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Separation of unitary representations of certain Cartan motion groups

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Abstract. Let G be a connected semisimple Lie group with finite center, K a maximal compact connected subgroup of G and G_0 the Cartan motion group associated to the Riemannian symmetric pair (G, K). Under two assumptions on the pair (G, K), we show that every irreducible unitary representation of G_0 is characterized by a single element in its generalized moment set.

Keywords: Cartan motion group, unitary representation, moment set

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Introduction

Let G be a Lie group with Lie algebra $\mathfrak{g}$. An interesting problem in harmonic analysis is to give a concrete description of the unitary dual $\widehat{G}$ of G, consisting of all equivalence classes of irreducible unitary representations of G. For several classes of Lie groups, such a description is obtained by using coadjoint orbits of the group in the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$. For example, if G is an exponential solvable Lie group, it is well known that the unitary dual $\widehat{G}$ is realized as the space $\mathfrak{g}^*/G$ of G-coadjoint orbits (see [5]).

Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of G and $\mathcal{H}_\pi^\infty$ the space of smooth vectors of π. In [11], N. Wildberger defined the moment map Ψ_π of π. For all $\xi \in \mathcal{H}_\pi^\infty \setminus \{0\}$ and X in $\mathfrak{g}$,

$$\Psi_\pi(\xi)(X) := \frac{1}{i} \frac{\langle d\pi(X)\xi, \xi\rangle}{\langle \xi, \xi\rangle},$$

where $d\pi$ is the derived representation. The moment set I_π of π is by definition the closure in $\mathfrak{g}^*$ of the image of the moment map $\Psi_\pi : \mathcal{H}_\pi^\infty \setminus \{0\} \longrightarrow \mathfrak{g}^*$. As shown in [11], the map $I : \widehat{G} \longrightarrow \mathcal{P}(\mathfrak{g}^*)$ which associates to π its moment set I_π is not necessarily injective even for a nilpotent connected simply connected Lie group. Therefore, the map I does not serve as a description of $\widehat{G}$. In order to obtain an injective map on $\widehat{G}$, A. Baklouti, J. Ludwig and M. Selmi extended the moment map to the dual of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ as follows: For all $A \in \mathcal{U}(\mathfrak{g}^{\mathbb{C}})$ and $\xi \in \mathcal{H}_\pi^\infty \setminus \{0\}$,

$$\widetilde{\Psi}_\pi(\xi)(A) := \mathfrak{Re}\Big(\frac{1}{i}\frac{\langle d\pi(A)\xi, \xi\rangle}{\langle \xi, \xi\rangle}\Big),$$

and considered the convex hull $J(\pi)$ of the image of this generalized moment map $\widetilde{\Psi}_\pi$:

$$J(\pi) := Conv(\widetilde{\Psi}_\pi(\mathcal{H}_\pi^\infty \setminus \{0\})).$$

Let $\mathcal{U}_n$ be the subspace of $\mathcal{U}(\mathfrak{g})$ consisting of elements of degree less or equal to n. By restriction to $\mathcal{U}_n$, one can define

$$J^n(\pi) := J(\pi)|_{\mathcal{U}_n} = \big\{F|_{\mathcal{U}_n};\ F \in J(\pi)\big\}.$$

In [4], A. Baklouti, J. Ludwig and M. Selmi shown that for all nilpotent Lie group, there exists an integer n such that, for any irreducible unitary representations π and ρ of G, we have

$$\pi \simeq \rho \Longleftrightarrow J^n(\pi) = J^n(\rho).$$

Later on, in [2], they shown with D. Arnal the following result:

Theorem A. (Separation of unitary representations of exponential Lie groups) Let $G = exp(\mathfrak{g})$ be an exponential Lie group. Let π and ρ be two irreducible unitary representations of G. Then

$$\pi \simeq \rho \Longleftrightarrow J(\pi) = J(\rho).$$

The injectivity of the map J is ultimately proved in [1] for any connected Lie group by using general and sophisticated analytic arguments.

