

$h^0(\mathcal{I}_W(k)) + e$. In the set-up of part (a) we have $h^1(\mathcal{I}_W(k)) = 0$ and hence $h^1(\mathcal{I}_X(k)) = 0$. In the set-up of part (c) we have $h^0(\mathcal{I}_W(k)) = 0$ and hence $h^0(\mathcal{I}_X(k)) \le e$. In the set-up of part (b) we have $h^1(\mathcal{I}_W(k)) = e(2t+1) + t(k+1) - e - \binom{n+k}{n}$ and hence $h^1(\mathcal{I}_X(k)) \le e(2k+1) + t(k+1) - \binom{n+k}{n} + e$. The equality " $h^0(\mathcal{I}_X(k)) = h^1(\mathcal{I}_X(k)) + \binom{n+k}{n} - e(2k+1) - t(k+1)$ " in part (b) is true, because $h^0(\mathcal{O}_X(k)) = e(2k+1) + t(k+1)$. $\boxed{QED}$

Lemma 4. *Fix integers $n \ge 2$ and $k > t > 0$, $s > 0$. Let $T \subset \mathbb{P}^n$ be an integral degree t hypersurface and let $Y \subset \mathbb{P}^n$ be a closed subscheme such that $h^1(\mathcal{I}_Y(k)) = 0$. Let S be a general subset of T with $\sharp(S) = s$. We have $h^1(\mathcal{I}_{Y\cup S}(k)) = 0$ if $h^0(\mathcal{I}_{Res_T(Y)}(k-t)) \le h^0(\mathcal{I}_Y(k)) - s$.*

Proof. Set $Y_0 := Y$. We order the points $P_1, \dots, P_s$ of S. For each integer $i \in \{1, \dots, s\}$ set $Y_i := Y \cup \{P_1, \dots, P_i\}$. Notice that $\mathrm{Res}_T(Y_i) = \mathrm{Res}_T(Y)$ for all i. Since $h^1(\mathcal{I}_Y(k)) = 0$, we have $h^1(\mathcal{I}_{Y\cup S}(k)) = 0$ if and only if $h^0(\mathcal{I}_{Y\cup S}(k)) = h^0(\mathcal{I}_Y(k)) - s$. Therefore by induction on s we may assume that $h^0(\mathcal{I}_{Y_{s-1}}(k)) = h^0(\mathcal{I}_Y(k)) - s + 1$. It is sufficient to prove that $h^0(\mathcal{I}_{Y_s}(k)) = h^0(\mathcal{I}_{Y_{s-1}}(k)) - 1$. Assume that the last inequality is not true. Since P_s is a general point of T, we get that T is in the base locus of $|\mathcal{I}_{Y_{s-1}}(k)|$. Hence $h^0(\mathcal{I}_Y(k)) - s + 1 = h^0(\mathcal{I}_{Y_{s-1}\cup T}(k)) = h^0(\mathcal{I}_{\mathrm{Res}_T(Y)}(k-t))$, a contradiction. $\boxed{QED}$

Lemma 5. *Fix a projective variety W, $R \in Pic(W)$, a closed subscheme U of W with $U = Y \sqcup A$, with A union of some of the connected components of U and A zero-dimensional. Let $B \subset W$ be a zero-dimensional scheme such that $A \subseteq B$ and $B \cap Y = \emptyset$. Set $V := Y \sqcup B$. Then $h^1(\mathcal{I}_{U,W} \otimes R) \le h^1(\mathcal{I}_{V,W} \otimes R)$ and $h^0(\mathcal{I}_{V,W} \otimes R) \le h^0(\mathcal{I}_{U,W} \otimes R)$.*

Proof. Since $U \subseteq V$, we obviously have $h^0(\mathcal{I}_{V,W} \otimes R) \le h^0(\mathcal{I}_{U,W} \otimes R)$. The ideal sheaf $\mathcal{I}_{U,V}$ of U in V is isomorphic to the ideal sheaf $\mathcal{I}_{A,B}$ of A in B. Since B is zero-dimensional and $\mathcal{I}_{A,B}$ is supported by B_{red}, we have $h^1(\mathcal{I}_{A,B} \otimes R) = 0$. Hence a trivial exact sequence gives $h^1(\mathcal{I}_{U,W} \otimes R) \le h^1(\mathcal{I}_{V,W} \otimes R)$. $\boxed{QED}$

2 +lines

For any $P \in \mathbb{P}^n$, $n \ge 1$, a tangent vector of $\mathbb{P}^n$ with P as its support is a degree two connected zero-dimensional scheme $v \subset \mathbb{P}^n$ such that $v_{red} = \{P\}$. If $X \subseteq \mathbb{P}^n$ and $P \in X$ we say that v is tangent to X or that it is a tangent vector of X if v is contained in the Zariski tangent space $T_P X \subseteq \mathbb{P}^n$ of X at P. Fix a line $L \subset \mathbb{P}^n$, $n \ge 2$, $O \in L$, and a tangent vector v of $\mathbb{P}^n$ at O which is not tangent to L (i.e. assume that L is not the line spanned by v). Set $A := L \cup v$. We say that A is a *+line*, that A is the support of L, that O is the support of the nilradical of $\mathcal{O}_A$ and that v is the tangent vector of A. The scheme A has $p(t) = t + 2$ as its Hilbert polynomial. For all $t > 0$ we have $h^1(\mathcal{O}_A(t)) = 0$ and $h^0(\mathcal{O}_A(t)) = t+2$. We say that L is the support of A and that O is the support of the nilradical of A. For all integers $r \geq 3$, $t \geq 0$ and $c \geq 0$ let $L(r, t, c)$ be the set of all disjoint unions of t lines and c +lines. Now assume $r \geq 4$ and fix a hyperplane $H \subset \mathbb{P}^n$. Let $L(r, t, c)_H$ be the set of all $A \in L(r, t, c)$ such that the nilradical $\mathcal{O}_A$ is supported by points of H (we do not impose that the c tangent vectors of A are contained in H; we only impose that their support is contained in H). The set $L(r, t, c)_H$ is an irreducible variety of dimension $(t + c)(2r - 2) + c(r - 1)$.

For all integers $r \geq 3$ and $k \geq 0$ define the integers $u_{r,k}$ and $v_{r,k}$ by the relations

$$(k + 1)m_{r,k} + n_{r,k} = \binom{r+k}{r}, \ 0 \leq n_{r,k} \leq k. \tag{3}$$

For all integers $r \geq 4$ and $k \geq 1$ consider the following assertions $B_{r,k}$:

$B_{r,k}, r \geq 4, k > 0$: A general $X \in L(r, m_{r,k} - n_{r,k}, n_{r,k})_H$ satisfies $h^0(\mathcal{I}_X(k)) = 0$

Remark 2. A statement like $B_{r,k}$, $r \geq 4$, $k \geq 1$, but with $X \in L(r, t, c)$ instead of $X \in L(r, t, c)_H$ was proved in [6], §4. $B_{r,1}$ is trivially true for all $r \geq 4$. It is easy to check that the quoted proof in all cases gives $X \in L(r, t, c)_H$ if as Y we take some $Y \in L(r, t', c')_H$, suitable t', c', i.e. if we use induction on k for a fixed r for the assertion $B_{r,k}$ as stated here, not as in [6].

Lemma 6. *Fix integers $r \geq 4$, $k > 0$, $t \geq 0$ and $c \geq 0$ such that $c \leq k$ and $(k + 1)t + (k + 2)c \leq \binom{r+k}{r}$. Then $h^1(\mathcal{I}_X(k)) = 0$ for a general $X \in L(r, t, c)_H$.*

Proof. We use induction on k, the case $k = 1$ being obvious. Assume $k \geq 2$ and that the lemma is true in $\mathbb{P}^r$ for the integer $k - 1$. Set $e := \binom{r+k}{r} - (k + 1)t - (k + 2)c$. Increasing if necessary t we may assume that $e \leq k$. Fix a general $Y \in L(r, m_{r,k} - n_{r,k}, n_{r,k})_H$. We have $h^i(\mathcal{I}_Y(k - 1)) = 0$, $i = 0, 1$ (Remark 2). Since $(k + 1)t + (k + 2)c + e = \binom{r+k}{r}$, we have

$$(k+1)(t-m_{r,k-1}+n_{r,k-1})+(k+2)(c-n_{r,k-1})+e+m_{r,k-1} = \binom{r+k-1}{r-1}. \tag{4}$$

Since $e \leq k$ and $c \leq k$, either $t = m_{r,k} - c$, $c \leq n_{r,k}$ and $e = n_{r,k} - c$ or $c > n_{r,k}$, $t = m_{r,k} - c - 1$ and $e = k + 1 - c + n_{r,k}$. The case $c > n_{r,k}$ is done in step (a) of the proof of [6, Theorem 1 for $r \geq 4$]. Now assume $c \leq n_{r,k}$. Fix any $W \in L(r, m_{r,k} - n_{r,k}, n_{r,k})_H$ with $h^i(\mathcal{I}_W(k)) = 0$, $i = 0, 1$. Take any $X \in L(r, t, c)$ with $W_{red} \subset X \subseteq W$ and use the surjectivity of the restriction map $H^0(W, \mathcal{O}_W(k)) \to H^0(X, \mathcal{O}_X(k))$. $\boxed{QED}$

3 $\mathbb{P}^3$

Example 1. Take $r = 3$, $k = 3$, $t = 2$ and $a = 3$. Fix a general $X \in Z(3,2,3)$. Then $h^0(\mathcal{I}_X(3)) = h^1(\mathcal{I}_X(3)) = 1$. Indeed, since $(k+1)t + (r+1)a = 20 = \binom{6}{3}$, we have $h^1(\mathcal{I}_X(3)) = h^0(\mathcal{I}_X(3))$. Write $X = U \sqcup A$ with $A = \cup_{O \in S} 2O$, $S \subset \mathbb{P}^3$ the union of 3 non-coplanar points and U a disjoint union of two lines. Let H be the plane spanned by S. Since U is general, there is a unique quadric surface Q containing $U \cup S$. We have $Q \cup H \in |\mathcal{I}_X(3)|$. Take any $T' \in |\mathcal{I}_X(3)|$. Since $T'|H$ has 3 non-collinear singular points, S, and contains two general points, $U \cap H$, of H, H must be a component of T'. Hence $h^0(\mathcal{I}_X(3)) = 1$.

Proposition 1. *Fix a general $X \in Z(3,t,a)$*

(i) If $(t,a) \in \{(1,4),(3,2),(4,1)\}$, then $h^0(\mathcal{I}_X(3)) = h^1(\mathcal{I}_X(3)) = 0$.

(ii) If $(t,a) \neq (2,3)$, then either $h^0(\mathcal{I}_X(3)) = 0$ or $h^1(\mathcal{I}_X(3)) = 0$.

Proof. First assume $(t,a) = (3,2)$. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Fix a general $S \subset Q$ with $\sharp(S) = 2$ and set $A := \cup_{O \in S} 2O$, where $2O$ is a 2-point of $\mathbb{P}^3$. Let $Y \subset \mathbb{P}^3$ be a general line. Let $E \subset Q$ be a general union of two lines of type $(0,1)$. We have $h^1(Q, \mathcal{I}_{Q \cap A}(3,1)) = 0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). Since $h^i(\mathcal{I}_{Y \cup S}(1)) = 0$, $i = 0,1$, we get $h^i(\mathcal{I}_{Y \cup E \cup A}(3)) = 0$, $i = 0,1$.

Now assume $(t,a) = (4,1)$. Let $H \subset \mathbb{P}^3$ be a plane. Fix a general union $Y \subset \mathbb{P}^3$ of 3 lines, a general line $L \subset H$ and a general $O \in H$. We obviously have $h^i(\mathcal{I}_{Y \cup \{O\}}(2)) = 0$, $i = 0,1$, and $h^1(H, \mathcal{I}_{(2O \cap H) \cup L}(3)) = 0$. Since $Y \cap H$ is a general union of 3 points, we have $h^i(H, \mathcal{I}_{(Y \cup L \cup 2O) \cap H}(3)) = 0$, $i = 0,1$. Hence $h^i(\mathcal{I}_{Y \cup L \cup 2O}(3)) = 0$, $i = 0,1$.

Now assume $(t,a) = (1,4)$. Let $Y \subset \mathbb{P}^3$ be a general union of one line and one 2-point. Take a general $S \subset H$ with $\sharp(S) = 3$ and let A be the union of the 2-points of $\mathbb{P}^3$ with the points of S as their support. It is sufficient to prove $h^1(\mathcal{I}_{Y \cup A}(3)) = 0$. We have $h^i(H, \mathcal{I}_{(Y \cup A) \cap H}(3)) = 0$, because $h^1(H, \mathcal{I}_{A \cap H}(3)) = 0$ and $\deg(H \cap (Y \cup A)) = \binom{5}{2}$. Since $h^1(\mathcal{I}_Y(2)) = 0$ and $h^0(\mathcal{I}_Y(1)) = 0$, Lemma 4 gives $h^i(\mathcal{I}_{Y \cup S}(2)) = 0$, $i = 0,1$.

Now assume $(t,a) = (2,4)$. Fix a general $Y \in Z(3,2,3)$ and a general $O \in \mathbb{P}^3$. Since $h^0(\mathcal{I}_Y(3)) = 1$ (Example 1), we have $h^0(\mathcal{I}_{Y \cup 2O}(3)) = 0$.

The other cases of part (ii) follow from part (i) and the case $(t,a) = (2,4)$ just done. $\boxed{QED}$

Proposition 2. *Fix a general $X \in Z(3,t,a)$.*

(a) If $(t,a) \in \{(1,7),(2,6),(3,5),(4,3),(5,2),(6,1),(7,0)\}$, then we have $h^1(\mathcal{I}_X(4)) = 0$.

(b) If $(t,a) \in \{(1,8),(2,7),(4,4),(5,3),(6,2)\}$, then $h^0(\mathcal{I}_X(4)) = 0$.

Proof. The case $(t,a)=(7,0)$ is true by [18]. Let $S_x \subset Q$ be a general union of x points and let $E_y \subset Q$ be a general union of y lines of type $(0,1)$. Set $A_x := \cup_{O\in S_x} 2O$. Let $H \subset \mathbb{P}^3$ be a plane and let $Q \subset \mathbb{P}^3$ be a smooth quadric surface.

(i) Assume $(t,a)=(1,7)$. Fix a general line $Y \subset \mathbb{P}^3$. We have $h^1(\mathcal{I}_Y(2)) = 0$ and $h^0(\mathcal{I}_Y)=0$. Hence $h^i(\mathcal{I}_{Y\cup S_7}(2))=0$, $i=0,1$ (Lemma 4). We have $h^1(Q,\mathcal{I}_{A_7\cap Q}(4))=0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]) and hence $h^1(Q,\mathcal{I}_{(Y\cup A_7)\cap Q}(4))=0$. Hence $h^1(\mathcal{I}_{Y\cup A_7}(4))=0$.

(ii) Assume $(t,a)=(2,6)$. Fix a general $Y\in Z(3,1,3)$. Proposition 1 gives $h^1(\mathcal{I}_Y(3))=0$. Since a double plane only contains a two-dimensional family of lines, we have $h^0(\mathcal{I}_Y(2))=0$. Hence $h^1(\mathcal{I}_{Y\cup S}(3))=0$ for a general $S\subset H$ with $\sharp(S)=3$ (Lemma 4). Set $A:=\cup_{O\in S}2O$. We have $h^1(H,\mathcal{I}_{A\cap H}(3))=0$ and hence $h^i(H,\mathcal{I}_{(Y\cup A)\cap H}(3))=0$, $i=0,1$. Hence $h^i(H,\mathcal{I}_{L\cup(Y\cup A)\cap H}(4))=0$, $i=0,1$, for a general line $L\subset H$. Therefore $h^1(\mathcal{I}_{Y\cup L\cup A}(4))=0$.

(iii) Assume $(t,a)=(3,5)$. Take a general $Y\in Z(3,3,1)$. We have $h^1(\mathcal{I}_Y(3))=0$ and $h^0(\mathcal{I}_Y(2))=0$. Hence $h^i(\mathcal{I}_{Y\cup S}(3))=0$, $i=0,1$, for a general $S\subset H$ with $\sharp(S)=4$. Set $A:=\cup_{O\in S}2O$. We have $h^1(H,\mathcal{I}_{A\cap H}(4))=0$ and hence $h^i(H,\mathcal{I}_{(Y\cup A)\cap H}(4))=0$, $i=0,1$. Therefore $h^1(\mathcal{I}_{Y\cup A}(4))=0$, $i=0,1$.

(iv) Assume $(t,a)=(4,3)$. Let $Y\subset\mathbb{P}^3$ be a general union of two lines. We have $h^1(\mathcal{I}_Y(2))=0$ and $h^0(\mathcal{I}_Y)=0$ and hence $h^1(\mathcal{I}_{Y\cup S_3}(2))=0$. We have $h^1(Q,\mathcal{I}_{Q\cap A_3}(4,2))=0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]) and hence $h^i(Q,\mathcal{I}_{Q\cap(Y\cup A_3)}(4))=0$, $i=0,1$. Hence $h^1(\mathcal{I}_{Y\cup E_2\cup A_3}(4))=0$.

(v) Assume $(t,a)=(5,2)$. Fix a general $Y\in Z(3,2,0)$. We have $h^1(\mathcal{I}_Y(2))$ $=0$ and hence $h^1(\mathcal{I}_{Y\cup S_2}(3))=0$. Since $h^1(Q,\mathcal{I}_{A_2\cap Q}(4,1))=0$ and $Y\cap Q$ is a general union of 4 points, we have $h^1(Q,\mathcal{I}_{(Y\cup A_2)\cap Q}(4,1))=0$, i.e. we have $h^1(Q,\mathcal{I}_{E_3\cup(Y\cup A_2)\cap Q}(4))=0$. Hence $h^1(\mathcal{I}_{Y\cup E_3\cup A_2}(4))=0$.

