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CONFORMALLY FLAT IMMERSIONS *

F. MERCURI **

Dedicated to Prof. Diana Benincasa

1. INTRODUCTION

A conformally flat manifold (C.F. manifold for short) is a differentiable manifold together with an atlas whose change of coordinate are conformal diffeomorphisms. In terms of Riemannian geometry a C.F. manifold is a Riemannian manifold such that each point has an open neighborhood conformally diffeomorphic to an open set of an euclidean space. We will take always this point of view. Since space forms (i.e. spaces of constant curvature) of the same dimension are all locally conformally diffeomorphic, C.F. manifolds may be considered as the analogue of such spaces in the context of conformal geometry.

Examples of C.F. manifolds are, besides space forms, all two dimensional Riemannian manifolds (due to the existence of isotermal coordinates) and products of two space forms of curvature c_1 and c_2 with $c_1 + c_2 = 0$. We observe explicitly that the product of two space forms is not in general a C.F. manifold (see remarks below 2.3) so the category of C.F. manifolds is not closed under products.

In this paper we will study some of the local and global geometry of C.F. manifolds which can be isometrically immersed in euclidean space with low codimension.

2. CHARACTERIZATIONS OF C.F. MANIFOLDS

In this section we will describe some characterizations of C.F. manifolds in terms of Riemannian invariants.

For a Riemannian manifold M^n of dimension n we will denote, as usual, by ∇ the Levi-Civita connection and by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ the curvature tensor. If $\{X_1, \ldots, X_n\}$ is an orthonormal basis of the tangent space of M at x, T_xM, the operator $Q(X) = \Sigma R(X, X_i)X_i$ is the Ricci tensor and the quadratic form Ricci $(X) = \langle Q(X), X \rangle$ is the Ricci curvature whose trace S is the scalar curvature. We define an operator γ : $T_xM \to T_xM$ by

$$\gamma(X) = \frac{1}{(n-2)} \left[Q(X) - \frac{SX}{2(n-1)} \right] \tag{2.1}$$

The best known characterization of conformal flatness is probably the following:

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Theorem 2.2. *M^n is a C.F. manifold is and only if*

$$(i) \qquad R(X,Y) = \gamma(X) \wedge Y + X \wedge \gamma(Y),$$

$$(ii) \qquad (\nabla_X \gamma)(Y) = (\nabla_Y \gamma)(X)$$

Moreover, if $n \geq 4$, (i) $\Rightarrow$ (ii).

Remark. In the above we have identified the bi-vector $X \wedge Y$ with the antisymmetric map $(X \wedge Y)(Z) = \langle X, Z\rangle Y - \langle Y, Z\rangle X$.

A particularly useful characterization of C.F. manifolds in terms of the sectional curvature k, is the following:

Theorem 2.3. *If $n \geq 4$, M^n is C.F. if and only if, given orthonormal vectors X_1, X_2, X_3, X_4, we have*

$$k(X_1, X_2) + k(X_3, X_4) = k(X_1, X_3) + k(X_2, X_4).$$

Remark. In particular, as we have anticipated in the introduction, if Q_1, Q_2 are space forms of curvatures c_1, c_2 and dimensions $n_i \geq 2, i = 1, 2$, then $Q_1 \times Q_2$ is C.F. if and only if $c_1 + c_2 = 0$.

For the above results and similar characterizations we refer to [6].

A further nice characterization of C.F. manifolds in terms of Lorentzian geometry is the following (see [1]): Let $\mathbf{L}^{n+2}$ be the Lorentz space of dimension $(n+2)$, i.e., $\mathbf{L}^{n+2} = \mathbf{R}^{n+2}$ with the indefinite metric

$$(X,Y) = -x_0 y_0 + \sum_{i=1}^{n+1} x_i y_i; \qquad X = (x_0, x_1, \ldots, x_{n+1}), Y = (y_0, y_1, \ldots, y_{n+1})$$

The light cone is defined as

$$\mathbf{V}^{n+1} = \{X \in \mathbf{L}^{n+2} : (X,X) = 0, x_0 > 0\}$$

The induced metric in $\mathbf{V}^{n+1}$ is semidefinite and the directions of null norm are the generatrices of the cone.