Let now G be a connected semisimple Lie group with finite center and K a maximal compact connected subgroup of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with $\mathfrak{k} = Lie(K)$. We can form the semidirect product $G_0 = K \ltimes \mathfrak{p}$ with respect to the adjoint action of K on $\mathfrak{p}$. The group G_0 is called the Cartan motion group associated to the pair (G, K). In this note, it is assumed that the Riemannian symmetric pair (G, K) has rank one. Furthermore, if $\mathfrak{a}$ is a fixed maximal abelian subspace of $\mathfrak{p}$, then we shall assume that the centralizer M of $\mathfrak{a}$ in K is connected. Our purpose is to give a simple and effective way to separate the irreducible unitary representations of G_0. More precisely, for any irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G_0, we associate a special vector ξ_π in the space $\mathcal{H}_\pi^\infty \setminus \{0\}$ and we show the following result:

Theorem B. Let π and ρ be two irreducible unitary representations of G_0. Then

$$\pi \simeq \rho \Longleftrightarrow \widetilde{\Psi}_\pi(\xi_\pi) = \widetilde{\Psi}_\rho(\xi_\rho).$$

1 Notation and preliminaries

Let G be a connected semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. We fix a maximal compact connected subgroup K of G and denote by θ the corresponding Cartan involution. Let B be the Killing form of $\mathfrak{g}$. For $X \in \mathfrak{g}$, we put $\|X\|^2 := -B(X, \theta X)$. Notice that $\|\cdot\|$ is a norm on the Lie algebra $\mathfrak{g}$. Setting $\mathfrak{p} := \{X \in \mathfrak{g};\ \theta X = -X\}$, we obtain the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = Lie(K)$. It is easy to see that the vector space $\mathfrak{p}$ is $Ad(K)$-invariant. The semidirect product $G_0 = K \ltimes \mathfrak{p}$ with respect to the adjoint action of K on $\mathfrak{p}$ is called the Cartan motion group of the pair (G, K). The multiplication rule in this group is given by

$$(k_1, X_1) \cdot (k_2, X_2) = (k_1 k_2, X_1 + Ad(k_1) X_2).$$

The corresponding Lie algebra of G_0 is denoted by $\mathfrak{g}_0$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. The dimension of the real vector space $\mathfrak{a}$ is called the rank of the Riemannian symmetric pair (G, K). Let $C^+(\mathfrak{a})$ be a fixed positive Weyl chamber in $\mathfrak{a}$. An important fact worth mentionning here is that every adjoint orbit of K in $\mathfrak{p}$ intersects the closure $\overline{C^+(\mathfrak{a})}$ in exactly one point. A proof of this fact can be found in the standard reference [6].

In the remainder of this note, we shall restrict ourselves to the case where the Riemannian symmetric pair (G, K) has rank one. In this case, we can find a unit vector $H_0 \in \mathfrak{a}$ such that $C^+(\mathfrak{a}) = \mathbb{R}_+^* H_0$. Furthermore, we shall assume that the stabilizer $M = \{k \in K;\ Ad(k)H_0 = H_0\}$ is connected.

2 Irreducible unitary representations of G_0

In the notation introduced above, we summarize the description of the unitary dual of the Cartan motion group $G_0 = K \ltimes \mathfrak{p}$ via Mackey's little group theory (see [8, 9]).

Let φ be a non-zero linear form on $\mathfrak{p}$. We denote by χ_φ the unitary character of the vector Lie group $\mathfrak{p}$ given by $\chi_\varphi = e^{i\varphi}$. We define the little group S_φ at φ to be the stabilizer of φ in K. Let σ be an irreducible unitary representation of S_φ on some vector space W. The map

$$\sigma \otimes \chi_\varphi : (k, X) \longmapsto e^{i\varphi(X)}\sigma(k)$$

is a representation of the semidirect product $S_\varphi \ltimes \mathfrak{p}$. Let $L^2(K, W)$ be the completion of the vector space of all continuous maps $\eta : K \longrightarrow W$ with respect to the norm