(vi) Assume $(t,a)=(6,1)$. Fix a general $Y\in Z(3,3,0)$. We obviously have $h^i(\mathcal{I}_{Y\cup S_1}(2))=0$, $i=0,1$. Since $h^1(Q,\mathcal{I}_{2O_1\cap Q}(4,1))=0$, we obtain $h^1(Q,\mathcal{I}_{(2O\cup Y\cup E_3)\cap Q}(4))=0$. Hence $h^1(\mathcal{I}_{Y\cup E_3\cup 2O_1}(4))=0$.

(vii) Assume $(t,a)=(1,8)$. Fix a general line $Y\subset\mathbb{P}^3$. We have $h^1(\mathcal{I}_Y(2))=$ 0 and $h^0(\mathcal{I}_Y)=0$. Hence $h^0(\mathcal{I}_{Y\cup S_8}(2))=0$, $i=0,1$ (Lemma 4). We have $h^1(Q,\mathcal{I}_{A_8\cap Q}(4))=0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]) and hence $h^0(Q,\mathcal{I}_{(Y\cup A_8)\cap Q}(4))=0$. Hence $h^0(\mathcal{I}_{Y\cup A_8}(4))=0$.

(viii) Assume $(t,a)=(2,7)$. Fix a general $Y\in Z(3,1,3)$. Lemma 1 gives $h^1(\mathcal{I}_Y(3))=0$. Since a double plane contains only a two-dimensional family of lines, we have $h^0(\mathcal{I}_Y(2))=0$. Hence $h^1(\mathcal{I}_{Y\cup S}(3))=0$ for a general $S\subset H$ with $\sharp(S)=4$ (Lemma 4). Set $A:=\cup_{O\in S}2O$. We have $h^0(H,\mathcal{I}_{A\cap H}(3))=0$. Hence $h^0(H,\mathcal{I}_{L\cup(A\cap H)}(4))=0$ for a general line $L\subset H$. Therefore $h^0(\mathcal{I}_{Y\cup L\cup A}(4))=0$.

(ix) Assume $(t,a)=(4,4)$. Let $Y\subset\mathbb{P}^3$ be a general union of two lines. We have $h^1(\mathcal{I}_Y(2))=0$ and $h^0(\mathcal{I}_Y)=0$ and hence $h^i(\mathcal{I}_{Y\cup S_4}(2))=0$, $i=0,1$. We have $h^1(Q, \mathcal{I}_{Q\cap A_4}(4,2)) = 0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]) and hence we get $h^0(Q, \mathcal{I}_{Q\cap(Y\cup A_4)}(4,2)) = 0$, i.e. $h^0(Q, \mathcal{I}_{E_2\cup(Y\cup A)_4\cap Q}(4)) = 0$. Hence $h^0(\mathcal{I}_{Y\cup E_2\cup A_4}(4)) = 0$.

(x) Assume $(t,a) = (5,3)$. Fix a general $Y \in Z(3,3,0)$. Notice that $h^0(\mathcal{I}_{Y\cup S_3}(2)) = 0$. Since $h^1(Q, \mathcal{I}_{A_3\cap Q}(4,2)) = 0$, then $h^0(Q, \mathcal{I}_{(Y\cup A_3)\cap Q}(4,2)) = 0$. Hence $h^0(Q, \mathcal{I}_{E_2\cup(Y\cup A_3)\cap Q}(4)) = 0$. Hence $h^0(\mathcal{I}_{Y\cup E_2\cup A_3}(4)) = 0$.

(xi) Assume $(t,a) = (6,2)$. Fix a general $Y \in Z(3,3,0)$. Notice that $h^0(\mathcal{I}_{Y\cup S_2}(2)) = 0$. We have $h^1(Q, \mathcal{I}_{A_2\cap Q}(4,1)) = 0$. Hence $h^0(Q, \mathcal{I}_{(Y\cup A_3)\cap Q}(4,1)) = 0$, i.e. $h^0(Q, \mathcal{I}_{E_3\cup(Y\cup A_3)\cap Q}(4)) = 0$. Hence $h^0(\mathcal{I}_{Y\cup E_3\cup A_2}(4)) = 0$. $\boxed{QED}$

Proposition 3. *Fix a general $X \in Z(3,t,a)$.*

(a) If $(t,a) \in \{(1,12),(2,11),(3,9),(4,8),(5,6),(6,5),(7,3),(8,2),(9,0)\}$, then $h^1(\mathcal{I}_X(5)) = 0$ and $h^0(\mathcal{I}_X(5)) = 56 - 6t - 4a$.

(b) If $(t,a) \in \{(1,13),(3,10),(5,7),(7,4),(9,1)\}$, then $h^0(\mathcal{I}_X(5)) = 0$.

Proof. The case $(t,a) = (9,0)$ is true by [18]. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Let $S_x \subset Q$ be a general union of x points and let $E_y \subset Q$ be a general union of y lines of type $(0,1)$. Set $A_x := \cup_{O\in S_x} 2O$. Let $H \subset \mathbb{P}^3$ be a plane.

(i) Assume $(t,a) = (e, 13-e)$ with $e = 1,2$. Fix a general $Y \in Z(3,e,4)$ and a general $S \sqcup S' \subset H$ such that $\sharp(S) = 6$, $\sharp(S') = 3-e$ and $S \cap S' = \emptyset$. Set $A := \cup_{O\in S} 2O$, $B := \cup_{O\in S'} 2O$ and $B' := B \cap H$. Since $h^1(H, \mathcal{I}_{A\cap H}(5)) = 0$ by the Alexander-Hirschowitz theorem, we have $h^i(\mathcal{I}_{S'\cup((Y\cup A)\cap H)}(5)) = 0$, $i = 0,1$. By the Differential Horace Lemma (Lemma 2) to prove that a general union W of $Y \cup A$ and $3-e$ 2-points satisfies $h^1(\mathcal{I}_W(5)) = 0$ it is sufficient to prove $h^1(\mathcal{I}_{Y\cup S\cup B'}(4)) = 0$. We have $h^0(\mathcal{I}_Y(3)) = 0$ by Lemma 1. Therefore Lemma 4 shows that it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup B'}(4)) = 0$. We have $h^1(H, \mathcal{I}_{B'\cup(Y\cap H)}(4)) = 0$, because $\sharp(B'_{red}) \le 3$ and $Y \cap H$ is general union of e points of H. Since $h^1(\mathcal{I}_Y(3)) = 0$, the Castelnuovo's sequence gives $h^1(\mathcal{I}_{Y\cup B'}(4)) = 0$.

(ii) Assume $(t,a) = (3,9)$. Let $S \subset H$ be a general subset with $\sharp(S) = 6$. Set $A := \cup_{O\in S} 2O$. Fix a general $Y \in Z(3,3,3)$. We have $h^1(\mathcal{I}_Y(4)) = 0$, e.g., by the case $(t,a) = (3,3)$ of Proposition 2. Since $h^0(\mathcal{I}_Y(3)) = 0$ by the case $(t,a) = (3,2)$ of Proposition 1, Lemma 4 gives $h^1(\mathcal{I}_{Y\cup S}(4)) = 0$. We have $h^1(H, \mathcal{I}_{A\cap H}(5)) = 0$ by the Alexander-Hirschowitz theorem and so $h^i(H, \mathcal{I}_{(Y\cup A)\cap H}(5)) = 0$, $i = 0,1$. The Castelnuovo's sequence gives $h^1(\mathcal{I}_{Y\cup A}(5)) = 0$.

(iii) Assume $(t,a) = (4,8)$. Fix a general $S \subset H$ such that $\sharp(S) = 4$ and a general line $L \subset H$. Fix a general $Y \in Z(3,3,4)$. We have $h^1(\mathcal{I}_Y(4)) = 0$ (case $(t,a) = (3,5)$ of Proposition 3) and $h^0(\mathcal{I}_Y(3)) = 0$ (Proposition 1). Hence $h^i(\mathcal{I}_{Y\cup S}(4)) = 0$, $i = 0,1$. We have $h^1(H, \mathcal{I}_{A\cap H}(4)) = 0$ by the Alexander-

Hirschowitz theorem. Hence $h^i(H, \mathcal{I}_{L\cup((Y\cup A)\cap H)}(5)) = 0$. The Castelnuovo's sequence gives $h^i(\mathcal{I}_{Y\cup L\cup A}(5)) = 0$, $i = 0, 1$.

(iv) Assume $(t, a) = (5, 6)$. Fix a general $Y \in Z(3, 2, 2)$. We have $h^1(\mathcal{I}_Y(3)) = 0$ (Proposition 1) and $h^0(\mathcal{I}_Y(1)) = 0$. Hence $h^1(\mathcal{I}_{Y\cup S_4}(3)) = 0$ (Lemma 4). Since $(Y \cup A_4) \cap Q$ is a general union of four 2-points of Q and 4 points, we have $h^1(Q, \mathcal{I}_{(Y\cup A_4)\cap Q}(5, 2)) = 0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). Hence $h^1(Q, \mathcal{I}_{(Y\cup A_4\cup E_3)\cap Q}(5)) = 0$. Hence $h^1(\mathcal{I}_{Y\cup A_4\cup E_3}(5)) = 0$.

(v) Assume $(t, a) = (6, 5)$. Fix a general $Y \in Z(3, 3, 1)$. We have $h^1(\mathcal{I}_Y(3)) = 0$ (Lemma 1) and $h^0(\mathcal{I}_Y(1)) = 0$. Hence $h^i(\mathcal{I}_{Y\cup S_4}(3)) = 0$, $i = 0, 1$ (Lemma 4). Since $(Y \cup A_4) \cap Q$ is a general union of four 2-points of Q and 6 points, we have $h^i(Q, \mathcal{I}_{(Y\cup A_4)\cap Q}(5, 2)) = 0$, $i = 0, 1$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). Hence $h^i(Q, \mathcal{I}_{(Y\cup A_4\cup E_3)\cap Q}(5)) = 0$, $i = 0, 1$. Hence $h^i(\mathcal{I}_{Y\cup A_4\cup E_3}(5)) = 0$, $i = 0, 1$.

(vi) Assume $(t, a) = (7, 3)$. Fix a general $Y \in Z(3, 4, 1)$. We have $h^1(\mathcal{I}_Y(3)) = 0$ (Proposition 1) and $h^1((Q, \mathcal{I}_{(Y\cup A_3)\cap Q}(5, 2)) = 0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). We get $h^1(\mathcal{I}_{Y\cup E_3\cup A_3}(5)) = 0$ and so $h^0(\mathcal{I}_{Y\cup E_3\cup A_3}(5)) = 2$.

(vii) Assume $(t, a) = (8, 2)$. Fix a general $Y \in Z(3, 3, 2)$. We have $h^i(\mathcal{I}_Y(3)) = 0$, $i = 0, 1$ (Proposition 1). Since $Y \cap Q$ is a general union of 6 points, we have $h^i(\mathcal{I}_{(Q\cap Y)\cup E_5}(5)) = 0$, $i = 0, 1$. Therefore $h^i(\mathcal{I}_{Y\cup E_5}(5)) = 0$.

(viii) Assume $(t, a) = (1, 13)$. Fix a general $Y \in Z(3, 1, 1)$. Since $h^1(\mathcal{I}_Y(3)) = 0$ and $h^0(\mathcal{I}_Y(1)) = 0$, we have $h^i(\mathcal{I}_{Y\cup S_{12}}(3)) = 0$, $i = 0, 1$ (Lemma 4). Since $h^i(Q, \mathcal{I}_{Q\cap A_{12}}(5)) = 0$, $i = 0, 1$, ([20, Propositions 4.1 and 5.2 and Theorem 7.2]), we have $h^0(Q, \mathcal{I}_{Q\cap(Y\cup A_{12})}(5)) = 0$. Therefore $h^0(\mathcal{I}_{Y\cup A_{12}}(5)) = 0$.

(ix) Assume $(t, a) = (3, 10)$. Fix a general $Y \in Z(3, 3, 0)$. Since $h^1(\mathcal{I}_Y(3)) = 0$ and $h^0(\mathcal{I}_Y(1)) = 0$, we have $h^0(\mathcal{I}_{Y\cup S_{10}}(3)) = 0$. Since $h^1(Q, \mathcal{I}_{A_{10}\cap Q}(5)) = 0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]), we have $h^i(Q, \mathcal{I}_{Q\cap(Y\cup A_{10})}(5)) = 0$, $i = 0, 1$. Hence $h^0(\mathcal{I}_{Y\cup A_{10}}(5)) = 0$.

(x) Assume $(t, a) = (5, 7)$. Fix a general $Y \in Z(3, 3, 1)$. Since $h^1(\mathcal{I}_Y(3)) = 0$ (Proposition 3) (i.e. $h^0(\mathcal{I}_Y(3)) = 4$) and $h^0(\mathcal{I}_Y(1)) = 0$, we have $h^0(\mathcal{I}_{Y\cup S_6}(3)) = 0$ (Lemma 4). Since $h^1(Q, \mathcal{I}_{Q\cap A_6}(5, 3)) = 0$ ([20]), we have $h^i(\mathcal{I}_{Q\cap(Y\cup A_6)}(5, 3)) = 0$, $i = 0, 1$. Therefore $h^i(\mathcal{I}_{Q\cap\cup E_2}(5)) = 0$. Hence $h^0(\mathcal{I}_{Y\cup E_2\cup A_6}(5)) = 0$.

(xi) Assume $(t, a) = (7, 4)$. Fix a general $Y \in Z(3, 3, 2)$. Since $h^i(\mathcal{I}_Y(3)) = 0$, $i = 0, 1$ (Proposition 3), we have $h^0(\mathcal{I}_{Y\cup S_2}(3)) = 0$. Since $h^1(Q, \mathcal{I}_{A_2\cap Q}(5, 1)) = 0$, we have $h^i(\mathcal{I}_{Q\cap(Y\cup A_2)}(3)) = 0$. Therefore $h^0(Q, \mathcal{I}_{E_4\cup(Y\cup A_2)}(5)) = 0$ and so $h^0(\mathcal{I}_{Y\cup E_2\cup A_2}(5)) = 0$.

(xii) Assume $(t, a) = (9, 1)$. Fix a general $Y \in Z(3, 5, 0)$. We have $h^i(\mathcal{I}_Y(3)) = 0$, $i = 0, 1$. Since $h^1(Q, \mathcal{I}_{A_1\cap Q}(5, 1)) = 0$, we have $h^0(Q, \mathcal{I}_{E_4\cup(Q\cap(Y\cup A_1))}(5)) = 0$ and so $h^0(\mathcal{I}_{Y\cup E_4\cup A_1}(5)) = 0$. $\boxed{QED}$

Lemma 7. *Fix a line $L \subset Q$ and take a general union $S' \subset L$ of 4 points of L. Fix a general $S \subset Q$ with $\sharp(S) = 4$. Take a general $W \in Z(3,3,3)$. Then $h^1(\mathcal{I}_{W\cup S\cup S'}(4)) = 0$.*

Proof. Since $h^0(\mathcal{I}_W(2)) = 0$ and S is general in Q, it is sufficient to prove $h^1(\mathcal{I}_{W\cup S'}(4)) = 0$ (Lemma 4). It is sufficient to prove that $h^1(\mathcal{I}_{W\cup L}(4)) = 0$. This is the case $(t,a) = (4,3)$ of Proposition 2. $\boxed{QED}$

Lemma 8. *Let (t,a) be one of the following pairs: $(9,5)$, $(8,7)$, (w,z) ($w \geq 9, z \geq 6$), $(10,4)$. Fix a general $X \in Z(3,t,a)$. Then either $h^1(\mathcal{I}_X(6)) = 0$ or $h^0(\mathcal{I}_X(6)) = 0$.*

Proof. We have $\binom{9}{3} = 84$. In all cases we will take integers $x \geq 0$ and y such that $2t+5x+3y$ is near to 49, $y \leq a$ and y is as large as possible. In all cases we will have $x \leq t$. Let $Y \subset \mathbb{P}^3$ be a general union of $t-x$ lines and $a-y$ 2-points. Fix a general $S \subset \mathbb{P}^3$ such that $\sharp(S) = y$ and set $A := \cup_{O\in S} 2O$. Let $E \subset Q$ be a general union of x lines of type $(0,1)$. We always use $Y \cup E \cup A \in Z(3,t,a)$.

First assume $(t,a) = (9,5)$. Take $(x,y) = (5,2)$. We have $h^1(Q, \mathcal{I}_{A\cap Q}(6,1)) = 0$ and hence $h^i(Q, \mathcal{I}_{(Y\cup E\cup A)\cap Q}(6)) = 0$. We have $h^0(\mathcal{I}_Y(2)) = 0$ (obvious) and $h^1(\mathcal{I}_Y(4)) = 0$ (Proposition 2). Hence $h^1(\mathcal{I}_{Y\cup S}(4)) = 0$ (Lemma 4). Hence $h^1(\mathcal{I}_{Y\cup E\cup A}(6)) = 0$.

Now assume $(t,a) = (w,z)$, with $z \geq 6$ and $w \geq 9$. Take a general $M \in Z(3,9,5)$. We proved that $h^0(\mathcal{I}_M(6)) = 1$. Hence $h^0(\mathcal{I}_{M\cup B}(6)) = 0$ if B contains either a general line or a general 2-point.

Now assume $(t,a) = (10,4)$. We take $(x,y) = (4,3)$. We have $h^1(\mathcal{I}_{A\cap Q}(6,2)) = 0$ and hence $h^i(Q, \mathcal{I}_{(Y\cup E\cup A)\cap Q}(6)) = 0$, $i = 0,1$. We have $h^0(\mathcal{I}_Y(2)) = 0$ (obvious) and $h^1(\mathcal{I}_Y(4)) = 0$ (Lemma 2). Hence $h^i(\mathcal{I}_{Y\cup S}(4)) = 0$, $i = 0,1$ (Lemma 4). Hence $h^i(\mathcal{I}_{Y\cup E\cup A}(6)) = 0$, $i = 0,1$.

