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NASH HARDY FIELDS IN SEVERAL VARIABLES

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1. PRELIMINARIES

We recall some notions about signed places and valuations over ordered fileds.

Let K and L be ordered fields. We define the algebraic operations over $L \cup \{\pm\infty\}$ in the obvious manner. We have then the following definition:

Definition 1.1. [1] *An application $p\colon K \to L \cup \{\pm\infty\}$ is said to be a signed place if:*

1) $p(1) = 1$

2) $p(x + y) = p(x) + p(y)$

3) $p(x \cdot y) = p(x) \cdot p(y)$

for any $x, y \in K$ if all the terms are defined.

The set $A_p = \{x \in K : p(x) \in L\}$ turns out to be a valuation ring over K with maximal ideal $M_p = \{x \in K : p(x) = 0\}$.

We denote by $U(A_p)$ the set $A_p - M_p$.

Let ν be the valuation over K generated by A_p.

ν is a function from $K^* = K - \{0\}$ in the ordered group $\Gamma = K^*/U(A_p)$ with the following properties:

1) $\nu(x \cdot y) = \nu(x) + \nu(y) \quad \forall x, y \in K$

2) $\nu(x + y) \geq g.l.b.\{\nu(x), \nu(y)\}$ with the equality if $\nu(x) \neq \nu(y)$. ν can be defined also for $x = 0$ by extending Γ to $\overline{\Gamma} = \Gamma \cup \{\infty\}$ and defining $\nu(0) = \infty$.

Particularly, we obtain the following equivalences between the signed place p and the valuation ν:

a) $x \in A_p$ iff $p(x) \neq \pm\infty$ iff $\nu(x) \geq 0$

b) $x \in M_p$ iff $p(x) = 0$ iff $\nu(x) > 0$

c) $x \in U(A_p)$ iff $p(x) \neq 0$ and $p(x) \neq \pm\infty$ iff $\nu(x) = 0$

d) $x \in K - A_p$ iff $p(x) = \pm\infty$ iff $\nu(x) < 0$.

Let I be a set of indeces bijectively corresponding to the set of principal convex sub-groups H of $\Gamma (H \neq \{0\})$.

If we denote by H_σ the convex sub-group corresponding to the index $\sigma \in I$, we can define a total ordering over I by: $\sigma \leq \tau$ in I iff $H_\sigma \supseteq H_\tau$.

Definition 1.2. [5] *The order type of I is said to be the rank of the valuation ν.*

Remark. If I is finite the order type of I coincides with the number of its elements.

2. RANK OF HARDY FIELDS IN SEVERAL VARIABLES

We denote by C any smoothness category of real valued functions of n real variables [2]. Let $\overline{0} \in \overline{\mathbb{R}^n}$. $\overline{\mathbb{R}^n}$ is the one-point compactification of the euclidean n-space $\mathbb{R}^n$ to a point $\alpha \notin \mathbb{R}^n$. Let $\mathcal{F}_{\overline{0}}$ be any filter of subsets of $\overline{\mathbb{R}^n}$ with connected basis converging to $\overline{0}$, constitued by open subsets of $\mathbb{R}^n$.

$C(\mathcal{F}_{\overline{0}})$ is the ring of germs in $\overline{0}$ following $\mathcal{F}_{\overline{0}}$ of the C-functions with the pointwise defined operations.

If there is not ambiguity we use the same symbol f for the germ $[f]$ and the function $f \in [f]$. Moreover we denote by $\underline{x}$ the vector $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Definition 2.1. [4] *A sub-ring K of $C(\mathcal{F}_{\overline{0}})$ is said to be a C-Hardy field in several variables in $\overline{0}$ for $\mathcal{F}_{\overline{0}}$ if:*

a) K is a sub-field of $C(\mathcal{F}_{\overline{0}})$

b) $f \in K \rightarrow \dfrac{\partial f}{\partial x_i} = f_i \in K$, $i = 1, 2, \ldots, n$.

From now forward K will denote any C-Hardy field in several variables ordered in the usual manner.

Proposition 2.1. *For every $f \in K$ there exists $\lim\limits_{\substack{\underline{x} \to \overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) = l$ where $l \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.*

Moreover the function $p: K \rightarrow \overline{\mathbb{R}}$ defined by $p(f) = l$ turns out to be a signed place.