Theorem 2.4. *M^n is a C.F. manifold if and only if there exists a local isometric immersion $f : M^n \to \mathbf{V}^{n+1}$. Moreover if M^n is simply connected, f is globally defined.*

Idea of the proof. Let $f : M^n \to \mathbf{V}^{n+1}$ be a (local) isometric immersion and

$$S^n = \{X \in \mathbf{V}^{n+1} : (X, e_0) = -1\}; e_0 = (1,0,\dots,0) \in \mathbf{L}^{n+2}.$$

Since M^n is Riemannian and f isometric, f is transverse to the lines $tX, X \in \mathbf{V}^{n+1}, t > 0$. We define a map

$$\pi : M^n \to S^n, \pi(x) = -f(x)/(f(x), e_0).$$

It is not difficult to see that π is a local conformal diffeomorphism, called the *developing map*, and therefore M is C.F.

If M^n is C.F. we will construct local isometric immersions $f : M^n \to \mathbf{V}^{n+1} \subseteq \mathbf{L}^{n+2}$ using the fundamental theorem of submanifolds (see 3.4. for the Riemannian version, which extends to our situation). For this we have to construct a Lorentzian 2-plane bundle $\mathbf{E}$, a bilinear map $\alpha : T_xM \oplus T_xM \to \mathbf{E}$ and a connection $\nabla^\perp$ in $\mathbf{E}$ such that those data satisfy the basic equations of Gauss, Codazzi-Mainardi and Ricci. The study of the local geometry of an isometric immersion $f : M^n \to \mathbf{V}^{n+1}$ suggest the following choises: We take $\mathbf{E} = M^n \times \mathbf{L}^2$ and let $e_0(x) = (x,1,0), e_1(x) = (x,0,1)$ be the basic sections so that $-(e_0,e_0) = 1 = (e_1,e_1)$ and $(e_0,e_1) = 0$. We define $A_0, A_1 : TM \to TM$ by $A_0 = \gamma - \frac{1}{2}I, A_1 = \gamma + \frac{1}{2}I$ and $\alpha(X,Y) = \langle A_0X, Y\rangle)e_0 + \langle A_1X, Y\rangle e_1$. Finally $\nabla^\perp$ is defined extending $\nabla^\perp_X e_0 = 0 = \nabla^\perp_X e_1, \forall X \in TM$. The Ricci equation is trivially satisfied and the equation of Gauss and Codazzi-Mainardi are essentially i) and ii) of theorem 2.2.

The above result may be used to give a proof of a classical theorem of Kuiper (see [5]):

Theorem 2.5. *Let M^n be a compact connected and simply connected C.F. manifold. Then there exists a conformal diffeomorphism $\pi : M^n \to S^n$.*

Proof. By 2.4 there is a (global) isometric immersion $f : M^n \to \mathbf{V}^{n+1}$. The developing map $\pi : M^n \to S^n$ is then a globally defined local conformal diffeomorphism. Since M^n is compact, π is a covering map and, being S^n simply-connected, a conformal diffeomorphism.

Classically, the construction of the developing map is based on Liuville's theorem (cf [5] [9]). If $n = 2$ a local diffeomorphism $\varphi : U \subseteq \mathbf{R}^2 \cong \mathbf{C} \to \mathbf{C}$ is conformal if and only if φ is holomorphic or anti-holomorphic. In dimension $n \geq 3$ Liuville's theorem guarantees the local conformal diffeomorphisms are essentially products of inversions. In particular, if $n \geq 3$, a conformal diffeomorphism of an open set $U \subseteq S^n = \mathbf{R}^n \cup \{\infty\}$ extends to a unique global conformal diffeomorphism of S^n.