Now assume $(t,a) = (8,7)$. Let $L \subset Q$ be a general line of type $(1,0)$ and $F \subset Q$ a general union of 4 lines of type $(0,1)$. Set $S' := F\cap L$ and $\Gamma := \cup_{O\in S'} 2O$. Notice that $E \cup L \cup \Gamma$ is a flat limit of a family of disjoint unions of 5 lines of $\mathbb{P}^3$. Let $W \subset \mathbb{P}^3$ be a general union of 3 lines and 3 2-points. Take $y = 4$. Set $M := W \cup L \cup F \cup \Gamma \cup A$. We have $M \in Z(3,8,7)'$ and $\mathrm{Res}_Q(M) = W \cup S \cup S'$. We have $h^0(\mathcal{I}_W(2)) = 0$ (obvious) and $h^1(\mathcal{I}_W(4)) = 0$ (Lemma 2). We have $h^i(\mathcal{I}_{W\cup S\cup S'}(4)) = 0$ (Lemma 7). Hence $h^i(\mathcal{I}_M(6)) = 0$, $i = 0,1$. $\boxed{QED}$

Proposition 4. *Fix integers $a \geq 0$, $t \geq 0$ and $k \geq 3$ such that $(k+1)t + 4a + 3k \leq \binom{k+3}{3}$. Let $X \subset \mathbb{P}^3$ be a general union of t lines and a 2-points of $\mathbb{P}^3$. Then $h^1(\mathcal{I}_X(k)) = 0$.*

Proof. Increasing if necessary a we may assume that $(k+1)t+4a+3k \geq \binom{k+3}{3}-3$. By [18] we may assume $a > 0$. By the Alexander-Hirschowitz theorem we may assume $t > 0$ at least if $k \geq 4$. Fix a smooth quadric surface $Q \subset \mathbb{P}^3$. The cases with $k = 3, 4, 5$ are true by Propositions 1, 2 and 3. Hence we may assume $k \geq 6$.

(a) Assume $k = 6$. We have $61 \leq 7t + 4a \leq 66$. Hence

$$(t, a) \in \{(1, 14), (2, 13), (3, 11), (4, 9), (5, 7), (6, 6), (7, 4), (8, 2)\}.$$

First assume $a \leq 8$ and $t \leq 7$. We specialize X to $X' = Y \sqcup Z$ with Y the union of t distinct lines of type $(1, 0)$ on Q. Since $t \leq 7$, we have $h^1(Q, \mathcal{I}_Y(6)) = 0$. Since $a \leq 8$, we have $h^1(\mathcal{I}_Z(4)) = 0$ by the Alexander-Hirschowitz theorem. Hence $h^1(\mathcal{I}_{X'}(6)) = 0$ by the Castelnuovo's sequence. Now assume $(t, a) = (8, 2)$. We specialize only 7 lines inside Q. We only need to use that $h^1(\mathcal{I}_{L \cup Z}(4)) = 0$ if L is a line and Z is a general union of two 2-points. Now assume $a \geq 9$. We have $2t + 3a \leq 49$. We specialize X to $Y \sqcup Z$ with Y a a general union of t lines and Z a general union of 9 2-points with support on Q. We have $h^1(Q, \mathcal{I}_{Q \cap Z}(6)) = 0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). Since $2t + 3a \leq h^0(Q, \mathcal{O}_Q(6))$ and $Y \cap Q$ is a general union of $2t$ points, we get $h^1(Q, \mathcal{I}_{(Y \cup Z) \cap Q}(6)) = 0$. By the Castelnuovo's sequence it is sufficient to prove $h^1(\mathcal{I}_{Y \cup S}(4)) = 0$, where S is a general union of a points of Q. Hence it is sufficient to prove that $h^1(\mathcal{I}_Y(4)) = 0$ and that $h^0(\mathcal{I}_Y(2)) \leq h^0(\mathcal{I}_Y(4)) - a$. This is true, because Y has maximal rank by [18].

(b) Now assume $k \geq 6$ and that the result is true for the integers $k' = k - 2$. Recall that $(k + 1)t + 4a + 3k \geq \binom{k+3}{3} - 3$. Set $u := \lfloor ((k + 1)^2 - 2t)/3 \rfloor$. Since $t \leq (k + 3)(k + 2)/6$ and $k \geq 7$, we have $u \geq 4$. Assume for the moment $a \geq u$. We specialize X to $X' = Y \sqcup Z' \sqcup Z$ with $Y \cup Z'$ a general union of t lines and $a - u$ 2-points and Z a general union of u 2-points of Q with support in Q. We have $h^1(Q, \mathcal{I}_{Z \cap Q}(k)) = 0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). Since $Y \cap Q$ is the union of $2t$ general points of Q and $2t + 3u \leq (k + 1)^2$, we have $h^1(Q, \mathcal{I}_{Q \cap X'}(k)) = 0$. We have $\mathrm{Res}_Q(X') = Y \cup Z' \cup S$, where $S := Z_{red}$ is a general union of u points of Q. By the Castelnuovo's sequence it is sufficient to prove that $h^1(\mathcal{I}_{Y \cup Z' \cup S}(k - 2)) = 0$. Since $(k + 1)t + 4a + 3k \leq \binom{k+3}{3}$ and $2t + 3u \geq (k + 1)^2 - 2$, we have $(k - 1)t + 4(a - u) + u + 3k - 2 \leq \binom{k+1}{3}$. Since $u \geq 4$, we get $(k - 1)t + 4(a - u) + 3(k - 2) \leq \binom{k+1}{3}$. Hence the inductive assumption gives $h^1(\mathcal{I}_{Y \cup Z'}(k - 2)) = 0$. Assume $h^1(\mathcal{I}_{Y \cup Z' \cup S}(k - 2)) > 0$. Let $u' \leq u$ be the first integer such that $h^1(\mathcal{I}_{Y \cup Z \cup S'}(k - 2)) > 0$ for some $S' \subseteq S$ with $\sharp(S') = u'$. Fix $P \in S'$ and set $S'' = S' \setminus \{P\}$. By the minimality of the integer u' we have $h^1(\mathcal{I}_{Y \cup Z' \cup S''}(k - 2)) = 0$ and $h^0(\mathcal{I}_{Y \cup Z' \cup S''}(k - 2)) = h^0(\mathcal{I}_{Y \cup Z' \cup S'}(k - 2))$. Since P is a general point of Q, we get that $|\mathcal{I}_{Y \cup Z' \cup S''}(k - 2)|$ has Q in its base locus. Since no irreducible component of $(Y \cup Z')_{red}$ is contained in Q, we get $h^0(\mathcal{I}_{Y \cup Z' \cup S''}(k - 2)) = h^0(\mathcal{I}_{Y \cup Z'}(k - 4))$. Since $u' \leq u$, we get $h^0(\mathcal{I}_{Y \cup Z'}(k - 2)) - u + 1 \leq h^0(\mathcal{I}_{Y \cup Z'}(k - 4))$. Since $h^1(\mathcal{I}_{Y \cup Z'}(k - 2)) = 0$, we have

$h^0(\mathcal{I}_{Y\cup Z'}(k-2)) = \binom{k+1}{3}-(k-1)t-4(a-u)$. Since $(k-1)t+4(a-u)+u+3k-2 \le \binom{k+1}{3}$, we get $h^0(\mathcal{I}_{Y\cup Z'}(k-4)) \ge 3k-2$. Since $(k+1)t+4a+3k \ge \binom{k+3}{3}-3$ and $(k+1)^2-2 \le 2t+3u \le (k+1)^2$, we get $h^1(\mathcal{I}_{Y\cup Z'}(k-4)) > 0$. Since Y has maximal rank by [18], we get $Z' \ne \emptyset$ and $(k-3)t \le \binom{k-1}{3}-3k+2$. Since $k-4 \ge 3$, we may use the inductive assumption for the integer $k' := k-4$. Let v be the maximal integer such that $v \le a-u$ and $(k-3)t+4v+3(k-4) \le \binom{k-1}{3}$. Assume for the moment $v \ge 0$ and take a union Z'' of v connected components of Z'. The inductive assumption gives $h^1(\mathcal{I}_{Y\cup Z''}(k-4)) = 0$. Since $h^1(\mathcal{I}_{Y\cup Z'}(k-4)) > 0$ we get $v < a-u$. Hence the maximality property of the integer v gives $(k-3)t+4v+3(k-4) \ge \binom{k-1}{3}-3$. Since $h^1(\mathcal{I}_{Y\cup Z''}(k-4)) = 0$, we get $h^0((\mathcal{I}_{Y\cup Z''}(k-4)) \le 3(k-4)+3 < 3k-2$. Since $Z' \supset Z''$ and $h^0(\mathcal{I}_{Y\cup Z'}(k-4)) \ge 3k-2$, we get a contradiction. If $v < 0$ we get the same contradiction taking Y instead of $Y \cup Z''$. $\boxed{QED}$

Proposition 5. *Fix integers $a \ge 0$, $t \ge 0$, $k \ge 3$ and e such that $0 \le e \le k$, and $(2k+1)e+(k+1)t+4a+5k \le \binom{k+3}{3}$. Let $X \subset \mathbb{P}^3$ be a general union of t lines, e reducible conics and a 2-points of $\mathbb{P}^3$. Then $h^1(\mathcal{I}_X(k)) = 0$.*

Proof. If $e = 0$, then we may apply Proposition 4. Hence we may assume $e > 0$. In particular we get $k \ne 3$. Since $(2k+2)e+(k+1)t \le \binom{k+3}{3}$, we may assume $a > 0$ by part (a) of Lemma 3. Increasing if necessary a we may assume $\binom{k+3}{3}-3 \le (2k+1)e+(k+1)t+4a+5k \le \binom{k+3}{3}$. Assume $k = 4$. Hence $e = a = 1$ and $t = 0$. This case is obvious. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface.

(a) Assume $k = 5$. We have $28 \le 11e+6t+4a \le 31$. The triples (t,e,a) are the following ones $(0,2,2)$, $(2,1,2)$, $(1,1,3)$. Fix a plane $H \subset \mathbb{P}^3$ and a reducible conic $T \subset H$. First assume $(t,e,a) = (0,2,2)$. Let $S \subset H$ be a general subset with $\sharp(S) = 2$. Set $A := \cup_{O\in S} 2O$. Let $Y \subset \mathbb{P}^3$ be a general reducible conic. Since $h^1(\mathcal{I}_Y(x)) = 0$, $x = 3,4$, we have $h^1(\mathcal{I}_{Y\cup S}(4)) = 0$ by Lemma 4. We have $h^1(H, \mathcal{I}_{(Y\cup T\cup A)\cap H}(5)) = 0$, because $h^1(H, \mathcal{I}_{(T\cup A)\cap H}(3)) = 0$. Hence $h^1(\mathcal{I}_{Y\cup A}(5)) = 0$. Now assume $(t,e,a) = (2,1,2)$. Let $Y_1 \subset \mathbb{P}^2$ be a general union of two lines. Since $h^1(\mathcal{I}_{Y_1\cup S}(4)) = 0$, we get $h^1(\mathcal{I}_{Y_1\cup A}(5)) = 0$. Now assume $(t,e,a) = (1,1,3)$. Let $Y' \subset \mathbb{P}^3$ be a general union of a line and a 2-point. Since $h^1(\mathcal{I}_{Y'}(x)) = 0$, $x = 3,4$ (e.g., by Proposition 4)), we first get $h^1(\mathcal{I}_{Y'\cup S}(4)) = 0$ and then $h^1(\mathcal{I}_{Y'\cup A}(5)) = 0$.

(b) Assume $k = 6$. We have $e > 0$ and $53 \le 13e+7t+4a \le 56$. Hence $e \le 4$. Let $E \subset Q$ be a union of e distinct lines of type $(0,1)$. For each integer $x \ge 0$ let $E_x \subset Q$ be a general union of x lines of type $(0,1)$. First assume $e+t \ge 5$. Fix a general $Y \in Z(3, e+t-5, a)$ and write $Y = Y' \cup Y''$ with $Y'' \in Z(3, 2e+t-5, a)$ and $Y' \in Z(3, e, 0)$. Since $Y \cap Q$ is a general union of $4e+2t-10 \ge e$ points of Q we may find E with $Y' \cup E$ a disjoint union of e

reducible conics, $E \cap Y'' = \emptyset$. We have $h^1(\mathcal{I}_Y(4)) = 0$ by Proposition 2, because $5(e+t-5)+4a \le 56-35-2(e+t-5)$. We claim that $h^1(\mathcal{I}_{Y\cup E\cup E_{5-e}}(6)) = 0$. The claim would prove Proposition 5 in this case. Since $h^1(\mathcal{I}_Y(4)) = 0$, it is sufficient to prove $h^1(Q, \mathcal{I}_{E\cup E_{5-e}\cup(Y\cap Q)}(6)) = 0$. Since each line of Y' meets E, $Y \cap (Q \setminus E)$ is the union, B, of $e+2(t-5+e) = 3e+2t-10$ general points of Q and some points of E. To prove the claim it is sufficient to prove that $h^1(Q, \mathcal{I}_B(6,1)) = 0$. Since B is general, it is sufficient to have $\sharp(B) \le 14$, which is true. Now assume $e+t \le 4$. Set $b := \min\{a, 6\}$ if $e+t = 4$, $b := 6$ if $(e,t) = (3,0)$ and $b := \min\{a, 9\}$ in all other cases. Fix a general $Y' \in Z(3, e, 0)$, a general $U \in Z(3, 0, a-b)$ and a general $S \subset Q$ such that $\sharp(S) = b$. Set $A := \cup_{O\in S} 2O$. We may assume that each line of E contains a point of $Y' \cap Q$ so that $Y' \cup E$ is a disjoint union of e reducible conics. The set $Y' \cap (Q \setminus E)$ is a general union of e points of Q. To prove Proposition 5 in these cases it is sufficient to prove that $h^1(\mathcal{I}_{Y'\cup U\cup A\cup E_t}(6)) = 0$. We have $h^1(\mathcal{I}_{Y'\cup U}(4)) = 0$ (Proposition 2) and $h^0(\mathcal{I}_{Y'\cup A}(4)) \ge h^0(\mathcal{I}_{Y'\cup U}(2)) + b$. Hence $h^1(\mathcal{I}_{Y\cup U\cup S}(4)) = 0$. Therefore it is sufficient to prove that $h^1(Q, \mathcal{I}_{Y'\cap(Q\setminus E)\cup(A\cap Q)}(6, 6-e-t)) = 0$. We have $h^1(Q, \mathcal{I}_{A\cap Q}(6, 6-e-t))$, because $3b \le 7(7-e-t)$ and, if $e+t = 4$, then $\sharp(S) \le 6$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). Therefore it is sufficient to check that $e+3b \le 7(7-e-t)$, i.e. $8e+7t+3b \le 49$. This is true, because $13e+7t+4a \le 56$, $e > 0$, $b \le a$ and $e+t \le 4$.

(c) From now on we assume $k \ge 7$ and that Proposition 5 is true for the integers $k-2$ and $k-4$. Increasing if necessary a we reduce to the case

$$\binom{k+3}{3} - 5k - 3 \le (2k+1)e + (k+1)t + 4a \le \binom{k+3}{3} - 5k. \qquad (5)$$

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Set $\epsilon := \max\{0, e-k+2\}$. Hence $0 \le \epsilon \le 2$ and either $\epsilon = 0$ or $e - \epsilon = k-2$. Since $(2k+1)e + (k+1)t \le \binom{k+3}{3}$, we have $4(e-\epsilon) + 2t \le (k+1)(k+1-\epsilon)$. Let $X_1 \subset Q$ be a general union of of ϵ lines of type $(0,1)$. Set $b := \min\{\lfloor((k+1)(k+1-\epsilon) - 4(e-\epsilon) - 2t)/3\rfloor, a\}$. Since $k+1-\epsilon \ge 3$, we have $h^1(Q, \mathcal{I}_B(k+1, k+1-\epsilon)) = 0$ for a general union $B \subset Q$ of b 2-points of Q ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). Set $S := B_{red}$. The set S is a general union of b points of Q. Let $A \subset \mathbb{P}^3$ be the union of b 2-points with S as its support. Notice that $Q \cap A = B$ (as schemes) and $\mathrm{Res}_Q(A) = S$.