Proof. Let be $Q_f = \{q \in \mathbb{Q} : f \geq q \text{ in } K\}$ then:

or $Q_f = \mathbb{Q}$ that is $\lim\limits_{\substack{\underline{x} \to \overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) = +\infty$

or $Q_f = \emptyset$ that is $\lim\limits_{\substack{\underline{x} \to \overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) = -\infty$

or $Q_f \neq \mathbb{Q}$ and $Q_f \neq \emptyset$. In the last case Q_f is upper bounded in $\mathbb{R}$ and $\lim\limits_{\substack{\underline{x} \to \overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) = l$ where $l = l.u.b. Q_f$.

The last claim of the proposition follows obviously from the definition 1.1. and the operations over K.

We define now the rank and the rational rank of a C-Hardy field, see for example [6].

Definition 2.2. *The rank of a C-Hardy field K in several variables is the rank of the valuation ν over K generated by the signed place p defined in the proposition 2.1.*

Remark. If $f \in K$, by the equivalences between a signed place p and the corresponding valuation ν, we obtain in this case:

a) $\lim\limits_{\substack{\underline{x}\to\overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) \neq \pm\infty$ iff $\nu(f) \geq 0$

b) $\lim\limits_{\substack{\underline{x}\to\overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) = 0$ iff $\nu(f) > 0$

c) $\lim\limits_{\substack{\underline{x}\to\overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) = \pm\infty$ iff $\nu(f) < 0$

d) $\lim\limits_{\substack{\underline{x}\to\overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) \neq 0$ and $\lim\limits_{\substack{\underline{x}\to\overline{0} \\ \mathcal{F}_{\overline{0}}}} f(\underline{x}) \neq \pm\infty$ iff $\nu(f) = 0$.

If K, K' are C-Hardy fields, $K \subset K'$, and p, p' the corresponding signed places, we have obviously: $p = p'|_K$. Moreover the valuation ν over K is the restriction to K of the valuation ν' over K'. Particularly, if K' denotes the real closure of K (K' is a C-Hardy field [4]), $\nu(K^*)$ turns out to be a sub-group of the ordered vector space, over $\mathbb{Q}$, $\nu(K'^*)$. So, we can give the following definition:

Definition 2.3. *The rational rank of a C-Hardy field K in several variables is the dimension of the vector sub-space over $\mathbb{Q}$ generated by $\nu(K^*)$ in $\nu(K'^*)$ where K' is the real closure of K.*

Proposition 2.2. *Let K, K' be C-Hardy fields in several variables such that $K \subset K'$ and deg.tr. $K'/K = r$ is finite, than, the rank of K' is obtained from the rank of K adding, at most, r distinct indeces.*

Proof. Suppose the contrary. Then there exist n convex principal sub-groups $H_{i_1} \subset H_{i_2} \subset \ldots \subset H_{i_n}$ of $\nu(K'^*)$, $n \geq r+1$, such that $i_j \notin \varphi(I)$, $j = 1, 2, \ldots, n$ where φ is the order preserving canonical injection of I in I'. Let a_j be the generator of H_{ij} and $f_j \in K'^*$ be such that $\nu(f_j) = a_j$, $j = 1, 2, \ldots, n$.

By the hypothesis, there exist some $c's \in K$ such that: $\sum\limits_{l_1+\ldots+l_n=0}^{s} c_{l_1\ldots l_n} \cdot f_1^{l_1} \ldots f_n^{l_n} = 0$, with $s, l_1, \ldots, l_n \in N$.

For the properties of valuation we have:

$$\nu(c_{l_1\ldots l_n} \cdot f_1^{l_1} \ldots f_n^{l_n}) = \nu(c_{t_1\ldots t_n} \cdot f_1^{t_1} \ldots f_n^{t_n}),$$

then

$$\nu\left(\frac{c_{l_1\ldots l_n}}{c_{t_1\ldots t_n}}\right) = \nu\left(f_1^{t_1-l_1} \ldots f_n^{t_n-l_n}\right) = \sum_{j=1}^{n}(t_j - l_j) \cdot \nu(f_j).$$

Thus $\nu\left(\frac{c_{l_1\ldots l_n}}{c_{t_1\ldots t_n}}\right)$ turns out to be a generator of $H_{i_{\bar{j}}}$ with $\bar{j}=\max\{j : t_j - l_j \neq 0, j=1,2,\ldots,n\}$, which is a contradiction.

3. AN INDUCTIVE CONSTRUCTION OF NASH HARDY FIELDS IN SEVERAL VARIABLES

We recall some definitions [7].