Let M^n be a C.F. manifold, $n \geq 3$, and $\varphi : U \subseteq M^n \to \varphi(U) \subseteq S^n$ a conformal diffeomorphism. Let $\gamma : [0,1] \to M$ be a curve with $p = \gamma(0) \in U$. Thus we can "developed" φ along γ as follows: Cover $\varphi([0,1])$ with open sets $U = U_1, U_2, \ldots, U_k$ such that $U_i \cap U_{i+1} \neq \phi$ and let $\varphi_i : U_i \to A \subseteq S^n$ be conformal diffeomorphisms. By Liouville's theorem $\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$ extends to a global conformal diffeomorphism $\Phi_{1,2}.S^n \to S^n$. The (local) diffeomorphism $\widetilde{\varphi}_2 = \Phi_{12} \circ \varphi_2$ agrees with φ_1 in $U_1 \cap U_2$ and therefore φ_1 can be extended to $U_1 \cup U_2$. If we go on this way we end up with a local conformal diffeomorphism φ_γ of a neighborhood of $\gamma([0,1])$ onto an open set of S^n. If γ_1, γ_2 are curves in M with $\gamma_i(0) = p\ \gamma_i(1) = q$ which are homotopic (with end-points fixed), a monodromy argument shows that $\varphi_{\gamma_1}(q) = \varphi_{\gamma_2}(q)$; therefore, if M is simply connected we can define globally a developing map $\pi : M^n \to S^n$.

The developing map although not defined on M, is certainly defined on the universal cover $\widetilde{M}$ of M and is a useful tool in the study of the geometry and the topology of C.F. manifolds.

3. CONFORMALLY FLAT HYPERSURFACES

Let M^n be a Riemannian manifold and $f : M^n \to \mathbf{R}^{n+p}$ an isometric immersion. The orthogonal decomposition $f^*T\mathbf{R}^{n+p} \cong TM \oplus \nu M$ induces, modulo the usual identifications, the decomposition

$$D_X Y = \nabla_X Y + \alpha(X,Y)$$

$$D_X \xi = -A_\xi X + \nabla^\perp_X \xi$$

where X, Y are smooth sections of TM and ξ of νM; D and ∇ denote the Levi-civita connections of $\mathbf{R}^{n+p}$ and M^n respectively and α is induced by a bilinear fibred map $TM \oplus TM \to \nu M$ wich is related to the Weingarten operator $A_\xi : TM \to TM$ by $\langle \alpha(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle$. Finally $\nabla^\perp$ is a metric connection in νM, called the normal connection whose curvature we will denote by $R^\perp$.

The basic relations of the geometry of an isometric immersion are the following:

Gauss equation 3.1. $k(X,Y) = \langle \alpha(X,X), \alpha(Y,Y) \rangle - \|\alpha(X,Y)\|^2$.

Codazzi-Mainardi equation 3.2. $(\tilde{\nabla}_X \alpha)(Y,Z) = (\tilde{\nabla}_Y \alpha)(X,Z)$ where $(\tilde{\nabla}_Z \alpha)$
$(V,W) = \nabla^\perp_Z \alpha(V,W) - \alpha(\nabla_Z V, W) - \alpha(V, \nabla_Z W)$.

Ricci equation 3.3. $\langle R^\perp(X,Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta] X, Y \rangle$.

Those relations are basic in the sense that they determine the geometry of the immersion in view of the fundamental theorem of submanifold theory:

Theorem 3.4. *Let M^n be a Riemannian manifold,* $\mathbf{E}$ *a p-dimensional Riemannian vector bundle with a metric connection ∇'. Let $\alpha' : TM \oplus TM \to \mathbf{E}$ be a symmetric bilinear map which verifies the analogue of 3.1, 3.2, 3.3. Then there exists a locally defined isometric immersion f of M^n into $\mathbf{R}^{n+p}$ with νM isometric to $\mathbf{E}, \nabla^{\perp}$ and α corresponding to ∇' and α' via this isometry. Moreover the immersion is unique up to rigid motions of $\mathbf{R}^{n+p}$ and, if M is simply connected is globally defined.*

We will discuss, in this section, the case of hypersurfaces, i.e., $p = 1$. We fix a unit normal field ξ (locally) and write A for A_ξ. A is a symmetric operator, hence diagonalizable. Let $\{X_1, \ldots, X_n\}$ be an orthogonal basis which diagonalizes A; the eigenvalues of A, $\lambda_i = \langle AX_i, X_i\rangle$ are called the principal curvatures and the Gauss equation gives $k(X_i, X_j) = \lambda_i \lambda_j$.