$$\|\eta\| = \Big(\int_K \|\eta(k)\|^2 dk \Big)^{\frac{1}{2}},$$

where dk is a normalized Haar measure on K. Define $L^2(K, W)^\sigma$ to be the subspace of $L^2(K, W)$ consisting of the maps ξ which satisfy the covariance condition

$$\xi(kh) = \sigma(h^{-1})\xi(k)$$

for $h \in S_\varphi$ and $k \in K$. The induced representation

$$\pi_{(\sigma,\chi_\varphi)} := Ind_{S_\varphi \ltimes \mathfrak{p}}^{G_0}(\sigma \otimes \chi_\varphi)$$

is realized on $L^2(K, W)^\sigma$ by

$$\pi_{(\sigma,\chi_\varphi)}((k, X))\xi(h) = e^{i\varphi(Ad(h^{-1})X)}\xi(k^{-1}h),$$

where $(k, X) \in G_0$, $\xi \in L^2(K, W)^\sigma$ and $h \in K$. By Mackey's theory, we know that the representation $\pi_{(\sigma,\chi_\varphi)}$ is irreducible and that every infinite dimensional irreducible unitary representation of G_0 is equivalent to some $\pi_{(\sigma,\chi_\varphi)}$. Furthermore, two representations $\pi_{(\sigma,\chi_\varphi)}$ and $\pi_{(\sigma',\chi_{\varphi'})}$ are equivalent if and only if φ and φ' belong to the same sphere centered at 0 and the representations σ and σ' are equivalent under the identification of the conjugate subgroups S_φ and $S_{\varphi'}$. In this way, we obtain all irreducible representations of G_0 which are not trivial on the normal subgroup $\mathfrak{p}$. On the other hand, every irreducible unitary representation τ of K extends trivially to an irreducible representation, also denoted by τ, of G_0 by $\tau(k, X) := \tau(k)$ for $k \in K$ and $X \in \mathfrak{p}$.

For $r \in \mathbb{R}_+^*$, we denote by χ_r the character associated with the linear form φ_r on $\mathfrak{p}$ which is defined by

$$\varphi_r(X) := rB(H_0, X).$$

The stabilizer S_{φ_r} of φ_r is the subgroup $M = Z_K(H_0)$. If σ_μ is an irreducible representation of M with highest weight μ, then we simply write $\pi_{(\mu,r)}$ instead of $\pi_{(\sigma_\mu,\chi_r)}$. From the obove description of $\widehat{G_0}$, we can state the following

Proposition 1. The unitary dual of G_0 is in bijection with the set

$$(\widehat{M} \times \mathbb{R}_+^*) \bigcup \widehat{K}.$$

Concluding this section, let us mention that $\widehat{G_0}$ has a complete orbital description. More precisely, Lipsman's orbit method tells us that $\widehat{G_0}$ is in bijection with the set of "admissible coadjoint orbits" of G_0 (see [7] for details).

3 Separation of irreducible unitary representations of G_0

We keep the notation of the previous section. Let us fix a positive real $r \in \mathbb{R}_+^*$ and take S and T to be maximal tori respectively in M and K such that $S \subset T$. Consider an irreducible unitary representation $\sigma_\mu : M \longrightarrow U(W)$ with highest weight μ. Then

$$\pi_{(\mu,r)} = Ind_{M \ltimes \mathfrak{p}}^{G_0}(\sigma_\mu \otimes \chi_r)$$

is an (infinite-dimensional!) irreducible unitary representation of G_0. Recall that $\pi_{(\mu,r)}$ is realized on the Hilbert space $\mathcal{H}_{\mu,r} := L^2(K,W)^{\sigma_\mu}$. Let for each $\gamma \in \widehat{K}$, (τ_γ, W'_γ) be a fixed representative. An application of the Peter-Weyl theorem (see, e.g., [10]) yields

$$\mathcal{H}_{\mu,r} \cong \widehat{\bigoplus_{\gamma \in \widehat{K}}} W'_\gamma \otimes Hom_M(W'_\gamma, W).$$