Definition 3.1. *A semi-algebraic subset A of $\mathbf{R}^n$ is said to be a semi-algebraic cell iff it is inductively obtained in the following manner:*

1) if $A = \{a\}$, $a \in \mathbb{R}$ then A is a cell and $\dim.(A) = 0$*; if $A = (a,b)$, $a,b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, then A is a cell and* $\dim.(A) = 1$

2) let $A \subset \mathbf{R}^n$ be cell with $\dim.(A) = k$ *and $f: A \to \mathbb{R}$ be a semi-algebraic continuous function then its graph $\Gamma(f)$ is a cell and* $\dim.(\Gamma(f)) = k$

3) let $A \subseteq \mathbb{R}^n$ be a cell with $\dim.(A) = k$ *and $f: A \to \mathbb{R}$, $g: A \to \mathbb{R}$ be semi-algebraic continuous functions such that $f(\underline{x}) < g(\underline{x})$ $\forall \underline{x} \in A$ then the set $(f,g)_A = \{(\underline{x}, y) : \underline{x} \in A, f(\underline{x}) < y < g(\underline{x})\}$ is a cell and* $\dim.((f,g)_A) = k+1$.

Let $\mathcal{F}_1$ be the filter of $\mathbf{R}$ with the basis $\mathcal{B}_1 = \{(0, 1/n) : n \in N\}$. We denote by K_1 the Hardy field of germs of 1-variable rational functions in 0 for $\mathcal{F}_1$ and by $\overline{K}_1$ its real closure.

$\overline{K}_1$ is the field of germs in 0 for $\mathcal{F}_1$ of 1-variable Nash functions [3];

Let $I_1 = \{f \in \overline{K}_1 : f > 0, \nu(f) > 0\}$. We denote by $C_2(f, U)$ the cell $(0, f)_U$ where $f \in I_1$ and U is any element of $\mathcal{B}_1$ such that $f(x) > 0$ $\forall x \in U$. Let $C_2(f, U, m) = C_2(f, U) \cap B_2(0, 1/m)$ where $B_2(0, 1/m)$ is the open ball of $\mathbb{R}^2$ with center in 0 and radius $1/m$.

Proposition 3.1. $\mathcal{B}_2 = \{C_2(f, U, m) : f \in I_1, U \in \mathcal{B}_1$ *with* $f|_U > 0, m \in N\}$ *is an open connected basis for a filter $\mathcal{F}_2$ of $\mathbb{R}^2$ converging to 0.*

Proof. Let $C_2(f,U,m), C_2(g,V,n) \in \mathcal{B}_2$. Then: $C_2(f,U,m) \cap C_2(g,V,n) \supset C_2(h,W,s)$ where $h = \min\{f,g\}$ in $\overline{K}_1$, $s = \max\{m,n\}$ and W is any cell of $\mathcal{B}_1$ such that $f(x) - g(x)$ has constant sign $(>0, =0, <0)$ $\forall x \in W$.

Theorem 3.1. *The ring K_2 of germs in 0 following $\mathcal{F}_2$ of 2-variables rational functions turns out to be a Nash Hardy field.*

Proof. Let $P(x, y) \in R[x, y]$. We prove, by induction over the degree of y that $P(x, y)$ has constant sign in some set of $\mathcal{B}_2$. If $\deg_y \cdot P(x, y) = 0$ then $P(x, y) \equiv P(x)$ and its germ is in $\overline{K}_1$.

If $\deg_y \cdot P(x, y) = n$ then $\left(\frac{\partial P}{\partial y}\right)(x, y)$ has constant sign in some $V \in \mathcal{B}_2$. If $\left(\frac{\partial P}{\partial y}\right)(x, y) = 0 \ \forall (x, y) \in V$ then $P(x, y) \equiv P(x)$.

Otherwise the semi-algebraic sets $Z(P(x,y)) \cap V$, where $(Z(P(x,y))$ denotes the zero set of $P(x,y)$, has a finite number of connected, semi-agebraic components [1].

By the implicit function theorem for Nash functions, $Z(P(x, y)) \cap V$ is stratified in the graphs of a finite number of Nash functions $\alpha_1, \alpha_2, \ldots, \alpha_k$.

If 0 is a cluster point of $Z(P(x, y)) \cap V$ then we consider the α_i's defined over some cell of $\mathcal{B}_1$, belonging to I_1.

So $P(x, y)$ has constant sign over any $C \in \mathcal{B}_2$ such that: $C \subset \bigcap_i C_2(\alpha_i, U_i, m_i)$.

Thus any 2-variables rational function $f(x, y)$ has constant sign over a suitable set of $\mathcal{B}_2$.