As an immediate consequence of 2.3 we have:

Lemma 3.5. *(Cartan). Let $f : M^n \to \mathbf{R}^{n+1}$ be an isometric immersion, $n \geq 4$. Then M^n is C.F. if and only if A has an eigenvalue of multiplicity at least $(n-1)$.*

Remark. If $n = 2$ the above condition is trivially satisfied. If $n = 3$ there are examples of isometric immersions of C.F. manifolds into $\mathbf{R}^4$ with distinct principal curvatures.

With respect to the basis $\{X_1, \ldots, X_n\}$ we have, up to reordering:

$$A = \begin{bmatrix} \mu & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{bmatrix}$$

(if $n \geq 4$ and M^n is C.F.). Let $U = \{x \in M : \lambda(x) = \mu(x)\}$ be the set of umbilic points. In the open set $M - U$ we have a well defined smooth codimension one distribution $D_\lambda = span\{X_2, \ldots, X_n\}$ whose leaves Σ_λ are totally umbilic in $\mathbf{R}^{n+1}$, complete if M is complete and λ is constant along the leaves. All those facts are direct consequence of 3.2 but for completeness. For the completeness we refer to [16].

If M^n is compact, the leaves Σ_λ are spheres in hyperplanes of $\mathbf{R}^{n+1}$.

The geometric structure of a compact C.F. manifold in this situation is described by the following result:

Theorem 3.6. *Let $f : M^n \to \mathbf{R}^{n+1}$ be an isometric immersion of a compact C.F. manifold, $n \geq 4$.*

(a) If $U = \{x : \mu(x) = \lambda(x)\} = \phi$ *then* M *is either diffeomorphic to* $S^{n-1} \times S^1$ *or to a generalized Klein bottle (i.e. a non orientable* S^{n-1} *bundle over* S^1 *).*

(b) If $U \neq \phi$ *any connected component* C *of* $M - U$ *is foliated by* $(n-1)$ *spheres and* ∂C *has at most two connected components each one being either a point, an* $(n-1)$ *sphere or two* $(n-1)$ *-spheres with a common point.*

(c) Each connected component of ∂U *is either a point, an* $(n-1)$ *sphere or an* n*-dimensional umbilic set bounded by a union of* $(n-1)$ *spheres and two such spheres have at most one common point.*

The following figures are taken from [3] (to which we refer for a proof of 3.6) and represent the closure of the possible connected components of $M - U$.