(c1) Here we assume $b = \lfloor((k+1)(k+1-\epsilon) - 4(e-\epsilon) - 2t)/3\rfloor$. Let $Y \subset \mathbb{P}^3$ be a general union of $a-b$ 2-points, $e-\epsilon$ reducible conics, t lines, ϵ lines, each of them intersecting one of the components of X_1, so that $Y \cup X_1 \cup A$ is a disjoint union of e reducible conics, t lines and a 2-points. Set $W := Y \cup X_1 \cup A$. By the semicontinuity theorem for cohomology it is sufficient to prove that $h^1(\mathcal{I}_W(k)) = 0$. We have $\mathrm{Res}_Q(W) = Y \cup S$. The scheme $W \cap Q$

is the disjoint union of X_1, B and $4(e-\epsilon)+2t$ general points of Q. Since $h^1(Q,\mathcal{I}_B(k+1,k+1-\epsilon))=0$ and $3b+4(e-\epsilon)+2t \le (k+1)(k+1-\epsilon)$, we have $h^1(Q,\mathcal{I}_{W\cap Q}(k))=0$. By the Castelnuovo's sequence it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup S}(k-2))=0$. Since X_1 is a general union of two lines of type $(0,1)$ of Q and every line of $\mathbb{P}^3$ meets Q, Y may be considered as a general union of $a-b$ 2-points, $t+\epsilon$ lines and $e-\epsilon$ reducible conics. We have $h^0(\mathcal{O}_{Y\cup S}(k-2)) = h^0(\mathcal{O}_W(k))-3b-4(e-\epsilon)-k\epsilon$, because for all $x\ge 0$ we have $h^0(\mathcal{O}_E(x))=2x+1$ if E is a reducible conic and $h^0(\mathcal{O}_E(x))=x+1$ if E is a line. Since $(k+1)(k+1-\epsilon)-4(e-\epsilon)-2t \ge (k+1)^2-2$ and $\binom{k+3}{3}-\binom{k+1}{3}=(k+1)^2$, we get $\binom{k+1}{3}-h^0(\mathcal{O}_{Y\cup S}(k-2)) \ge \binom{k+3}{3}-h^0(\mathcal{O}_W(k))-2 \ge 5k-2$. Since Y is a general union of $e-\epsilon$ reducible conics, $t+\epsilon$ lines and $a-b$ 2-points and $h^0(\mathcal{O}_Y(k-2))+5(k-2)\le\binom{k+1}{3}$, the inductive assumption gives $h^1(\mathcal{I}_Y(k-2))=0$. We also have $h^0(\mathcal{I}_Y(k-2))\ge \sharp(S)+5k-2$. Hence if either $h^0(\mathcal{I}_Y(k-4))=0$ or $h^0(\mathcal{I}_Y(k-2))\ge\sharp(S)+h^0(\mathcal{I}_Y(k-4))$, then we may apply Lemma 4. Assume $h^0(\mathcal{I}_Y(k-4))>0$. We have $h^0(\mathcal{O}_Y(k-2))-h^0(\mathcal{O}_Y(k-4))=4(e-\epsilon)+2(t+\epsilon)$, while $\binom{k+1}{3}-\binom{k-1}{3}=(k-1)^2$. However, we cannot claim that $h^1(\mathcal{I}_Y(k-4))=0$, because we cannot claim that $h^0(\mathcal{O}_Y(k-4))+5(k-4)\le\binom{k-1}{3}$. Assume for the moment $h^0(\mathcal{O}_Y(k-4))+5(k-4)>\binom{k-1}{3}$. Take $E\subset Y$ with E a union of all 2-points of Y, all degree 1 connected components of Y, some of the reducible conics of Y, at least one the components of each reducible conic of Y so that E is minimal with the property that $h^0(\mathcal{O}_E(k-4))\ge\binom{k-1}{3}-5(k-4)$. Set $x:=\deg(Y)-\deg(E)$. First assume $h^0(\mathcal{O}_E(k-4))=\binom{k-1}{3}-5(k-4)$. In this case the inductive assumption gives $h^1(\mathcal{I}_E(k-4))=0$. Hence $h^0(\mathcal{I}_E(k-4))=5(k-4)$. Each line contained in Y, but not contained in E may be considered as a general line intersecting one of the lines of E. Hence this line may contain a general point of $\mathbb{P}^3$. Hence $h^0(\mathcal{I}_Y(k-4))\le\max\{0,h^0(\mathcal{I}_E(k-4))-x\}=5(k-4)-x$. Since $h^0(\mathcal{I}_Y(k-2))\ge\sharp(S)+5k-2$, we get $h^1(\mathcal{I}_{Y\cup S}(k-2))=0$ in this case by Lemma 4. Now assume $h^0(\mathcal{O}_E(k-4))>\binom{k-1}{3}-5(k-4)$. The scheme $E\subset Y$ is minimal, either because it contains no reducible conic or because it contains at least one reducible conic, but $h^0(\mathcal{O}_E(k-4))\le\binom{k+1}{3}-5(k-4)+(k-5)$. Assume that E contains at least one conic, but $h^0(\mathcal{O}_E(k-4))\le\binom{k+1}{3}-5(k-4)+(k-5)$. Let $F\subset E$ be obtained from E taking only one component of one of the conics of E. We have $h^0(\mathcal{O}_F(k-4))=h^0(\mathcal{O}_E(k-4))-(k-4)<\binom{k+1}{3}-5(k-4)$ and hence we may apply the inductive assumption to F (with a separate analysis of the case $k=6$) and get $h^0(\mathcal{I}_Y(k-4))\le -x-1+h^0(\mathcal{I}_F(k-4))\le 5(k-4)+(k-5)-x-1$. Now assume that E is maximal, because it contains no reducible conic, i.e. Y is the union of E and $e-\epsilon$ suitable lines. If $h^0(\mathcal{O}_E(k-4))\le\binom{k-1}{3}-3(k-4)$, then Proposition 4 gives $h^1(\mathcal{I}_E(k-4))=0$. Now assume $h^0(\mathcal{O}_E(k-4))>\binom{k-1}{3}-3(k-4)$. Let $G\subset Y$ the union of all of the 2-points of E and of some of the lines of E with $\deg(G)$ maximal among all subschemes with $h^0(\mathcal{O}_G(k-4))\le$

$\binom{k-1}{3} - 3(k-4)$. The maximality of G gives $h^0(\mathcal{O}_G(k-4)) \geq \binom{k-1}{3} - 4(k-4)$. Proposition 4 gives $h^1(\mathcal{I}_G(k-4)) = 0$ and hence $h^0(\mathcal{I}_G(k-4)) \leq 4(k-4)$. Since $h^0(\mathcal{I}_Y(k-4)) \leq h^0(\mathcal{I}_G(k-4))$ and $h^0(\mathcal{I}_Y(k-2)) \geq \sharp(S) + 5k - 2$, we get $h^1(\mathcal{I}_{Y\cup S}(k-2)) = 0$ by Lemma 4.

(c2) Here we assume $b < \lfloor((k+1)(k+1-\epsilon) - 4(e-\epsilon) - 2t)/3\rfloor$. Hence $b = a$ in this case. Since $e \leq k$ and $3b \leq (k+1)^2 - 4e - 2t$, the first inequality in (5) gives $t \geq k-1$ (otherwise one has to adapt the definition of f below). Let f be maximal integer such that $0 \leq f \leq k-1-\epsilon$ and $4+2(t-f)+3b \leq (k+1)(k+1-\epsilon-f)$. Let $E \subset Q$ be a general union of $f+\epsilon$ lines of type $(0,1)$. In this step we take as Y a general union of e reducible conics and $t' := t - f - \epsilon$ lines. Set $W := Y \cup A \cup E$. Since $k - (f+\epsilon) > 0$ and $e > 0$, we have $h^1(Q, \mathcal{I}_{(Y\cap Q)\cup B\cup E}(k)) = 0$ ([20, Propositions 4.1 and 5.2 and Theorem 7.2]). Since $\mathrm{Res}_Q(W) = Y \cup S$, it is sufficient to prove $h^1(\mathcal{I}_{Y\cup S}(k-2)) = 0$. Y is a general union of t' lines and e conics. We first check that $(t'+e)(k-1) \leq \binom{k+3}{3}$. We have $\binom{k+3}{3} - h^0(\mathcal{O}_W(k)) = \binom{k+1}{3} - h^0(\mathcal{O}_{Y\cup S}(k-2)) + \delta$, where $\delta = h^0(Q, \mathcal{O}_Q(k)) - (k+1)(f+\epsilon) - 3b - 4e - 2(t-f-\epsilon)$. We have $h^0(\mathcal{O}_Q(k)) = (k+1)^2 = \binom{k+3}{3} - \binom{k+1}{3}$. The maximality of the integer f gives the inequality $\delta \leq 3k+2$. Hence $h^0(\mathcal{O}_{Y\cup S}(k-2)) \leq \binom{k+1}{3} - 5k + 3k + 2 \geq e$. Part (a) of Lemma 3 gives $h^1(\mathcal{I}_Y(k-2)) = 0$. Since $h^0(\mathcal{O}_{Y\cup S}(k-2)) \leq \binom{k+1}{3} - 5k + 3k + 2$, we have $h^0(\mathcal{I}_Y(k-2)) \geq \sharp(S) + 2k - 2$. By Lemma 4 to prove that $h^1(\mathcal{I}_{Y\cup S}(k-2)) = 0$ it is sufficient to prove the inequality $h^0(\mathcal{I}_Y(k-4)) \leq h^0(\mathcal{I}_Y(k-2)) - \sharp(S)$. Lemma 3 gives that either $h^0(\mathcal{I}_Y(k-4)) \leq e$ or $h^1(\mathcal{I}_Y(k-4)) \leq e$. Since $e \leq k$, $h^0(\mathcal{I}_Y(k-2)) \geq \sharp(S) + 2k - 2$, $\binom{k+1}{3} - \binom{k-1}{3} = (k-1)^2$ and $h^0(\mathcal{O}_Y(k-2)) - h^0(\mathcal{O}_Y(k-4)) = 2t' + 4e$, we are done. $\boxed{QED}$

4 $\mathbb{P}^4$

In this section we handle the case $r = 4$. Let $H \subset \mathbb{P}^4$ be a hyperplane.

Lemma 9. *Fix a hyperplane $H \subset \mathbb{P}^4$, $O \in H$, a general $Y \in Z(4,1,0)$ and a general $P \in \mathbb{P}^4$. Set $B' := 2O \cap H$. Then $h^1(\mathcal{I}_{Y\cup B'\cup 2P}(2)) = 0$.*

Proof. Let $\ell : \mathbb{P}^4 \setminus \{P\} \to \mathbb{P}^3$ denote the linear projection from P. For general P, Y, O, the scheme $\ell(Y \cup B')$ is a general element of $Z(3,1,1)$. Therefore we have $h^1(\mathbb{P}^3, \mathcal{I}_{\ell(Y\cup B')}(2)) = 0$, i.e. $h^0(\mathbb{P}^3, \mathcal{I}_{\ell(Y\cup B')}(2)) = 3$. Since $|\mathcal{I}_{Y\cup B'\cup 2P}(2)|$ is the set of all quadrics cones containing $Y \cup B'$ and with vertex containing P, we get $h^0(\mathcal{I}_{Y\cup B'\cup 2P}(2)) = 3$, i.e. $h^1(\mathcal{I}_{Y\cup B'\cup 2P}(2)) = 0$. $\boxed{QED}$

Lemma 10. *Fix integers $t \geq 0$, $a \geq 0$ such that $(t,a) \neq (0,7)$. Fix a general $X \in Z(4,t,a)$. Then either $h^0(\mathcal{I}_X(3)) = 0$ or $h^1(\mathcal{I}_X(3)) = 0$.*

Proof. All cases with $a = 0$ are true by [18]. Hence we may assume $a > 0$. Among the X's with $h^0(\mathcal{O}_X(3)) = 4t + 5a \le \binom{7}{4} = 35$ it is sufficient to check the pairs $(t, a) \in \{(1, 6), (5, 3)\}$. Among the pairs (t, a) with $4t + 5a > 35$ it is sufficient to check the pairs $(t, a) \in \{(0, 8), (1, 7)\}$. All cases with $t = 0$ are covered by the Alexander-Hirschowitz theorem, because we excluded the case $(t, a) = (0, 7)$.

We need to check all pairs (t, a) with $a > 0$, $t > 0$, and $32 \le 4t + 5a \le 38$. We may also exclude all cases with $4t + 5a \ge 37$ and $t > 0$, because any two general points of $\mathbb{P}^4$ are contained in a line and hence to prove the case (t, a) it is sufficient to check the case $(t', a') = (t-1, a)$. Hence (t, a) is one of the following pairs $(7, 1)$, $(6, 2)$, $(5, 3)$, $(4, 4)$, $(3, 4)$, $(2, 5)$. All cases with either $a \ge 9$ or $a = 8$ and $t > 0$ or $a = 7$ and $t \ne 1$ are obvious and the remaining ones are reduced (increasing or decreasing t, but with a fixed a) to a case with $32 \le 4t+5a \le 36$ or to the case $(t, a) = (2, 6)$.

Let $H \subset \mathbb{P}^4$ be a hyperplane. Let $E \subset H$ be a general union of f lines and let $S_x \subset H$ be a general union of x points. Set $A_x := \cup_{P \in S_x} 2O$.

(a) In this step we assume $a = 1$ and $t = 7$. Notice that $|30 - 4t| = 2$. Let $Y \subset \mathbb{P}^4$ be a general union of 4 lines. We have $h^1(\mathcal{I}_{Y \cup S_1}(2)) = 0$ and $h^0(\mathcal{I}_{Y \cup S_1}(2)) = 2$. Take $f = 3$. Proposition 1 gives $h^i(\mathcal{I}_{(Y \cap H) \cup E \cup (A_1 \cap H)}(3)) = 0$, $i = 0, 1$. Use $Y \cup E \cup A_1$.

(b) In this step we assume $a = 2$ and $t = 6$. Let Y be a general union of 4 lines. We have $h^1(\mathcal{I}_{Y \cup S_2}(2)) = 0$. Take $f = 2$. e $h^1(H, \mathcal{I}_{(Y \cap H) \cup E \cup (A_2 \cap H)}(3)) = 0$. Use $Y \cup E \cup A_2$ and the Castelnuovo's sequence.

(c) In this step we assume $a = 3$ and $t = 5$. Let $Y \subset \mathbb{P}^4$ be a general union of 4 lines. Take $f = 1$. Since $Y \cap H$ is general in H, we have $h^i(H, \mathcal{I}_{(Y \cap H) \cup E \cup (A_3 \cap H)}(3)) = 0$, $i = 0, 1$ (Proposition 1). We have $h^i(\mathcal{I}_{Y \cup S_3}(2)) = 0$, $i = 0, 1$, by [18] and the case $t = 1$ of Lemma 4. Use $Y \cup E \cup A_3$ and the Castelnuovo's sequence.

(d) Assume $a = 4$ and $t \in \{3, 4\}$. Take $f = 0$. Let $Y \subset \mathbb{P}^4$ be a union of t lines. Use $Y \cup A_4$. We have $h^1(H, \mathcal{I}_{(Y \cup A_4) \cap H}(3)) = 0$ and $h^0(\mathcal{I}_{(Y \cup A_4) \cap H}(3)) = t - 4$ (Proposition 1). We have $h^1(\mathcal{I}_{Y \cup S_4}(2)) = \max\{0, 11 - 3t\}$.

(e) Assume $a = 5$ and $t = 2$. Let $Y \subset \mathbb{P}^4$ be a general union of 2 lines and one 2-point. Use $Y \cup A_4$. We have $h^i(\mathcal{I}_{Y \cup S_4}(2)) = 0$, $i = 0, 1$ (Lemmas 1 and 4). We have $h^1(H, \mathcal{I}_{H \cap (Y \cup A_4)}(3)) = 0$ (Proposition 1).

(f) Assume $a = 6$ and $t = 1$. Fix $O \in S_5$ and set $S_4 := S_5 \setminus \{O\}$. Set $B' := 2O \cap H$. Fix a general $Y \in Z(4, 1, 1)$. We have $h^1(\mathcal{I}_{Y \cup B'}(2)) = 0$ (Lemma 9). We have $h^1(H, \mathcal{I}_{A_4 \cap H}(3)) = 0$ and hence $h^1(H, \mathcal{I}_{(Y \cup A_4) \cap H \cup \{O\}}(3)) = 0$. Since $h^0(\mathcal{I}_Y(1)) = 0$ and $h^0(\mathcal{O}_{Y \cup B'}(2)) = 16$, Lemma 4 gives $h^1(\mathcal{I}_{Y \cup B' \cup S_4}(2)) = 0$. The Differential Horace Lemma (Lemma 2) gives that a general union W of $Y \cup A_4$ and a 2-point satisfies $h^1(\mathcal{I}_W(3)) = 0$.

(g) Assume $a = 6$ and $t = 2$. Let $Y \subset \mathbb{P}^4$ a general union of two lines and one 2-point. We have $h^0(\mathcal{I}_{Y\cup S_5}(2)) = 0$ and $h^0(H, \mathcal{I}_{(Y\cup A_5)\cap H}(3)) = 0$ (even without the two points of $Y \cap H$). $\boxed{QED}$

Lemma 11. *Fix integers $a \ge 0$ and $t \ge 0$ with $(t, a) \ne (0, 14)$. Fix a general $X \in Z(4, t, a)$. Then either $h^0(\mathcal{I}_X(4)) = 0$ (case $a + t \ge 14$) or $h^1(\mathcal{I}_X(4)) = 0$ (case $a + t \le 14$).*

Proof. Since $\binom{8}{4} = 70$, it would sufficient to do the case $a + t = 14$, but the case $(t, a) = (0, 14)$ is an exceptional case in the Alexander-Hirschowitz list (it has $h^0(\mathcal{I}_X(4)) = h^1(\mathcal{I}_X(4)) = 1$ by [15], Proposition 2.1). By the Alexander-Hirschowitz theorem all cases $(0, a)$ with $a \ne 14$ are true. Hence it is sufficient to do the cases with $a = 14 - t$ and $1 \le t \le 14$. The case $a = 0$ is true by [18] and hence we may assume $t \le 13$. For all $(x, y) \in \mathbb{N}^2$ let $S_x \cup S'_y \subset H$ denote a general union of points with $\sharp(S_x) = x$, $\sharp(S'_y) = y$ and $S_x \cap S'_y = \emptyset$. Set $A_x := \cup_{O\in S_x} 2O$, $B_y := \cup_{O\in S'_y} 2O$ and $B'_y := B_y \cap H$. Let $E_x \subset H$, $x \ge 0$, be a general union of x lines.