Now, we fix an irreducible unitary representation $\tau_\mu : K \longrightarrow U(W')$ with highest weight μ and we realize the representation space W of σ_μ as the smallest M-invariant subspace of W' that contains the μ-weight space of W'. Choosing a normalized highest weight vector w_μ in W' and an orthonormal basis $\{w_j\}_{j=1,\dots,d}$ of W, we define a smooth function $\xi_{\mu,r} \in C^\infty(K,W)$ by

$$\xi_{\mu,r}(k) := \left(\frac{d'}{d}\right)^{\frac{1}{2}} \sum_{j=1}^{d} \langle w_\mu, \tau_\mu(k) w_j \rangle w_j,$$

where $d' = dim(W')$. One easily verify that $\xi_{\mu,r}$ is a smooth norm-one vector of the representation $\pi_{(\mu,r)}$.

Remark. Define a linear form $\theta_\mu \in \mathfrak{k}^*$ by

$$\theta_\mu(A) := -i\langle d\tau_\mu(A) w_\mu, w_\mu \rangle$$

for all $A \in \mathfrak{k}$. If we set $\phi_{\mu,r} := (\theta_\mu, \varphi_r)$, then we can see that the stabilizer $G_0(\phi_{\mu,r})$ of $\phi_{\mu,r}$ in G_0 is equal to $G_0(\phi_{\mu,r}) = K(\phi_{\mu,r}) \ltimes \mathfrak{p}(\phi_{\mu,r})$. Hence, $\phi_{\mu,r}$ is aligned in the sense of Lipsman (see [7]). A linear functional $\phi \in \mathfrak{g}_0^*$ is called admissible, if there exists a unitary character χ of the connected component of $G_0(\phi)$, such that $d\chi = i\phi|_{\mathfrak{g}_0}$. Notice that the linear functional $\phi_{\mu,r}$ is admissible and so, according to Lipsman [7], the representation of G_0 obtained by holomorphic induction from $\phi_{\mu,r}$ is equivalent to the representation $\pi_{(\mu,r)}$.

To simplify notation, denote by $\widetilde{\Psi}_{\mu,r}$ the generalized moment map of the representation $\pi_{(\mu,r)}$.

Lemma 1. For all $(A, X) \in \mathfrak{g}_0$, we have:

$$\widetilde{\Psi}_{\mu,r}(\xi_{\mu,r})((A,X)) = -i\langle d\tau_\mu(A)w_\mu, w_\mu\rangle + \int_K \varphi_r(Ad(k^{-1})X)\langle \xi_{\mu,r}(k), \xi_{\mu,r}(k)\rangle dk.$$

Proof. For $(A, X) \in \mathfrak{g}_0$ and $h \in K$, we have:

$$\begin{aligned} d\pi_{\mu,r}((A,X))\xi_{\mu,r}(h) &= \frac{d}{dt}\Big|_{t=0} \pi_{\mu,r}((exp_K(tA), tX))\xi_{\mu,r}(h) \\ &= \frac{d}{dt}\Big|_{t=0} \Big\{ e^{it\varphi_r(Ad(h^{-1})X)}\xi_{\mu,r}(h) + \xi_{\mu,r}(exp_K(-tA)h) \Big\} \\ &= i\varphi_r(Ad(h^{-1})X)\xi_{\mu,r}(h) + (A.\xi_{\mu,r})(h). \end{aligned}$$

Using the Schur orthogonality relations for the compact Lie group K, we get the following equality:

$$\begin{aligned} \langle A.\xi_{\mu,r}, \xi_{\mu,r}\rangle &= \frac{d'}{d}\sum_{j=1}^{d}\int_K \langle \tau_\mu(k)w_j, w_\mu\rangle \overline{\langle \tau_\mu(k)w_j, d\tau_\mu(A)w_\mu\rangle} dk \\ &= \langle d\tau_\mu(A)w_\mu, w_\mu\rangle. \end{aligned}$$