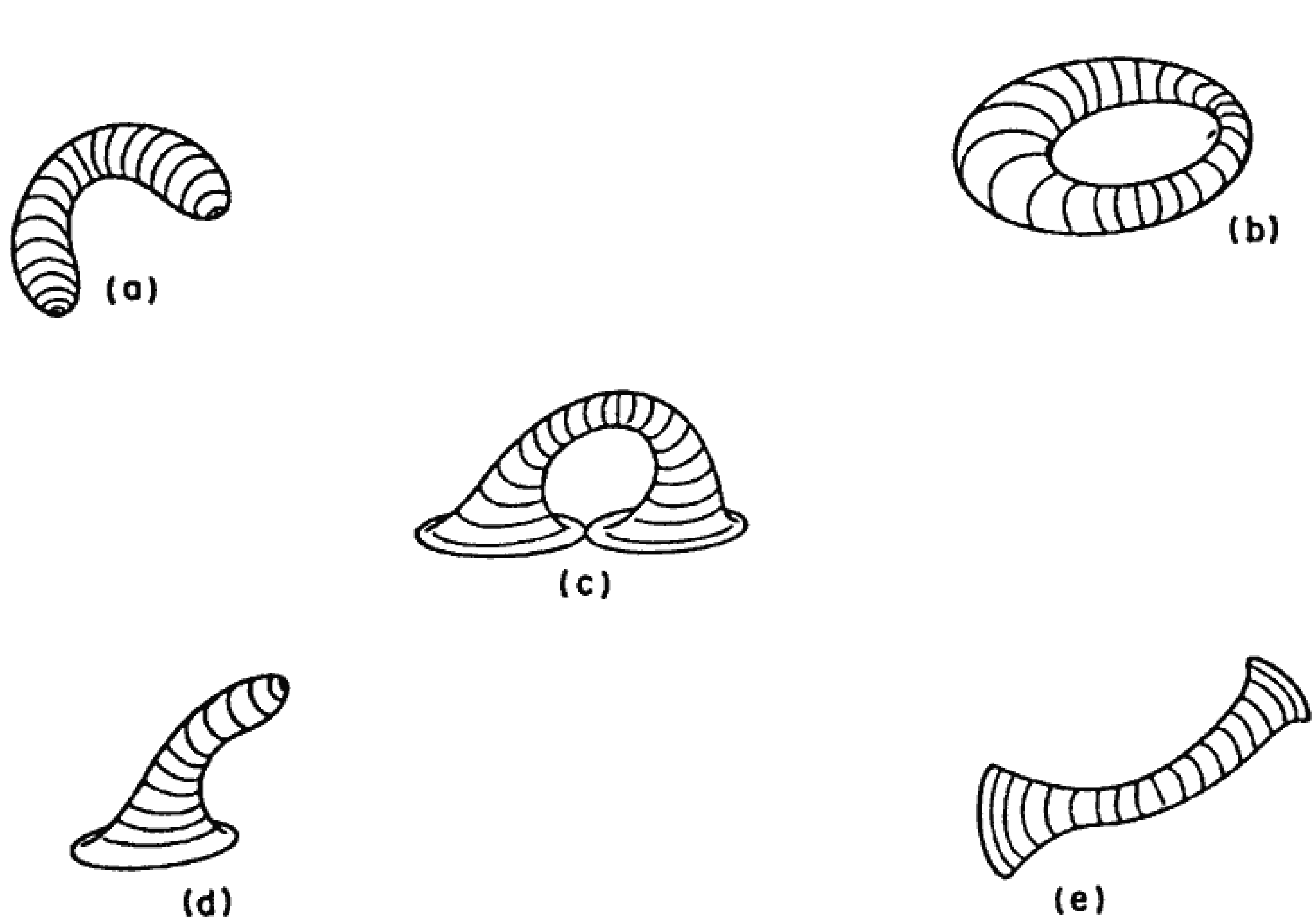

Figure 1

The classical description of "generic" C.F. hypersurfaces (i.e. $\mu \neq \lambda \neq 0$) was given in terms of envelopes of a 1-parameter family of n-spheres (see [2]).

This means that we start with a regular curve $c : (a,b) \to \mathbf{R}^{n+1}$ and a positive function $r : (a,b) \to \mathbf{R}$ with $\|\dot{c}(t)\| > |r'(t)|$. To describe the envelope of the spheres centred at $c(t)$ of radius $r(t)$ we consider first the functions

$$S = rr'/\|\dot{c}\|^2; \quad R = r[1 - (r'/\|\dot{c}\|)^2]^{1/2} \tag{3.7}$$

and a function $\phi : S^{n-1} \times (a,b) \to \mathbf{R}^{n+1}$ such that $\phi = \phi(\cdot, t)$ is an immersion $\forall t \in (a,b)$, $\|\phi_t(x)\| = 1$ and $\langle \phi_t(x), \dot{c}(t) \rangle = 0$. Let $g : S^{n-1} \times (a,b) \to \mathbf{R}^{n+1}$ be the function

$$g(x,t) = [c(t) - S\dot{c}(t)] + R(t)\phi(x,t) \tag{3.8}$$

Proposition 3.9. *If g is an immersion then $S^{n-1} \times (a, b)$ with the induced metric is C.F. and g realized it as an hypersurface with $\mu \neq \lambda \neq 0$. Conversely given a C.F. hypersurface with $\mu \neq \lambda \neq 0$, this is of the form 3.8.*

On time! This means that we can look at "generic" C.F. hypersurfaces either as envelopes of n-spheres centred at $c(t)$ of radius $r(t)$ or as foliated by a family of $(n-1)$ spheres centred at $\gamma(t) = c(t) - S(t)\dot{c}(t)$ of radius $R(t)$ and contained in the hyperplane perpendicular to $\dot{c}(t)$.