(a) Assume $1 \le t \le 7$. Define the integers u, v by the relations $t+4u+v = 35$, $0 \le v \le 3$. The quadruples (t, a, u, v) are the following ones: $(1, 13, 8, 2)$, $(2, 12, 8, 1)$, $(3, 11, 8, 0)$, $(4, 10, 7, 3)$, $(5, 9, 7, 2)$, $(6, 8, 7, 1)$, $(7, 7, 7, 0)$. In all cases we have $a \ge u+v$ and $u \ge v$. Let $Y \subset \mathbb{P}^4$ be a general element of $Z(4, t, a-u-v)$. Since $v \le 3$ and any 3 points of $\mathbb{P}^4$ are contained in a hyperplane, $Y \cup B_v$ may be considered as a general element of $Z(4, t, a - u)$. In all cases we have $h^1(\mathcal{I}_Y(3)) = h^1(\mathcal{I}_{Y\cup B_v}(3)) = 0$ by Lemma 10 (when $t = 1$ we have $a - u - v = 3$ and $a - u = 5$). Lemma 5 gives $h^1(\mathcal{I}_{Y\cup B'_v}(3)) = 0$, i.e. $h^0(\mathcal{I}_{Y\cup B'_v}(3)) = u$. Since obviously $h^0(\mathcal{I}_Y(2)) = 0$, we get $h^i(\mathcal{I}_{Y\cup B'_v\cup S_u}(3)) = 0$, $i = 0, 1$. Since $a \le 8$, in all cases we have $h^1(H, \mathcal{I}_{A_u\cap H}(4)) = 0$ by the Alexander-Hirschowitz theorem. Hence $h^i(H, \mathcal{I}_{H\cap(Y\cup A_u)\cup S'_v}(4)) = 0$. The Differential Horace Lemma for double points (Lemma 2) gives that a general union X of $Y \cup A_u$ and v 2-points satisfies $h^i(\mathcal{I}_X(4)) = 0$, $i = 0, 1$.

(b) Assume $8 \le t \le 13$. Let f be the minimal integer such that $t + 4f + 4a \ge 35$. Set $f' := t + 4f + 4a - 35$. Notice that $0 \le f' \le 3$ and that $t - f - f' + 4a + 5f = 35$. The quadruple (t, a, f, f') are the following ones: $(8, 6, 1, 1)$, $(9, 5, 2, 0)$, $(10, 4, 3, 3)$, $(11, 3, 3, 0)$, $(12, 2, 4, 1)$, $(13, 1, 5, 2)$. In all cases we have $f \ge f'$ and $t \ge f + f'$. Fix $E_f \subset H$ and write $E_f = E_{f-f'} \sqcup E_{f'}$. Let $Y \subset \mathbb{P}^4$ be a general union of $t - f$ lines, with the only restriction that f' of the lines of Y meet H in a point of a different component of $E_{f'}$, so that $Y \cup E_f$ is a disjoint union $W \sqcup F$ of $W \in Z(4, t - 2f', 0)$ and a disjoint union F of f' reducible conics. Let $G \subset \mathbb{P}^4$ be general sundials with $F = G_{red}$. The scheme $\mathrm{Res}_H(G)$ is a general union of f' +lines of $\mathbb{P}^4$ with nilradical supported by a point of H. Proposition 2 gives $h^1(H, \mathcal{I}_{E_f\cup(A_a\cap H)}(3)) = 0$, i.e.

$h^0(H, \mathcal{I}_{E_f \cup (A_a \cap H)}(3)) = t - f - f'$. Hence $h^i(H, \mathcal{I}_{(W \cup G \cup A_a) \cap H}(4)) = 0$, $i = 0, 1$. Since $f' \leq 3$, Lemma 6 gives $h^1(\mathcal{I}_{W \cup \mathrm{Res}_H(G)}(3)) = 0$. Since $h^0(\mathcal{I}_Y(2)) = 0$, Lemma 4 gives $h^i(\mathcal{I}_{W \cup \mathrm{Res}_H(G) \cup S_a}(3)) = 0$, $i = 0, 1$. Hence $h^i(\mathcal{I}_{W \cup G \cup A_a}(4)) = 0$, $i = 0, 1$. Since $W \cup A_a \in Z(4, t - 2f', a)$ and G is a disjoint union of f' sundials, we are done. $\boxed{QED}$

Remark 3. Fix integers $t > 0$, $r \geq 4$ and $k \geq 4$ and $a \geq \lceil k/r \rceil + r - 1$. Let f be the minimal integer such that $t - f + (k+1)f + ru' + v' = \binom{r+k-1}{r-1}$ (i.e. such that $t + kf + ru' + v' = \binom{r+k-1}{r-1}$) for some integers u', v' with $u' \geq 0$, $0 \leq v' \leq r-1$ and $u' + v' \leq a$. There is such an integer, because $a \geq \lceil k/r \rceil + r - 1$ and hence among the integers $ru' + v'$ with $u' + v' \leq a$ we may realize all integers x with $0 \leq x < k$.

Claim : We have $a - u' \leq \lfloor k/r \rfloor + r - 1$.

Proof of the Claim: Write $k = e_1 r + e_2$ with $e_1 = \lfloor k/r \rfloor$ and $0 \leq e_2 \leq r-1$. Let u'', v'' be the only integers such that $0 \leq v'' \leq r-1$ and $k(f-1) + ru'' + v'' = kf + ru' + v'$, i.e.

$$-k + ru'' + v'' = ru' + v'. \tag{6}$$

Since $f - 1 < f$, the minimality of f gives that either $u'' < 0$ or $a < u'' + v''$. Since $u' \geq 0$ and $v'' \leq r - 1$, we have $u'' \geq 0$. Hence $a \leq u'' + v'' - 1$. Since $a \geq u' + v'$, we get $u'' + v'' \geq u' + v' + 1$. From (6) we get that either $v'' = e_2 + v'$ and $u'' = u' + e_1$ (case $v' + e_2 \leq r - 1$) or $v'' = e_2 + v' - r$ and $u'' = u' + e_1 + 1$ (case $v' + e_2 \geq r$). Since $a \leq u'' + v'' - 1$, $0 \leq v'' \leq r - 1$ and $0 \leq v' \leq r - 1$, we get $a - u' \leq e_1 + r - 2$ in the first case and $a - u' \leq e_1 + r - 1$ in the second case.

To conclude the proof of the case $r = 4$ of Theorem 1 it is sufficient to prove the following result.

Proposition 6. *Fix non-negative integers k, a, t such that $k \geq 4$. Let $X \subset \mathbb{P}^4$ be a general union of t lines and a 2-points. If $k = 4$, then assume $(t, a) \neq (0, 14)$. Then either $h^1(\mathcal{I}_X(k)) = 0$ (case $(k+1)t + 5a \leq \binom{k+4}{4}$) or $h^0(\mathcal{I}_X(k)) = 0$ (case $(k+1)t + 5a \geq \binom{k+4}{4}$).*

Proof. If $k = 4$, then the result is true (Lemma 11). We fix an integer $k \geq 5$ and we almost use induction on k, in the sense that we use the result for the integers $k' := k - 1$ and $k' = k - 2$, except that if $2 \leq k' \leq 3$, then we quote Lemma 1 and Proposition 1.

(a) In this step we assume $t(k+1) + 5a \leq \binom{k+4}{4}$. Increasing if necessary the integer a we may assume $t(k+1) + 5a \geq \binom{k+4}{4} - 4$. Set $u := \lfloor (\binom{k+3}{3} - t)/4 \rfloor$ and $v := \binom{k+3}{3} - t - 4u$. Notice that $0 \leq v \leq 3$.

Claim 1: $u + 4v \leq \binom{k+2}{3} - t$ if $k \geq 8$.

Proof of Claim 1: Assume $u+4v \geq \binom{k+2}{3} - t + 1$. Since $4u+v = \binom{k+3}{3} - t$, we get $15v \geq 4\binom{k+2}{3} - \binom{k+3}{3} - 3t + 4$. Since $v \leq 3$, we get

$$41 + 3t \geq (k+2)(k+1)(3k-3)/6. \tag{7}$$

Since $(k+1)t \leq (k+4)(k+3)(k+2)(k+1)/24$, we get $41+(k+4)(k+3)(k+2)/8 \geq (k+2)(k+1)(k-1)/2$. The last inequality is false if $k \geq 8$.

(a1) Here we assume $a \geq u+v$.

(a1.1) We check here Claim 1 (with the assumption $a \geq u+v$) for $k = 5, 6, 7$. First assume $k = 5$. From (7) we get $t \geq 14$. Since $6t + 5a \leq 126$, we get $a \leq 8$. Since $t + 4u + v = 56$ and $a \geq u+v$, we get $t \geq 24$. Hence $6t > 126$, a contradiction. Now assume $k = 6$. From (7) we get $t \geq 33$. Hence $7t > 210 = \binom{10}{4}$, a contradiction. Now assume $k = 7$. From (7) we get $t \geq 65$. Hence $8t > 330 = \binom{11}{4}$.

Let $Y \subset \mathbb{P}^4$ be a general union of t lines and $a-u-v$ 2-points. Set $J := Y \cap H$. J is a general union of t points of H. Fix a general $S \cup S' \subset H$ with $\sharp(S) = u$, $\sharp(S') = v$ and $S \cap S' = \emptyset$. Let $A \subset \mathbb{P}^4$ (resp. A') be the union of the 2-points of $\mathbb{P}^4$ with S (resp. S') as its support. Let $B \subset H$ (resp. $B' \subset H$) be the union of the 2-points of H supported by the points of S (resp. S'). The definition of the integers u and v gives $h^0(H, \mathcal{O}_{J\cup B\cup S'}(k)) = \binom{k+3}{3} - t$.

Claim 2: We have $u \geq v$ and $h^i(H, \mathcal{I}_{J\cup B\cup S'}(k)) = 0$, $i = 0, 1$.

Proof Claim 2: We have $v \leq 3$. Assume $u < v$ and hence $u \leq 2$. We get $t \geq \binom{k+2}{3} - 11$. Since $t \leq (k+4)(k+3)(k+2)/24$ and $k \geq 5$, we get a contradiction. Since S' is general in H and $h^0(H, \mathcal{O}_{J\cup B\cup S'}(k)) = \binom{k+3}{3} - t$, it is sufficient to check that $h^1(H, \mathcal{I}_{J\cup B\cup S'}(k)) = 0$. Since $k \geq 5$, then Claim 2 follows from the Alexander-Hirschowitz theorem, because $v' \geq 0$.

By the Differential Horace Lemma (Lemma 2) and Claim 2 to get $h^1(\mathcal{I}_X(k)) = 0$ it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup S\cup B'}(k-1)) = 0$. We have $\binom{k+3}{4} - h^0(\mathcal{O}_{Y\cup S\cup B'}(k-1)) = \binom{k+4}{4} - h^0(\mathcal{O}_X(k))$. By Lemma 4 it is sufficient to prove $h^1(\mathcal{I}_{Y\cup B'}(k-1)) = 0$ and $h^0(\mathcal{I}_Y(k-1)) \geq \deg(S \cup B') + h^0(\mathcal{I}_Y(k-2))$. Lemma 5 gives $h^1(\mathcal{I}_{Y\cup B'}(k-1)) \leq h^1(\mathcal{I}_{Y\cup A'}(k-1))$. Since $\sharp(S') \leq 3$, S' is general in H and any 3 points of $\mathbb{P}^4$ are contained in a hyperplane, $Y \cup A'$ may be considered as a general union of t lines and $a-u$ 2-points. Assuming for the moment $k \geq 6$ the inductive assumption gives that $h^1(\mathcal{I}_{Y\cup A'}(k-1)) = 0$ if $h^0(\mathcal{O}_{Y\cup A'}(k-1)) \leq \binom{k+3}{4}$. We have $h^0(\mathcal{O}_{Y\cup S\cup B'}(k-1)) = \binom{k+3}{4} - (\binom{k+4}{k} - h^0(\mathcal{O}_X(k)) \leq \binom{k+3}{4}$ and $\deg(A') = \deg(B') + v$. Hence to prove that $h^1(\mathcal{I}_{Y\cup B'}(k-1)) = 0$ it is sufficient to note that $u \geq v$ (Claim 2). In the case $k = 5$ we also need that either $t > 0$ or $a - u < 9$; assume $k = 5$ and $t = 0$; in this case $h^1(\mathcal{I}_X(k)) = 0$ by the Alexander-Hirschowitz theorem (alternatively, in this case we have $a = 25$, $u = 19$ and $v = 0$).

(a1.2) In this step we check that $h^0(\mathcal{I}_Y(k-2)) = 0$.

Claim 3: We have $h^0(\mathcal{O}_Y(k-2)) \ge \binom{k+2}{4}$.

Proof Claim 3: We have $\binom{k+4}{4} = \binom{k+2}{4} + \binom{k+3}{3} + \binom{k+2}{3}$. We have $h^0(\mathcal{O}_Y(k-2)) = (k-1)t+5(a-u-v)$. We have $(k+1)t+5a \ge \binom{k+4}{4} - 4$ and $t+4u+v = \binom{k+3}{3}$. Hence $2t+8u+2v = 2\binom{k+3}{3}$. Hence it is sufficient to check that $3u-3v \ge \binom{k+2}{2}+4$, i.e $4u-4v \ge 2(k+2)(k+1)/3+16/3$. We have $4u = \binom{k+3}{3} - t - v$ and hence it is sufficient to check that $2(k+2)(k+1)/3+16/3+5v \le \binom{k+3}{3} - t$. Since $v \le 3$, it is sufficient to check that $t \le (k+2)(k+1)(k-1)/6 - 15 - 16/3$. We have $(k+1)((k+2)(k+1)(k-1)/6 - 15 - 16/3) > \binom{k+4}{4}$ for all $k \ge 6$. Now assume $k = 5$. First of all Y is a general union of t lines and $a-u-v$ 2-points. To get $h^0(\mathcal{I}_Y(3)) = 0$ it is sufficient to have $4t+5(a-u-v) \ge 35$ and either $t \ne 0$ or $a-u-v \ge 8$ (Proposition 10). Hence we may assume $t \le 8$. Since $122 \le 6t+5a \le 126$ and $t+4u+v = 56$ we get the following values for the quadruples (t,a,u,v): $(8,14,2,0)$, $(7,16,12,1)$, $(6,18,12,2)$, $(5,19,12,3)$, $(4,20,13,0)$, $(3,21,13,1)$, $(2,22,13,2)$, $(1,24,13,3)$. All the values for $(t,a-u-v)$ gives $h^0(\mathcal{I}_Y(3)) = 0$ by Proposition 10.

(a2) Now assume $k \le a < u+v$. Let f be the minimal integer such that $t-f+(k+1)f+4u'+v' = \binom{k+3}{3}$ for some integers u', v' with $u' \ge 0$, $0 \le v' \le 3$ and $u'+v' \le a$. There is such an integer, because $a \ge k \ge 3+\lfloor k/4 \rfloor$ and hence among the integers $4u'+v'$ we may realize all integers x with $0 \le x \le k$. Since $(k+1)t+5a \ge \binom{k+4}{4} - 4$, we have

$$\binom{k+3}{4} - 4 \le k(t-f) + 5a - 4u' - v' \le \binom{k+3}{4}. \tag{8}$$

The minimality of the integer f gives $a - u' \le 3 + \lfloor k/4 \rfloor$ (Remark 3). Since $a \ge k \ge 3 + u' + \lfloor k/4 \rfloor$, we have $u' \ge v'$.

Claim 4: We have $f \le t$.

Proof of Claim 4: Assume $f \ge t+1$. From the first inequality in (8) we get $\binom{k+3}{4} \le 4 + 5a - 4u' - v' - k$. Since $5a - 5u' \le 15 + 5k/4$ and $v' \ge 0$, we get $u' + 11 + k/4 \ge \binom{k+3}{4}$. Since $4u' \le \binom{k+3}{3}$, we get $(k+3)(k+2)(k+1)/24 + 11 + k/4 \ge \binom{k+3}{4}$ (false for all $k \ge 5$).

Claim 5: If $k \ge 7$, then $t - f + v' \ge 3k$.

Proof of Claim 5: Assume $t - f \le 3k - v' - 1$. From the first inequality in (8) we get $3k^2 - k - kv' + 5a - 4u' - v' \ge \binom{k+3}{4} - 4$. Since $5a - 5u' \le 15 + 5k/4$, we get $3k^2 + u' - (k+1)v' + 15 + 5k/4 \ge \binom{k+3}{4} - 4$. Since $4u' + v' \le \binom{k+3}{3}$ and $v' \ge 0$, we get $3k^2 + (k+3)(k+2)(k+1)/24 + 5k/4 \ge \binom{k+3}{4} - 4$, which is false for all $k \ge 7$.

Assume for the moment $t - f + v' \ge 3k$ (e.g., assume $k \ge 7$). Fix a general $S \cup S' \subset H$ with $\sharp(S) = u'$, $\sharp(S') = v'$ and $S \cap S' = \emptyset$. Let $E \subset H$ be a general union of f lines. Set $A := \cup_{O \in S} 2O$, $Z := A \cap H$, $B := \cup_{O \in S'} 2O$ and $B' := B \cap H$.