K_2 is then a Nash Hardy field and its real closure $\overline{K}_2$ turns out to be the Hardy field of germs in 0 following $\mathcal{F}_2$ of 2-variables Nash functions.

Remark. rank $\overline{K}_2$ = rank $K_2 = 2$. In this case the principal convex sub-groups of $\nu(K_2^*)$ and $\nu(\overline{K}_2^*)$ are the sub-groups H_1 and H_2, $H_1 \subset H_2$, generated respectively by $\nu(x)$ and $\nu(y)$.

Inductively, we denote by K_n the Hardy field of germs in 0 following the filter $\mathcal{F}_n$ (with basis $\mathcal{B}_n$ converging to 0) of n-variables rational functions and by $\overline{K}_n$ its real closure. So we define:

$$I_n = \{f \in \overline{K}_n : f > 0, \nu(f) > 0\}.$$

Moreover $C_{n+1}(f, U)$ is the cell $(0, f)_U$ where $f \in I_n$, $U \in \mathcal{B}_n$ with $f|_U > 0$ and $C_{n+1}(f, U, m) = C_{n+1}(f, U) \cap B_{n+1}(0, 1/m)$ where $B_{n+1}(0, 1/m)$ is the open ball of $\mathbb{R}^{n+1}$.

As in the 2-variables case we can prove the following proposition:

Proposition 3.2. $\mathcal{B}_{n+1} = \{C_{n+1}(f, U, m) : f \in I_n, U \in \mathcal{B}_n$ *with* $f|_U > 0, m \in N\}$ *is an open connected basis for a filter* $\mathcal{F}_{n+1}$ *of* $\mathbb{R}^{n+1}$ *converging to* 0.

We obtain then the following theorem:

Theorem 3.2. *The ring* K_{n+1} *of germs in* 0 *following* $\mathcal{F}_{n+1}$ *of* $(n+1)$*-variables rational functions turns out to be a Nash Hardy field.*

Proof. Let $P(x_1, x_2, \ldots, x_n, y) = P(\underline{x}, y) \in R[\underline{x}, y]$. Then $P(\underline{x}, y) = \sum_{i=0}^{s} P_i(\underline{x}) y^i$.

We need to modify the proof of theorem 3.1. in the case $\left(\frac{\partial P}{\partial y}\right)(\underline{x}, y) = 0$ once we have stratified the set $Z(P(\underline{x}, y)) \cap V$ in the graphs of a finite number of Nash functions $\alpha_1(\underline{x}), \alpha_2(\underline{x}), \ldots, \alpha_k(\underline{x})$ and 0 is a cluster point of $\Gamma(\alpha_t)$ with dom. $(\alpha_t) \subset U \in \mathcal{B}_n$ for some $t \in \{1, 2, \ldots, k\}$.

Choosing a suitable $W \in \mathcal{B}_n$, $P(\underline{x}, y)$ can be considered as an element of $\overline{K}_n[y]$. So:

$$P(\underline{x}, y) = P_s(\underline{x}) \prod_{l=1}^{q} (y - \gamma_l(\underline{x})) \prod_{r=1}^{p} \left[(y + \beta_r(\underline{x}))^2 + \delta_r^2(\underline{x}) \right]$$

with $\gamma_l, \beta_r, \delta_r \in \overline{K}_n$ and $\delta_r \neq 0$, $x \in W$, $l = 1, 2, \ldots, q$ and $r = 1, 2, \ldots, p$.

Thus $P(\underline{x}, \alpha_t(\underline{x})) = 0$ $\forall x \in$dom. $(\alpha_t) \cap W$ if it is different from the empty set. So $\alpha_t(\underline{x}) = \gamma_l(\underline{x})$ $\forall \underline{x} \in$ dom. $(\alpha_t) \cap W$ for a certain $l \in \{1, 2, \ldots, q\}$. Then $\nu(\gamma_l) > 0$.

If $h = \min\{\gamma_l\colon\ l = 1, 2, \ldots, q\}$ in $\overline{K}_n$, then $P(\underline{x}, y)$ has constant sign over $C_{n+1}(h, W, m)$.

K_{n+1} is then a Nash Hardy field and its real closure $\overline{K}_{n+1}$ turns out to be the Hardy field of germs in 0 following $\mathcal{F}_{n+1}$ of $(n+1)$-variables Nash functions.

Then, utilizing inductively proposition 3.2. and theorem 3.1., we can construct from K_2 and $\overline{K}_2$ the Nash Hardy fields K_n and $\overline{K}_n$ for every $n \in N$.

Remark. rank $\overline{K}_n$ = rank $K_n = n$.

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