Examples 3.10. If c is a straight line we get a rotation hypersurface. If r is a small constant we get a tube around $c = \gamma$. Let $c : S^1 \to \mathbf{R}^2 \subseteq \mathbf{R}^{n+1}$ be a circle with $||\dot{c}(t)|| = L$ and $r(t) = \beta^{-1}\sin(t/L) + 1$. If $\beta > 1$ and L is sufficiently large we get a C.F. hypersurface which is neither rotational or a tube.

An example when singularities appear (i.e., g is not an immersion) is given by taking $c(t)$ as a straight line and $r(t)$ in such a way that $R'(t)$ changes sign along a leaf when $r'(t) \neq 0$. This given an hypersurface obtained rotating a curve as in the figure below.

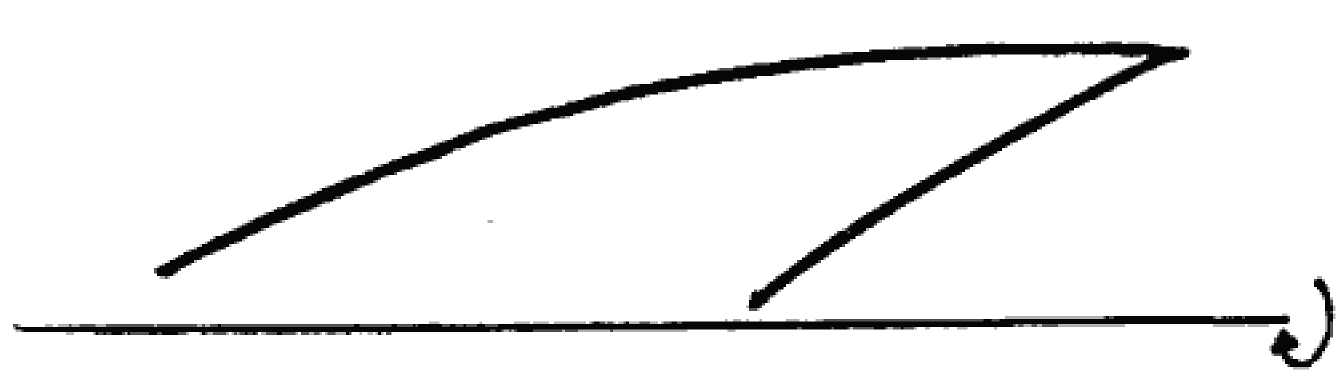

Figure 2

Anyhow, with a little patience, we can write down conditions on c and r such that the map g is an immersion.

Proposition 3.11. *A point $(x, t) \in S^{n-1} \times (a, b)$ is singular for g if and only if*

$$(1 - S') = ||\dot{c}||^{-2}\{R\langle\phi, \ddot{c}\rangle + S\langle\dot{c}, \ddot{c}\rangle\}$$

which, if $S \neq 0$ is equivalent to $R' = S\langle\ddot{c}, \phi\rangle$.

We will now describe the global structure of compact C.F. hypersurface ($n \geq 4$ as always). For this we will use the Morse theory for the height functions whose basic facts we will rapidly recall (also for use in the next section).

Let $f : M^n \to \mathbf{R}^{n+p}$ be an immersion and $\xi \in S^{n+p-1} \subseteq \mathbf{R}^{n+p}$ be a fixed unit vector. The height function in the ξ direction is the function

$$h_\xi : M^n \to \mathbf{R}, \quad h_\xi(x) = \langle f(x), \xi\rangle.$$

Then $h_\xi(x)$ is the height of $f(x)$ with respect to the (oriented) hyperplane $\xi^\perp$ or, equivalently, the projection of $f(x)$ on the line $t\xi, t \in \mathbf{R}$. It is well known that $x_0 \in M$ is a critical point for h_ξ if and only if $\xi \in \upsilon_{x_0} M$ and at such a point the hessian of h_ξ is given by A_ξ.