Let $Y \subset \mathbb{P}^4$ be a general union of $t-f$ lines and $a-u'-v'$ 2-points. As in step (a1) it is sufficient to prove that $h^1(H, \mathcal{I}_{(Y\cap H)\cup S'\cup Z}(k)) = 0$ and $h^1(\mathcal{I}_{Y\cup S\cup B'}(k-1)) = 0$. Since $(Y \cup H) \cup S'$ is a general union of $t-f+v' \geq 3k$ points of H and $\deg((Y \cap H) \cup S' \cup Z) = \binom{k+3}{3}$, Proposition 4 gives $h^1(\mathcal{I}_{Y\cup S\cup B'}(k-1)) = 0$. We first check that $h^1(\mathcal{I}_{Y\cup B'}(k-1)) = 0$. Since $B' \subset B$ and B' is a union of connected components of $Y \cup B'$, we have $h^1(\mathcal{I}_{Y\cup B'}(k-1)) \leq h^1(\mathcal{I}_{Y\cup B}(k-1))$. Hence to prove that $h^1(\mathcal{I}_{Y\cup B'}(k-1)) = 0$ it is sufficient to prove $h^1(\mathcal{I}_{Y\cup B}(k-1)) = 0$. Since $v' \leq 3$ and any 3 points of $\mathbb{P}^4$ are contained in a hyperplane, $Y \cup B$ may be considered as a general union of $t-f$ lines and $a-u'$ 2-points. Since $u' \geq v'$ (Claim 3) and $h^0(\mathcal{O}_{Y\cup B}(k)) = h^0(\mathcal{O}_{Y\cup S'\cup Z'}(k)) + v' - u' = \binom{k+3}{3} + \binom{k+4}{4} - (k+1)t - 5a$, the inductive assumption (see below for the case $k=5$) gives $h^1(\mathcal{I}_{Y\cup B}(k-1)) = 0$ and hence $h^1(\mathcal{I}_{Y\cup B'}(k-1)) = 0$; if $k=5$ we need that either $a-u' \leq 13$ or $t-f > 0$; the latter inequality is obvious, because we assumed $t-f \geq 3k$. Therefore $h^0(\mathcal{I}_{Y\cup Z'}(k-1)) \geq u'$. Since S is general in H, to get $h^1(\mathcal{I}_{Y\cup S\cup B'}(k-1)) = 0$, it is sufficient to prove that either $h^0(\mathcal{I}_Y(k-2)) = 0$ or $h^0(\mathcal{I}_Y(k-2)) \leq h^0(\mathcal{I}_Y(k-1)) - u' - 4v'$ (Lemma 4). We assume for the moment that Y has maximal rank (this is true by the inductive assumption and Proposition 10 if either $k \geq 6$ or $k=5$ and $(t-f, a-u-f) \neq (0,7)$). Hence the inequality $h^0(\mathcal{I}_Y(k-2)) \leq h^0(\mathcal{I}_Y(k-1)) - u' - 4v'$ is equivalent to $t-f+u'+4v' \leq \binom{k+2}{3}$. Assume $t-f+u'+4v' \geq \binom{k+2}{3}+1$. Since $t-f+(k+1)f+4u'+v' = \binom{k+3}{3}$, we get $(k+1)f + 3u' - 3v' \leq \binom{k+2}{2} + 1$. Since $a < u+v$, we have $f > 0$. Hence $u' \leq v' + (k^2+k+2)/2$. Hence $t-f+5v' + (k^2+k+2)/2 \geq \binom{k+2}{3} + 1$. Since $v' \leq 3$, we get $t-f \geq (k^3-k-90)/6$, contradicting the second inequality in (8) for all $k \geq 7$. Now assume $k = 5, 6$. We assumed $t-f+u'+4v' \geq \binom{k+2}{3}+1$. Since $a \geq u'+v'$, we have $5a - 4u' - v' \geq u' + 4v'$. Hence the last inequality in (8) gives $(k-1)(t-f) + \binom{k+2}{3} + 1 \leq \binom{k+3}{4}$. Since $t-f \geq (k^3-k-90)/6$, we get a contradiction even if $k = 5, 6$.

(a3) Now assume $a \leq k-1$. Since $(k+1)t + 5a \geq \binom{k+4}{4} - 4$, we have $(k+1)t + 5k - 1 \geq \binom{k+4}{4}$. Let f be the minimal integer such that $t - f + (k+1)f + 4a \geq \binom{k+3}{3}$. Set $f' := t - f + (k+1)f + 5a - \binom{k+3}{3}$. We have $0 \leq f' \leq k-1$ and

$$t - f - f' + (k+1)f + 4a = \binom{k+3}{3}. \tag{9}$$

From (9) we get

$$\binom{k+3}{4} - 4 \leq (k+1)(t-f-f') + kf' + a \leq \binom{k+3}{4}. \tag{10}$$

Claim 6: We have $f \geq 0$.

Proof of Claim 6: Assume $f \leq -1$, i.e. assume $t - k + 4a \geq \binom{k+3}{3}$. Since

$a \le k-1$, we get $t \ge (k^3+6k^2-19k+36)/6$. Since $k \ge 5$, we get $(k+1)t > \binom{k+4}{4}$, a contradiction.

Claim 7: We have $t - f - f' \ge 0$.

Proof of Claim 7: Since $(k+1)t + 5a \ge \binom{k+4}{4} - 4$, we have $f \le t$. Assume $t - f - f' \le -1$. Since $f' \le k-1$ we also get $f \ge t - k + 2$. Hence (9) gives $-1 + (k-2)(k+1) + (k+1)t + 4a \le \binom{k+3}{3}$. Since $a \le k-1$ and $k \ge 5$, we get $(k+1)t + 5a < \binom{k+4}{4} - 4$, a contradiction.

Claim 8: If $k \ge 7$, then $t - f - f' \ge 3k$.

Proof of Claim 8: Assume $t-f-f' \le 3k-1$. Since $a \le k-1$ and $f' \le k-1$, from the first inequality in (10) we get $\binom{k+3}{4} - 4 \le (k+1)(3k-1) + k^2 - 1$, which is false for all $k \ge 7$.

Claim 9: Take $k = 5, 6$ and take a pair (t, a) for which $t - f - f' \le 3k - 1$ and $0 < a \le k - 1$. The pair (t_1, a_1) with $t_1 := f$ and $a_1 := a$ is covered by Lemmas 3 and 8.

Proof of Claim 9: First assume $k = 5$. We have $122 \le 6t + 5a \le 126$, $t + 5f + 4a - f' = 56$ and $1 \le a \le 4$. Since $1 \le a \le 4$, we get that (t, a, f, f') is one of the following quadruples: $(17, 4, 5, 2)$, $(18, 3, 6, 4)$, $(19, 2, 6, 1)$, $(20, 1, 7, 3)$. The pairs $(5, 4)$, $(6, 3)$, $(6, 2)$ and $(7, 1)$ are covered by Proposition 3.

Now assume $k = 6$. We have $206 \le 7t + 5a \le 210$ and $t + 6f + 5a - f' = 84$ and $1 \le a \le 5$, the quadruple (t, a, f, f') is one of the following quadruples: $(26, 5, 7, 4)$, $(27, 4, 7, 1)$, $(27, 3, 8, 3)$, $(28, 2, 8, 0)$, $(29, 1, 9, 3)$. The pairs $(7, 4)$, $(8, 3)$ and $(9, 3)$ are covered by Lemma 8. Fix a general $S \subset H$ and set $A := \cup_{O \in S}(A)$. Let $E \subset H$ be a general union of f lines. Write $E = E_1 \sqcup E_2$ with $\deg(E_2) = f'$ and $\deg(E_1) = t-f-f'$. We have $h^1(H, \mathcal{I}_{E\cup(A\cap H}(k)) = 0$ (Claims 7 and 8 and Proposition 4).

Since $f' \le k-1$, either $h^1(\mathcal{I}_Y(k-1)) = 0$ or $h^0(\mathcal{I}_Y(k-1)) = 0$ for a general $Y \in L(4, t-f-f', f')_H$ by $B_{4,k-1}$. Since $h^0(\mathcal{O}_Y(k-1)) = \binom{k+3}{4} - a - (\binom{k+4}{4} - (k+1)t - 5a)$, we get $h^1(\mathcal{I}_Y(k-1)) = 0$ and $h^0(\mathcal{I}_Y(k-1)) \ge a$. For a general $Y \in L(4, t-f-f', f')_H$ we have $h^0(\mathcal{I}_Y(k-2)) \le h^0(\mathcal{I}_{Y_{red}}(k-2)) = 0$ ([18]). Hence $h^1(\mathcal{I}_{Y\cup S}(k-1)) = 0$ (Lemma 4). Write $Y = Y_1 \sqcup Y_2$ with $Y_1 \in L(4, t-f-f', 0)$, $Y_2 \in L(4, 0, f')$. Any f' general lines of H have the property that picking a general point of each of them we get a general union of f' points of H. Hence for fixed Y, S, E_1 we may assume that E_2 of f' lines of H pass through f' general points of H. Hence we may find S, E and Y so that $h^1(\mathcal{I}_{Y\cup S}(k-1)) = 0$, $h^1(H, \mathcal{I}_{E\cup(A\cap H}(k)) = 0$, S, E_1 and Y_1 are general, but each line of E_2 contains one of the points of $Y_2 \cap H$. In this case $Y \cup E$ is a disjoint union of $t - 2f'$ lines and f' sundials. Hence $Y \cup E \cup A \in Z(4, t, a)'$. The Castelnuovo's sequence gives $h^1(\mathcal{I}_{Y\cup E\cup A}(k)) = 0$.

(b) In this step we assume $t(k+1) + 5a > \binom{k+4}{4}$. Set $\delta := t(k+1) + 5a - \binom{k+4}{4}$. Decreasing if necessary a we reduce to the cases in which either $\delta \le 4$ or

$a = 0$ and $t(k+1) \le \binom{k+4}{4} + k$. In the latter case we have $h^0(\mathcal{I}_X(k)) = 0$ by [18]. Hence we may assume $1 \le \delta \le 4$. We may also assume $a > 0$ ([18]) and $t > 0$ (by the Alexander-Hirschowitz theorem). Since $h^0(\mathcal{I}_W(k)) = 5 - \delta$ for a general $W \in Z(4, t, a-1)$ and any 2-point contains a point, the case $\delta = 4$ follows from the case $(t, a-1)$ proved in step (a). Every 2-point contains a tangent vector. Hence in characteristic zero if $h^0(\mathcal{I}_W(k)) = 2$, then $h^0(\mathcal{I}_{W \cup 2O}(k)) = 0$ for a general $O \in \mathbb{P}^4$ ([16], [8], Lemma 1.4). Hence in characteristic zero we may assume $\delta \in \{1, 2\}$. We will easily adapt step (a3) to the case $t(k+1)+5a > \binom{k+4}{4}$. To adapt steps (a1) and (a2) we need the following observations. In steps (a1) and (a2) we needed to prove that $h^1(\mathcal{I}_{Y \cup B'}(k-1)) = 0$, where B' is a general union of v (or v') 2-points of H. In our set-up we need $h^0(\mathcal{I}_{Y \cup S \cup B'}(k)) = 0$. Of course, if $h^0(\mathcal{I}_{Y \cup B'}(k-1)) = 0$, then we are done; however we always have $h^0(\mathcal{I}_{Y \cup B'}(k-1)) \ge v - \delta$ (case $a \ge u + v$) or $h^0(\mathcal{I}_{Y \cup B'}(k-1)) \ge v' - \delta$ (case $k \le a < u + v$). To get $h^0(\mathcal{I}_{Y \cup S \cup B'}(k)) = 0$ it is sufficient to prove that $h^0(\mathcal{I}_Y(k-2)) = 0$ (easy) and that $h^1(\mathcal{I}_{Y \cup B'}(k)) = 0$ (the difficult part). Let $A' \subset \mathbb{P}^4$ the the union of the 2-points of $\mathbb{P}^4$ with the points B'_{red} as their support. We saw that it is sufficient to prove that $h^1(\mathcal{I}_{Y \cup A'}(k)) = 0$. We have $h^0(\mathcal{O}_{Y \cup A'}(k)) = \delta + v - u$ (case $a \ge u + v$) or $h^0(\mathcal{O}_{Y \cup A'}(k)) = \delta + v' - u'$ (case $k \le a < u+v$) by the inductive assumption (see below for cases $k = 5, 6$). Hence we introduce the integers u, v as in step (a1) and if $a \ge u + v$ we need to check the following numerical condition £:

£: $u \ge v + \delta$.

If $k \le a < u + v$ we introduce the integers u', v', f as in step (a2). We need to check the following numerical condition ££:

££: $u' \ge v' + \delta$.

(b1) Assume $k = 5$. We have $127 \le 6t+5a \le 129$. We get that (t, a, δ, u, v) is one of the following quintuples: $(2, 23, 1, 13, 2)$, $(3, 22, 2, 13, 1)$, $(4, 21, 3, 13, 0)$, $(7, 17, 1, 12, 1)$, $(8, 16, 2, 12, 0)$, $(9, 15, 3, 11, 3)$, $(12, 11, 1, 11, 0)$, $(13, 10, 2, 10, 3)$, $(14, 9, 3, 10, 2)$, $(17, 5, 1, 9, 3)$, $(18, 4, 2, 9, 2)$, $(19, 3, 3, 9, 1)$. In all cases we have $u \ge v + \delta$. Among the previous quintuples with $5 \le a < u+v$ we get the following sextuples $(t, a, \delta, f, u', v')$: $(13, 10, 2, 2, 8, 1)$, $(14, 9, 3, 2, 8, 0)$, $(17, 5, 1, 5, 3, 2)$. We always have $u' \ge v'+\delta$. We get the following pairs (f, u): $(2, 8)$, $(5, 3)$. Proposition 3 covers these cases. Now assume $1 \le a \le 4$, i.e. $(t, a, \delta) = (19, 3, 3)$. We have $(t, a, \delta, f, f') = (19, 3, 3, 5, 0)$. We have $t - f - f' \ge 0$ and $f \ge f'$. The pair $(f, a') = (5, 3)$ is covered by Proposition 3.

(b2) Assume $k = 6$. We have $211 \le 7t+5a \le 213$. We get that (t, a, δ, u, v) is one of the following quintuples: $(2, 40, 2, 20, 2)$, $(4, 37, 1, 20, 0)$,$(5, 36, 3, 19, 3)$, $(6, 34, 2, 19, 2)$, $(8, 31, 1, 19, 0)$, $(9, 30, 3, 18, 3)$, $(11, 29, 2, 18, 1)$, $(13, 24, 1, 17, 3)$, $(14, 23, 3, 17, 2)$, $(16, 22, 2, 17, 0)$, $(18, 19, 1, 16, 2)$, $(19, 18, 3, 16, 1)$, $(21, 15, 2, 15, 3)$, $(23, 11, 1, 15, 1)$, $(24, 10, 3, 15, 0)$, $(26, 8, 2, 14, 2)$, $(28, 3, 1, 14, 0)$, $(29, 2, 3, 13, 3)$.

The ones with $a \geq u + v$ (i.e. the ones with $t \leq 19$) have $u \geq v + \delta$ and hence we may quote [18] in these cases. The quintuples with $6 \leq a < u + v$ give the following sextuples $(t, a, \delta, f, u', v')$: $(21, 15, 2, 1, 4, 1)$, $(23, 11, 1, 4, 9, 1)$, $(24, 10, 3, 9, 0)$, $(26, 8, 2, 7, 3, 0)$. We always have $u' \geq v' + \delta$ and $t - f \geq 18$. The latter inequality allows us to apply Proposition 4 (alternatively, use Proposition 8). The ones with $1 \leq a \leq 5$, give the following quintuples: (t, a, δ, f, f'): $(28, 3, 1, 8, 4)$, $(29, 2, 3, 8, 1)$, $(29, 2, 3, 8, 1)$. In these cases we always have $f \geq f'$ and $t - f - f' \geq 0$ and hence we may apply Lemma 2.

(b3) From now on we assume $k \geq 7$. To repeat steps (a1) (case $a \geq u+v$) or (a2) (case $k \leq a < u+v$), we need to check the numerical conditions used in those steps plus £ and ££.

Claim 10 Assume $a \leq u + v$. Then Claim 1 is true.

Proof of Claim 10: Since $a > 0$ and $\delta \leq 4$, we have $(k+1)t \leq (k+4)(k+3)(k+2)(k+1)/24$. By (7) we get a contradiction if $k \geq 8$, while if $k = 7$ we get $t \geq 58$. Since $58 \cdot 6 > 330 = \binom{11}{4}$, Claim 10 is true.

Claim 11: Assume $a \geq u + v$. Then £ is true

Proof of Claim 11: We have $(k+1)t = \binom{k+2}{3}(k+1) - 4(k+1)u - (k+1)v$. Since $a \geq u + v$, we have $(k+1)t \leq \binom{k+4}{4} - 5u - 5v + \delta$. Hence $\binom{k+2}{3}(k+1) - 4(k+1)u - (k+1)v \leq \binom{k+4}{4} - 5u - 5v + \delta$, i.e. $u(1-4k) + v(4-k) - \delta \leq (k+2)(k+1)(-5k^2+k+12)/24$. Assume $u \leq v - 1 + \delta$. Since $v \leq 3$ and $\delta \leq 2$, we get $-4 - 16k + 8 - 2k - 2 \leq (k+2)(k+1)(-5k^2+k+12)/24$, which is false for all $k \geq 7$.

Claim 12: Assume $k \leq a < u + v$. Then ££ is true.

Proof of Claim 12: Assume $u' \leq v' + \delta' - 1$. Hence $u' \leq 6$. Since $a - u' \leq 3 + \lfloor k/4 \rfloor$ (Remark 3), we get $\lfloor k/4 \rfloor \leq k - 3$, which is false for all $k \geq 7$.