Thus we have

$$\begin{aligned} \widetilde{\Psi}_{\mu,r}(\xi_{\mu,r})((A,X)) =& \Psi_{\mu,r}(\xi_{\mu,r})(A,X) \\ =& - i\langle d\tau_\mu(A)w_\mu, w_\mu\rangle + \int_K \varphi_r(Ad(k^{-1})X)\langle \xi_{\mu,r}(k), \xi_{\mu,r}(k)\rangle dk. \end{aligned}$$

□

Let us fix an orthonormal basis $\{X_1, ..., X_p\}$ of $\mathfrak{p}$ with respect to the scalar product $\langle\ ,\ \rangle_{\mathfrak{p}} := B\big|_{\mathfrak{p}\times\mathfrak{p}}$, and put

$$\Delta_{\mathfrak{p}} := -\sum_{j=1}^{p} X_j^2.$$

Obviously, $i\Delta_{\mathfrak{p}}$ is an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_0^{\mathbb{C}})$ of $\mathfrak{g}_0^{\mathbb{C}}$. Given a unit vector $\xi \in \mathcal{H}_{\mu,r}^{\infty} \setminus \{0\}$, we have

$$\begin{aligned} d\pi_{(\mu,r)}(\Delta_{\mathfrak{p}})\xi(h) &= \Big(\sum_{j=1}^{p}\langle Ad(h)H_0, X_j\rangle_{\mathfrak{p}}^2\Big) r^2\xi(h) \\ &= r^2\xi(h), \end{aligned}$$

and hence

$$\widetilde{\Psi}_{\mu,r}(\xi)(i\Delta_{\mathfrak{p}}) = r^2.$$

Definition 1. For $\pi \in \widehat{G_0}$, define

$$\xi_\pi := \begin{cases} \xi_{\mu,r} & \text{if } \pi \simeq \pi_{(\mu,r)}; \\ w_\lambda & \text{if } \pi \simeq \tau_\lambda. \end{cases}$$

Theorem 1. Let π and ρ be two irreducible unitary representations of G_0. Then

$$\pi \simeq \rho \Longleftrightarrow \widetilde{\Psi}_\pi(\xi_\pi) = \widetilde{\Psi}_\rho(\xi_\rho).$$

Proof. Assume that $\widetilde{\Psi}_\pi(\xi_\pi) = \widetilde{\Psi}_\rho(\xi_\rho)$. Since

$$\widetilde{\Psi}_\pi(\xi_\pi)(i\Delta_{\mathfrak{p}}) = \begin{cases} r^2 & \text{if } \pi \simeq \pi_{(\mu,r)}; \\ 0 & \text{if } \pi \simeq \tau_\lambda, \end{cases}$$

we conclude that the irreducible representation (π,ρ) of $G_0 \times G_0$ is unitarily equivalent to a representation either of type $(\tau_\lambda, \tau_{\lambda'})$ or of type $(\pi_{(\mu,r)}, \pi_{(\mu',r)})$.

Case 1. If $(\pi,\rho) \simeq (\tau_\lambda, \tau_{\lambda'})$, then

$$\widetilde{\Psi}_\pi(\xi_\pi) = \widetilde{\Psi}_\rho(\xi_\rho) \Leftrightarrow \theta_\lambda = \theta_{\lambda'},$$

and hence $\lambda = \lambda'$.

Case 2. If $(\pi,\rho) \simeq (\pi_{(\mu,r)}, \pi_{(\mu',r)})$, then we can write

$$\widetilde{\Psi}_{\mu,r}(\xi_{\mu,r})((A,0)) = \widetilde{\Psi}_{\mu',r}(\xi_{\mu',r})((A,0))$$

for all $A \in \mathfrak{k}$. This implies that $\theta_\mu(A) = \theta_{\mu'}(A)$ for all $A \in \mathfrak{k}$. Thus we get $\mu = \mu'$. □

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