(b3.1) Assume $0 < a \leq k - 1$. In this case the proof of step (a3) works verbatim (here we prove that $h^0(\mathcal{I}_{Y \cup S}(k-1)) = 0$, because $h^0(\mathcal{I}_Y(k-1)) = \max\{0, a - \delta\}$ and hence $h^0(\mathcal{I}_Y(k-2)) = 0$). $\boxed{QED}$

5 $\mathbb{P}^5$

Lemma 12. *Fix a hyperplane $H \subset \mathbb{P}^r$, $r \geq 5$, $O \in H$ and set $B' := 2O \cap H$. Fix an integer $t \geq 0$. Let X be a general element of $Z(5, t, 1)$. Then either $h^0(\mathcal{I}_{X \cup B'}(2)) = 0$ or $h^1(\mathcal{I}_{X \cup B'}(2)) = 0$.*

Proof. Write $X = Y \sqcup 2P$ with $Y \in Z(r, t, 0)$. Let $\ell_P : \mathbb{P}^r \setminus \{P\} \to \mathbb{P}^{r-1}$. For general H, O, Y, P, the map ℓ sends isomorphically $Y \cup B'$ onto its image $\ell(Y \cup B')$ and $\ell(Y \cup B')$ is a general element of $Z(r-1, t, 1)$. Lemma 1 gives $h^0(\mathbb{P}^{r-1}, \mathcal{I}_{\ell(Y \cup B')}(2)) = \max\{0, \binom{r+1}{2} - r - 3t\}$ and $|\mathcal{I}_{Y \cup B' \cup 2P}(2)| \cong |\mathcal{I}_{\ell(Y \cup B')}(2)|$. Therefore $h^0(\mathcal{I}_X(2)) = \max\{0, \binom{r+2}{2} - h^0(\mathcal{O}_X(2))\}$. $\boxed{QED}$

Lemma 13. *Fix integers $t \geq 0$, $t' \geq 0$. Fix a hyperplane $H \subset \mathbb{P}^5$, a general $O \in H$, a general $P \in \mathbb{P}^5$ and a general $U \in L(5,t,t')_H$. Set $B' := 2O \cap H$ and $W := U \cup 2P \cup B'$. Then either $h^0(\mathcal{I}_W(2)) = 0$ (case $3t + 4t' \geq 10$) or $h^1(\mathcal{I}_W(2)) = 0$ (case $3t + 4t' \leq 10$).*

Proof. Let $E \subset \mathbb{P}^3$ be the image of the linear projection of U by the line ℓ spanned by O and P. For general O, P, U the scheme E is a general element of $L(3,t,t')$ and it is general (we use that P is general, so that the condition that the nilradical of U is supported by t' points of H give no restriction to E). By [6] E has maximal rank. Since the linear system $|\mathcal{I}_{2P \cup B'}(2)|$ is the projective space of all quadric cones with vertex containing ℓ, we have $h^0(\mathcal{I}_W(2)) = h^0(\mathbb{P}^3, \mathcal{I}_E(2))$. $\boxed{QED}$

Lemma 14. *Fix $(t,a) \in \mathbb{N}^2 \setminus \{(0,0)\}$. Fix a general $X \in Z(5,t,a)$. Then either $h^0(\mathcal{I}_X(3)) = 0$ or $h^1(\mathcal{I}_X(3)) = 0$.*

Proof. We have $\binom{8}{3} = 56$ and $\binom{7}{3} = 35$. Set $e := 56 - 4t - 6a$. Increasing or decreasing if necessary t it is sufficient to cover all pairs (t,a) with $-3 \leq e \leq 3$. By [18] and the Alexander-Hirschowitz theorem all cases with either $t = 0$ or $a = 0$ are true. Let $H \subset \mathbb{P}^5$ be a hyperplane. Let $S_x \subset H$ denote a union of x general points of H. Set $A_x := \cup_{O \in S_x} 2O$.

Assume for the moment $t > 0$ and $e \in \{-3,-2\}$. We have $56 - 4(t-1) - 6a \in \{1,2\}$. Assume that Lemma 14 is true for the pair $(t-1,a)$ and take $Y \in Z(5,t-1,a)$ such that $h^1(\mathcal{I}_Y(3)) = 0$, i.e. such that $h^0(\mathcal{I}_Y(3)) = 4 + e \leq 2$. Since any two points of $\mathbb{P}^5$ are contained in a line, we get $h^0(\mathcal{I}_{Y \cup L}(3)) = 0$ for a general line L. Hence Lemma 14 is true for the pair (t,a). Therefore it is sufficient to prove Lemma 14 for all pairs (t,a) with $t > 0$, $a > 0$ and $-1 \leq e \leq 3$, i.e. for the triples (t,a,e): $(2,8,0)$, $(3,7,2)$, $(5,6,0)$, $(6,5,2)$, $(8,4,0)$, $(9,3,2)$, $(11,2,0)$, $(12,1,2)$.

(a) Take $(t,a) = (2,8)$. Fix a general line $L \subset \mathbb{P}^5$ and a general line $R \subset H$ containing the point $L \cap H$. Let $U \subset \mathbb{P}^5$ a general sundial with $L \cup R$ as its support. Set $L' := \mathrm{Res}_H(U)$. The scheme L' is a general +line with L as its support and $R \cap L$ as the support of its nilradical. Fix a general $P \in \mathbb{P}^5$ and set $Y := L' \cup 2P$. Fix a general $S \cup S' \subset H$ with $\sharp(S) = 6$, $\sharp(S') = 1$ and $S \cap S' = \emptyset$. Let A (resp. B) the union of the 2-points of $\mathbb{P}^5$ with the points of S (resp. S') as their support. Set $B' := B \cap H$. Since $h^1(H, \mathcal{I}_{R \cup (A \cap H)}(3)) = 0$ (case $(t,a) = (1,6)$ of Lemma 10), we have $h^i(H, \mathcal{I}_{R \cup (A \cup H) \cup S'}(3)) = 0$, $i = 0,1$. To prove the case $(t,a) = (2,8)$ it is sufficient to prove that a general union W of A, the sundial U and a general 2-point satisfies $h^0(\mathcal{I}_W(3)) = 0$. By the Differential Horace Lemma (Lemma 2), this is the case if $h^0(\mathcal{I}_{Y \cup S \cup B'}(2)) = 0$. Since Y contains a 2-point, we have $h^0(\mathcal{I}_Y(1)) = 0$. Hence by Lemma 4 it is sufficient to prove $h^1(\mathcal{I}_{Y\cup B'}(2)) = 0$. This is true by the case $(t,t') = (0,1)$ of Lemma 13.

(b) Take $(t,a,e) = (3,7,2)$. Let $Y \subset \mathbb{P}^5$ be a general union of 3 lines and one 2-point. We have $h^1(\mathcal{I}_Y(2)) = 0$ (Lemma 1), i.e. $h^0(\mathcal{I}_Y(2)) = 6$. Since $h^0(\mathcal{I}_Y(1)) = 0$, we get $h^i(\mathcal{I}_{Y\cup S_6}(2)) = 0$, $i = 0,1$. Since $h^1(H,\mathcal{I}_{A_6\cap H}(3)) = 0$ by the Alexander-Hirschowitz theorem, we have $h^0(H,\mathcal{I}_{A_6\cap H}(3)) = 6$ and hence $h^1(H,\mathcal{I}_{(Y\cup A)\cap H}(3)) = 0$. Hence $h^1(\mathcal{I}_{Y\cup A}(3)) = 0$.

(c) Take $(t,a,e) = (5,6,0)$. Let $Y \subset \mathbb{P}^5$ be a general union of 5 lines and one 2-point. We have $h^i(\mathcal{I}_Y(2)) = 0$, $i = 0,1$ (Lemma 1). We have $h^1(H,\mathcal{I}_{A\cap H}(3)) = 0$ and hence $h^i(\mathcal{I}_{(Y\cup A)\cap H}(3)) = 0$. Hence $h^i(\mathcal{I}_{Y\cup A}(3)) = 0$, $i = 0,1$.

(d) Take $(t,a,e) = (6,5,2)$. Let $Y \subset \mathbb{P}^5$ be a general union of one 2-point and 2 lines. We have $h^1(\mathcal{I}_Y(2)) = 0$ (Lemma 1), i.e. $h^0(\mathcal{I}_Y(2)) = 6$ Obviously $h^0(\mathcal{I}_Y(1)) = 0$. Hence $h^1(\mathcal{I}_{Y\cup S_4}(2)) = 0$ (Lemma 4). We have $h^1(H,\mathcal{I}_{E_3\cup(A_4\cap H)}(3)) = 0$ (Lemma 1) and hence $h^i(H,\mathcal{I}_{(Y\cap H)\cup E_3\cup(A_4\cap H)}(3)) = 0$, $i = 0,1$. Therefore we have $h^1(\mathcal{I}_{Y\cup E_3\cup A_4}(3)) = 0$.

(e) Take $(t,a,e) = (8,4,0)$. Let $Y \subset \mathbb{P}^5$ be a general union of 4 lines and 2 +lines with nilradical supported by points of H. We have $h^1(\mathcal{I}_Y(2)) = 0$ (Lemma 6). Obviously $h^0(\mathcal{I}_Y(1)) = 0$. Hence $h^i(\mathcal{I}_{Y\cup S_4}(2)) = 0$, $i = 0,1$ (Lemma 4). Since E_2 contains two general points of H, without loss of generality we may assume that each line of E_2 contains the support of the nilradical of one of the +lines of Y, so that $Y \cup E_2$ a disjoint union of 4 lines and two sundials. We have $h^1(H,\mathcal{I}_{E_2\cup(A_4\cap H)}(3)) = 0$ (Lemma 10) and hence $h^1(H,\mathcal{I}_{(Y\cap H)\cup E_2\cup(A\cap H)}(3)) = 0$. Therefore $h^1(\mathcal{I}_{Y\cup A_4\cup E_2}(3)) = 0$. Since $Y \cup A_4 \cup E_2$ is a disjoint union of an element of $Z(5,4,4)$ and two sundials, we are done.

(f) Take $(t,a,e) = (9,3,2)$. Let $Y \subset \mathbb{P}^5$ be a general union of one 2-point and 4 lines. We have $h^1(\mathcal{I}_Y(2)) = 0$ (Lemma 1) and hence $h^1(\mathcal{I}_{Y\cup S_2}(2)) = 0$. We have $h^1(H,\mathcal{I}_{E_5\cup(A_5\cap H)}(3)) = 0$ (Lemma 10). Hence $h^1(\mathcal{I}_{A_5\cup E_5\cup Y}(3)) = 0$.

(g) Take $(t,a,e) = (11,2,0)$. We have $h^1(H,\mathcal{I}_{(A_2\cap H)\cup E_5}(3)) = 0$ (Lemma 10). Fix a general $Y \in L(5,5,1)_H$. We have $h^1(\mathcal{I}_Y(2)) = 0$ (Lemma 6). Obviously $h^0(\mathcal{I}_{Y_{red}}(1)) = 0$. Hence $h^i(\mathcal{I}_{Y\cup S_2}(2)) = 0$, $i = 0,1$ (Lemma 4). Since E_3 is general, we may assume that $D \cap H$ is a point of E. Therefore we have $h^i(H,\mathcal{I}_{(Y\cap H)\cup E_5\cup(A\cap H)}(3)) = 0$, $i = 0,1$. Hence $h^i(\mathcal{I}_{Y\cup A\cup E_4}(3)) = 0$, $i = 0,1$. Since $Y \cup A \cup E_4$ is a disjoint union of an element of $Z(5,9,2)$ and a sundial, we are done.

(h) Take $(t,a,e) = (12,1,2)$. Let $Y \subset \mathbb{P}^5$ be a general union of 5 lines and one 2-point. We have $h^i(\mathcal{I}_Y(3)) = 0$, $i = 0,1$. We have $h^1(H,\mathcal{I}_{E_7}(3)) = 0$ and hence $h^1(H,\mathcal{I}_{(Y\cap H)\cup E_7}(3) = 0$. Hence $h^1(\mathcal{I}_{Y\cup E_7}(3)) = 0$. $\boxed{QED}$

Lemma 15. *Fix any t,a and a general $X \in Z(5,t,a)$. Then either $h^0(\mathcal{I}_X(4)) = 0$ or $h^1(\mathcal{I}_X(4)) = 0$.*

Proof. We have $\binom{9}{5} = 126$ and $\binom{8}{4} = 70$. Set $e := 126 - 5t - 6a$. For a fixed t and any $f \in \mathbb{N}$ define the integers u, v, u_f, v_f by the relations $t+5u+v = 70$, $0 \le v \le 4$, $t + 4f + 5u_f + v_f = 70$, $0 \le v_f \le 4$. Increasing or decreasing if necessary t it is sufficient to cover all pairs (t, a) with $-4 \le e \le 4$. By [18] and the Alexander-Hirschowitz theorem all cases with either $t = 0$ or $a = 0$ are true. Assume for the moment $t > 0$ and $e \in \{-4, -3\}$. We have $70-5(t-1)-6a \in \{1, 2\}$. Assume that Proposition 15 is true for the pair $(t - 1, a)$ and take $Y \in Z(5, t - 1, a)$ such that $h^1(\mathcal{I}_Y(4)) = 0$, i.e. such that $h^0(\mathcal{I}_Y(4)) = 5 + e \le 2$. Since any two points of $\mathbb{P}^5$ are contained in a line, we get $h^0(\mathcal{I}_{Y\cup L}(4)) = 0$ for a general line L. Hence Lemma 15 is true for the pair (t, a). Therefore it is sufficient to prove Lemma 15 for all pairs (t, a) with $t > 0$, $a > 0$ and $-2 \le e \le 4$. The quintuples (t, a, e, u, v) are the following ones: $(1, 20, 1, 13, 4)$, $(2, 19, 2, 13, 3)$, $(3, 18, 3, 13, 2)$, $(4, 17, 4, 13, 1)$, $(4, 18, -2, 13, 1)$, $(5, 17, -1, 13, 0)$, $(6, 16, 0, 12, 4)$, $(7, 15, 1, 12, 3)$, $(8, 14, 2, 12, 2)$, $(9, 13, 3, 12, 1)$, $(10, 12, 4, 12, 0)$, $(10, 13, -2, 12, 0)$, $(11, 12, -1, 11, 4)$, $(12, 11, 0, 11, 3)$, $(13, 10, 1, 11, 2)$, $(14, 9, 2, 11, 1)$, $(15, 8, 3, 11, 0)$, $(16, 7, 4, 10, 4)$, $(16, 8, -2, 10, 4)$, $(17, 7, -1, 10, 3)$, $(18, 6, 0, 10, 2)$, $(19, 5, 1, 10, 1)$, $(20, 4, 2, 10, 0)$, $(21, 3, 3, 9, 4)$, $(22, 2, 4, 9, 3)$, $(22, 3, -2, 9, 3)$, $(23, 2, -1, 9, 1)$.

Let $H \subset \mathbb{P}^5$ be a hyperplane. For any positive integer x let $S_x \cup S_y \subset H$ be general subsets of H with $\sharp(S_x) = x$, $\sharp(S'_y) = y$ and $S_x \cap S'_y = \emptyset$. Set $A_x := \cup_{O\in S_x} 2O$, $B_y := (\cup_{O\in S'_y} 2O$ and $B'_y := H \cap B_y$.

(a) Assume $1 \le t \le 10$. In all cases we have $a \ge u + v$. Let $Y \subset \mathbb{P}^5$ be a general union of t lines and $a-u-v$ 2-point. In all cases we have $h^1(\mathcal{I}_{Y\cup B_v}(3)) = 0$ by Lemma 14, because $u \ge v$, $u \ge v-e$ if $e < 0$ and if $t = 1, 2$, then $a-u < 8$. Hence $h^1(\mathcal{I}_{Y\cup B'_v}(3)) = 0$. In all cases we have $h^0(\mathcal{I}_Y(2)) = 0$. Lemma 4 gives that either $h^0(\mathcal{I}_{Y\cup B'_v\cup S_u}(3)) = 0$ (case $e \le 0$) or $h^1(\mathcal{I}_{Y\cup B'_v\cup S_u}(3)) = 0$ (case $e \ge 0$). The Differential Horace Lemma (Lemma 2) gives all these cases.

(b) Assume $(t, a, e) = (11, 12, -1)$. Let $Y \subset \mathbb{P}^5$ be a general union of 11 lines. We have $h^1(\mathcal{I}_Y(3)) = 0$, $h^0(\mathcal{I}_Y(2)) = 0$ and hence $h^i(\mathcal{I}_{Y\cup S_{12}}(3)) = 0$. We have $h^1(H, \mathcal{I}_{A_{12}\cap H}(4)) = 0$, i.e., $h^0(H, \mathcal{I}_{A_{12}\cap H}(4)) = 10$. Therefore we obtain $h^0(H, \mathcal{I}_{(Y\cup A_{12})\cap H}(4)) = 0$. Hence $h^0(\mathcal{I}_{Y\cup A_{12}}(4)) = 0$.

(c) Assume $11 \le t \le 24$ and $(t, a, e) \ne (11, 12, -1)$. Let f be the minimal integer such that $t + 4f + 5a \ge 70$. Set $f' := t + 4f + 5a - 70$. The quadruples (t, a, e, f, f') are the following ones: $(23, 2, -1, 10, 3)$, $(23, 1, 5, 11, 4)$, $(22, 3, -2, 9, 3)$, $(22, 2, 4, 10, 2)$, $(21, 3, 3, 9, 2)$, $(20, 4, 2, 8, 2)$, $(19, 5, 1, 7, 2)$, $(18, 6, 0, 6, 2)$, $(17, 7, -1, 5, 2)$, $(17, 6, 5, 6, 1)$, $(16, 8, -2, 4, 2)$, $(16, 7, 4, 5, 1)$, $(15, 8, 3, 4, 1))$, $(14, 9, 2, 3, 1)$, $(13, 10, 1, 2, 1)$, $(12, 11, 0, 1, 1)$, $(11, 11, 5, 1, 0)$. In all cases we have $f' \le f$ and $t \ge f + f'$ and $t - f \ge 5$. Let $Y \subset \mathbb{P}^5$ be a general union of $t-f-f'$ lines and f' +lines with nilradical supported by a general point of H. Since $f \ge f'$ and E_f is a general union of f' lines of H, we may assume that each +line of Y meets H in a line of E_f so that $Y \cup E_f$ is a disjoint union of $t-2f'$ lines and f' sundials. In all cases we have $h^1(H, \mathcal{I}_{E_f\cup(A_a\cap H)}(4)) = 0$ by Proposition 7. Since $t-f-f'+5f+u_f+v_f = 70$, we get $h^i(H, \mathcal{I}_{(Y\cup E_f\cup A_a)\cap H}(4)) = 0$, $i = 0, 1$. Lemma 6 gives $h^1(\mathcal{I}_Y(3)) = 0$. Since $t - f \geq 5$, we have $h^0(\mathcal{I}_Y(2)) = 0$ ([18]). Lemma 4 gives that either $h^0(\mathcal{I}_{Y\cup S_a}(3)) = 0$ (case $e \leq 0$) or $h^1(\mathcal{I}_{Y\cup S_a}(3)) = 0$ (case $e \geq 0$). Therefore either $h^0(\mathcal{I}_{Y\cup E_f\cup A_a}(k)) = 0$ or $h^1(\mathcal{I}_{Y\cup E_f\cup A_a}(k)) = 0$. Since $Y \cup E_f \cup A_a$ is a disjoint union of a 2-points, $t-2f'$ lines and f' sundials, the lemma is true in these cases. $\boxed{QED}$

Proof of Theorem 1 for $r = 5$:

By Lemmas 14 and 15 we may assume $k \geq 5$. Let $H \subset \mathbb{P}^5$ be a hyperplane.

(a) In this step we assume $(k + 1)t + 6a \leq \binom{k+5}{5}$. Set $\delta' := \binom{k+5}{5} - (k + 1)t - 6a$. Increasing if necessary a we may assume that $0 \leq \delta' \leq 5$. Set $u := \lfloor (\binom{k+4}{4} - t)/5 \rfloor$ and $v := \binom{k+4}{4} - t - 5u$. Notice that $0 \leq v \leq 4$. See step (a4) for the case $k = 5$.

(a1) Here we assume $a \geq u + v$.

Claim 1: $u + 5v \leq \binom{k+3}{4} - t$.

Proof of Claim 1: Assume $u+5v \geq \binom{k+3}{4} - t+1$. Since $5u+v = \binom{k+4}{4} - t$, we get $4u - 4v \leq \binom{k+3}{3}$, i.e. $u \leq v + \binom{k+3}{3}/4$. Since $a \geq u+v$, we have $6u+6v \leq \binom{k+5}{5} - \delta' - (k+1)t = \binom{k+5}{5} - \delta' + (k+1)5u + (k+1)v - (k+1)\binom{k+4}{4}$, i.e $(k+1)\binom{k+4}{4} - \binom{k+5}{5} + \delta' \leq u(5k-1) + v(k-5)$. Hence $(k+1)\binom{k+4}{4} - \binom{k+5}{5} \leq (5k-1)\binom{k+3}{3}/4 + v(4k+4)$. Since $v \leq 4$, we get $(k+1)\binom{k+4}{4} - \binom{k+5}{5} \leq (5k-1)\binom{k+3}{3}/4 + 4(3k+4)$. Hence $(k+4)(k+3)(k+2)(k+1)k/30 \leq (5k-1)(k+3)(k+2)(k+1)/24 + 4(3k+4)$, which is false for all $k \geq 6$.

Fix a general $Y \in Z(5, t, a-u-v)$. Fix a general $S \cup S' \subset H$ with $\sharp(S) = u$, $\sharp(S') = v$ and $S \cap S' = \emptyset$. Let $A \subset \mathbb{P}^r$ (resp. A') be the union of the 2-points of $\mathbb{P}^r$ with S (resp. S') as its support. Let $B \subset H$ (resp. $B' \subset H$) be the union of the 2-points of H supported by the points of S (resp. S').

Claim 2: We have $u \geq v$.

Proof Claim 2: Since $5u + v = \binom{k+4}{4} - t$, Claim 2 follows from Claim 1.

Claim 3: We have $h^i(H, \mathcal{I}_{(Y\cap H)\cup B\cup S'}(k)) = 0$, $i = 0, 1$.

Proof of Claim 3: The definitions of the integers u and v are done to get $h^0(\mathcal{O}_{(Y\cap H)\cup B\cup S'}(k)) = \binom{k+4}{4}$. Hence to prove Claim 3 it is sufficient to prove that $h^1(H, \mathcal{I}_{(Y\cap H)\cup B\cup S'}(k)) = 0$. We first check that $h^1(H, \mathcal{I}_{(Y\cap H)\cup B}(k)) = 0$. Since $h^0(\mathcal{O}_{(Y\cap H)\cup B}(k)) = \binom{k+4}{4} - v - \delta' \leq \binom{k+4}{4}$ and $Y \cap H$ is a general subset of H with cardinality t, it is sufficient to prove that $h^1(H, \mathcal{I}_B(k)) = 0$. This is true by the Alexander-Hirschowitz theorem. Lemma 5 gives $h^1(\mathcal{I}_{Y\cup B'}(k-1)) \leq h^1(\mathcal{I}_{Y\cup A'}(k-1))$. Since $\sharp(S') \leq 4$, S' is general in H and any 4 points of $\mathbb{P}^5$ are contained in a hyperplane, $Y \cup A'$ may be considered as a general union of t lines and $a-u$ 2-points. Let z be the maximal integer such that $kz+5v \leq \binom{k+4}{4}$. Since $u \geq v$ (Claim 2) we have $z \geq t$. Hence we have $h^1(\mathcal{I}_{Y\cup A'}(k-1)) = 0$ by the inductive assumption on k. Therefore $h^1(\mathcal{I}_{Y\cup B'}(k-1)) = 0$. By Lemma 4 to prove Claim 3 it is sufficient to prove that $h^0(\mathcal{I}_Y(k-2)) = 0$. Since $k \ge 5$, the inductive assumption on k and Lemmas 14 and 15 give that either $h^0(\mathcal{I}_Y(k-2)) = 0$ or $h^1(\mathcal{I}_Y(k-2)) = 0$. Assume $h^1(\mathcal{I}_Y(k-2)) = 0$. By Claim 3 we have $h^1(H, \mathcal{I}_{(Y\cap H)\cup S\cup B'}(k-1)) = 0$. Since $u \ge v$, we have $5u+v \ge u+5v$. Hence the inductive assumption on k, the relation $t+5u+v = \binom{k+4}{4}$, the theorem in $\mathbb{P}^4$ and the Castelnuovo's sequence

$$0 \to \mathcal{I}_Y(k-2) \to \mathcal{I}_{Y\cup S\cup B'}(k-1) \to \mathcal{I}_{Y\cup S\cup B',H}(k-1) \to 0$$

give $h^1(\mathcal{I}_{Y\cup S\cup B'}(k-1)) = 0$.

(a2) Now assume $\lfloor k/5 \rfloor + 8 \le a < u+v$. Let f be the minimal integer such that $t - f + (k+1)f + 5u' + v' = \binom{k+4}{4}$ (i.e. $t + kf + 5u' + v' = \binom{k+4}{4}$) for some integers u', v' with $u' \ge 0$, $0 \le v' \le 4$ and $u'+v' \le a$. There is such an integer by Remark 3. We have

$$k(t-f) + 6(a-u'-v') + u' + 5v' = \binom{k+4}{5} - \delta'. \tag{11}$$

We have $a - u' \le \lfloor k/5 \rfloor + 4$ (Remark 3). Since $a - u' \le \lfloor k/5 \rfloor + 4$ and $a \ge \lfloor k/5 \rfloor + 8$, we have $u' \ge 4$. Hence $u' \ge v'$.

Claim 4: We have $f \le t$.

Proof of Claim 4: Assume $f \ge t+1$. We get $-k+(k+1)t+5u'+v' \ge \binom{k+4}{4}$ with $0 \le v' \le 4$ and $a \ge u'+v'$. Since $\delta' \le 5$, (11) gives $\binom{k+4}{4} \le 5 + 6a - 5u' - v' - k$, i.e., $6a - 5u' - v' \ge \binom{k+4}{4} + k - 5$. Since $a - u' \le 4 + \lfloor k/5 \rfloor$, we get $u' + 24 + 6k/5 + 5 - k \ge \binom{k+4}{4}$. Hence $5u' \ge 5\binom{k+4}{4} - k - 145$. Since $5u' \le \binom{k+4}{4}$ and $k \ge 6$, we get a contradiction.

Fix a general $S \cup S' \subset H$ with $\sharp(S) = u'$, $\sharp(S') = v'$ and $S \cap S' = \emptyset$. Let $E \subset H$ be a general union of f lines. Let $A \subset \mathbb{P}^5$ (resp. A') be the union of the 2-points of $\mathbb{P}^5$ with S (resp. S') as its support. Let $B \subset H$ (resp. $B' \subset H$) be the union of the 2-points of H supported by the points of S (resp. S').

Claim 5: We have $h^1(\mathcal{I}_{Y\cup B'}(k-1)) = h^1(\mathcal{I}_{Y\cup A'}(k-1)) = 0$.

Proof of Claim 5: Since B' is a union of connected components of $Y \cup B$, A' is zero-dimensional and $A' \supseteq B'$, to prove Claim 5 it is sufficient to prove that $h^1(\mathcal{I}_{Y\cup A'}(k-1)) = 0$. Since S' is general in H and any 4 points of $\mathbb{P}^5$ are contained in a hyperplane, $Y \cup A'$ may be considered as a general element of $Z(5, t-f, a-u')$. We have $k(t-f)+6(a-u') = \binom{r+k-1}{r} - \delta' - u' + v' \le \binom{k+4}{5} - \delta'$ by (11). Hence we may use the inductive assumption on k; the case $k=5$ is done below in step (a4), but it needs the cases $k=3$ and $k=4$ done in Lemmas 14 and 15.

Claim 6: We have $h^i(H, \mathcal{I}_{(Y\cap H)\cup S'\cup(A\cap H)\cup E}(k)) = 0$, $i = 0, 1$.

Proof of Claim 6: Since $h^0(\mathcal{O}_{(Y\cap H)\cup S'\cup(A\cap H)\cup E}(k)) = \binom{k+4}{4}$, $Y\cap H$ is formed by $t-f$ general points of H and S' is general, it is sufficient to use the inductive assumption.

Since $h^1(\mathcal{I}_{Y\cup B'}(k-1)) = 0$ (Claim 5), we have $h^0(\mathcal{I}_{Y\cup B'}(k-1)) = u'+\delta' \ge u'$. As in step (a1) to prove Theorem 1 in the case (a2) it is sufficient to use Claims 5 and 6 and check that $h^0(\mathcal{I}_Y(k-2)) = 0$. We have $h^0(\mathcal{O}_Y(k-2)) = (k-1)(t-f)+6(a-u'-v') = \binom{k+4}{5} - (t-f) - u' - 5v' - \delta'$. By the inductive assumption either $h^0(\mathcal{I}_Y(k-2)) = 0$ or $h^1(\mathcal{I}_Y(k-2)) = 0$. Hence it is sufficient to prove that $h^0(\mathcal{O}_Y(k-2)) \ge \binom{k+3}{5}$, i.e. that $t-f+\delta'+u'+5v' \le \binom{k+3}{4}$. Assume that $t-f-1+\delta'+u'+5v' \ge \binom{k+3}{4}$. Since $t-f+5u'+v' = \binom{k+4}{4}$, $\delta' \ge 0$ and $u' \ge v'$, we get $-1+\delta' \ge \binom{k+3}{3}$, contradicting the inequality $\delta' \le 5$.

(a3) Now assume $a \le \lfloor k/5 \rfloor + 7$. Let f be the minimal integer such that $t-f+(k+1)f+5a \ge \binom{k+4}{4}$. Set $f' := t-f+(k+1)f+5a-\binom{k+4}{4}$. We have $0 \le f' \le k-1$ and

$$t-f-f'+(k+1)f+5a = \binom{k+4}{4}. \tag{12}$$

From (12) we get

$$(k+1)(t-f-f')+kf'+a = \binom{k+4}{5} - \delta'. \tag{13}$$

Claim 7: We have $f \ge f'$.

Proof of Claim 7: Assume $f \le f'-1$. We get $t-f'+1+k(f'-1)+5a \ge \binom{k+4}{4}$. Hence $t+36+(k-1)f' \ge \binom{k+4}{4}$. Hence $t+36+k(k-1) \ge \binom{k+4}{4}$. Hence $(k+1)t+(k-1)(k+1)(k-1)+36(k+1) \ge (k+1)\binom{k+4}{4}$. Since $\binom{k+5}{5}+36(k+1)+(k+1)(k-1)(k-1) < (k+1)\binom{k+4}{4}$, we get a contradiction.

Claim 8: We have $t-f-f' \ge 0$.

Proof of Claim 8: Since $(k+1)t+6a \ge \binom{k+5}{5}-5$, we have $f \le t$. Assume $t-f-f' \le -1$. Since $f' \le k-1$ we also get $f \ge t-k+2$. Hence (12) gives $-1+(k-2)(k+1)+(k+1)t+5a \le \binom{k+4}{4}$. Since $a \le k-1$, $k \ge 6$, we have $-1+(k-2)(k+1)+k+1 \le \binom{k+4}{r}-6$. Therefore $(k+1)t+6a < \binom{k+4}{4}-5$, a contradiction.

The inequalities $f \ge f'$ and $t \ge f+f'$ allows everybody to copy the proof given in $\mathbb{P}^4$.

(a4) Now assume $k=5$. Increasing or decreasing a we reduce to the case $6t+6a = \binom{10}{5} = 256$ (i.e. to the cases $t+a=42$), even without the assumption “ $(k+1)t+6a \le \binom{k+5}{5}$ ”. We may assume $t>0$ and $a>0$, i.e. $1 \le t \le 41$ and $a = 42-t$.

First assume $1 \le t \le 21$. We have the following quadruples (t,a,u,v): $(1,41,25,0)$, $(2,40,24,4)$, $(3,39,24,3)$, $(4,38,24,2)$, $(5,37,24,1)$, $(6,36,24,0)$,

$(7,35,23,4)$, $(8,34,23,3)$, $(9,33,23,2)$, $(10,32,23,1)$, $(11,31,23,0)$, $(12,30,22,4)$, $(13,29,22,3)$, $(14,28,22,2)$, $(15,27,24,1)$, $(16,26,22,0)$, $(17,25,21,4)$, $(18,24,21,3)$, $(19,23,21,2)$, $(20,22,21,1)$, $(21,21,21,0)$.

In all cases we have $a \ge u+v$ and $u \ge v$.

Now assume $22 \le t \le 36$. Since $t+5f+5u'+v' = 126$, v' is the only integer such that $0 \le v' \le 4$ and $v' \equiv 126-t \pmod 5$, while $u'+f = (126-t-v')/5$. The quintuples (t,a,f,u',v') are the following ones:
$(22,20,4,16,4)$, $(23,19,4,16,3)$, $(24,18,4,16,2)$, $(25,17,4,16,1)$, $(26,16,4,16,0)$, $(27,15,8,11,4)$, $(28,14,8,11,3)$, $(29,13,8,11,2)$, $(30,12,8,11,1)$, $(31,11,8,11,0)$, $(32,10,12,6,4)$, $(33,9,12,6,3)$, $(34,8,12,6,2)$, $(35,7,12,6,1)$, $(36,6,12,6,0)$. We always have $t \ge f$ and $u \ge v$.

Now assume $37 \le t \le 41$. The quadruples (t,a,f,f') are the following ones: $(37,5,13,1)$, $(38,4,14,2)$, $(39,3,15,0)$, $(40,2,16,4)$, $(41,1,16,0)$. In all cases we have $f \ge f'$ and $t \ge f+f'$.

(b) In this step we assume $(k+1)t+(r+1)a > \binom{k+5}{5}$. Set $\delta := (k+1)t+6a-\binom{k+5}{5}$. Decreasing a if necessary we reduce to the case $\delta \le 5$. Hence by step (a4) we may assume $k \ge 6$. By [18] we may assume $a > 0$. By part (a) we have $h^1(\mathcal{I}_W(k)) = 0$ (i.e. $h^0(\mathcal{I}_W(k)) = 6-\delta$) for a general $W \in Z(5,t,a-1)$. Since a 2-point contains a point, we get $h^0(\mathcal{I}_X(k)) = 0$ if $\delta = 5$. Hence we may assume $1 \le \delta \le 4$. A general 2-point contains a general tangent vector. Hence in characteristic zero we may even get for free the case $\delta = 4$. As in the case of $\mathbb{P}^4$ we need to check the inequalities considered in steps (a1), (a2) and (a3) and, in (a1), that $u \ge v+\delta$, in (a2) that $u' \ge v'+\delta$.

(b1) Assume $a \ge u+v$. Claims 1, 2, 3 are true (and easier) using $-\delta$ instead of δ'. See step (b2) below for a proof (taking $f = 0$) of the inequality $u \ge v+\delta$.

(b2) Assume $\lfloor k/5 \rfloor + 8 \le a < u+v$. Claims 4 and 5 OK (and easier) with $-4 \le \delta \le -1$ instead of $\delta' \le 5$. As in the case of $\mathbb{P}^4$ to get Claim 6 it is sufficient to prove that $u' \ge v'+\delta$. Assume $u' \le v'+\delta-1$. Since $u' \ge 4$, we get $5-v' \le \delta$. Let $S_1 \subset H$ be a general set with $\sharp(S_1) = u+1$. Set $A_1 := \cup_{O \in S_1} 2O$. Fix a general $Y_1 \in Z(5,t-f,a-u-1)$. It is sufficient to prove that $h^0(\mathcal{I}_{Y_1 \cup A_1}(k)) = 0$. Hence it is sufficient to prove that $h^0(\mathcal{I}_{Y_1 \cup S_1}(k-1)) = 0$ and that $h^0(H, \mathcal{I}_{(Y_1 \cup A_1) \cap H}(k)) = 0$. The latter vanishing is true by the theorem in $\mathbb{P}^4$, because $\sharp(Y_1)+5(u'+1) = t-f+5u'+5 > t-f+5u'+v' = \binom{k+4}{4}$. The former vanishing is true for the following reasons. We checked that $h^0(\mathcal{I}_{Y_1}(k-2)) = 0$. Since S_1 is general in H, by the inductive assumption on k it is sufficient to prove that $h^0(\mathcal{O}_{Y_1}(k-1)) + u'+1 \ge \binom{k+4}{5}$. We have $h^0(\mathcal{O}_{Y_1}(k-1)) + u'+1 = \binom{k+4}{5} + \delta - (5-v') \ge \binom{k+4}{5}$.

(b3) This easy step is similar to step (a3). It is sufficient to use δ instead of $-\delta'$ in (13). $\boxed{QED}$

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