Let $\upsilon^1 M = \{(x,\eta) \in \upsilon M : \|\eta\| = 1\}$ be the normal sphere bundle and $\Gamma : \upsilon^1 M \to S^{n+p-1}, \Gamma(x,\eta) = \eta$ be the (generalized) Gauss map; if ξ is a regular value of Γ the hessian of h_ξ is non degenerate at each critical point of h_ξ, i.e., all critical points are non degenerate and h_ξ is what is called a Morse function. Recal that the index of a function at a critical point is the number of negative eigenvalues of the hessian.

The main result of Morse theory may be stated as follows:

Theorem 3.12. *Let h be a Morse function on a compact manifold M. Then M has the homotopy type of a cell complex with one cell of dimension m for each critical point of h of index m.*

The first consequence of the above theorem and Cartan's Lemma (3.5) is that a compact C.F. hypersurface in $\mathbf{R}^{n+1}, n \geq 4$ is homotopy equivalent to a cell complex with cells only in dimensions 0, 1, $(n-1)$ and n. In particular the homology groups $H_i(M;\mathbf{Z})$ are zero for $1 < i < (n-1)$.

Let now M^n be a compact C.F. manifold, $n \geq 4$, and $f : M^n \to \mathbf{R}^{n+1}$ an isometric immersion. Let ξ be a regular value of Γ so that h_ξ is a Morse function.

If h_ξ does not have critical points of index 1 then M is simply connected (by 3.12) and therefore diffeomorphic to S^n by Kuiper's theorem 2.5. Suppose $x_0 \in M$ is a critical point of index 1. Since x_0 is non degenerate, $\mu < 0 < \lambda$, so by 3.6 x_0 has a neighborhood V diffeomorphic to $S^{n-1} \times (a,b)$ via a diffeomorphism which sends the slices $S^{n-1} \times \{t\}$ onto the leaves of the foliation Σ_λ and, say, (p,t_0) in x_0. We can perform the following construction that we will call a conformal surgery at x_0.

Delete from M the image of $S^{n-1} \times (t_0 - \varepsilon, t_0 + \varepsilon)$ and fill up the "holes" with two disks. Playing with the local description in terms of the curve $c(t)$ and the function $r(t)$ and using 3.11 to avoid singularities, it is possible to show that the above conformal surgery can be performed in such a way that:

a) the resulting manifold is still a C.F. hypersurfaces

b) the Gauss map, restricted to the added disks, omits $\pm\xi$ (in particular h_ξ does not have critical points in those disks).

In the language of differential topology, the manifold M is obtained from the modified manifold by attaching an handle of type $S^{n-1} \times (a,b)$. We can now prove one of the main results of [3]:

Theorem 3.13. *Let M^n be a compact C.F. hypersurface in $\mathbf{R}^{n+1}, n \geq 4$. Then M^n is diffeomorphic to a sphere with b_1 handles (of type $S^{n-1} \times (a,b)$) attached. Conversely any*

Figure 3. Conformal surgery at x_0.

such manifold may be immersed in $\mathbf{R}^{n+1}$ *in such a way that the induced metric is C.F.*

Remark. The number b_1 is the first Betti number of M, i.e. $b_1 = \dim_Q H_1(M,Q)$.

Proof. Let ξ be a regular value fo Γ and $x_1,\ldots,x_k$ the critical points of h_ξ of index 1. Suppose we have ordered the x_i's in such a way that after performing conformal surgeries at $x_1,\ldots,x_{b_1}$ we obtain a connected manifold $\overline{M}$ and any surgery at $x_i, i > b_1$ disconnects $\overline{M}$. It is then sufficient to prove that $\overline{M}$ is diffeomorphic to S^n. Performing conformal surgeries in $\overline{M}$ at $x_{b_{1+1}},\ldots,x_k$ we obtain a C.F. manifold M' on which h_ξ has only critical points of index $0,(n-1)$ or n. The connected components of M' are simply connected (by 3.12) and hence diffeomorphic to spheres by Kuiper's theorem 2.5. It is not diffucult to see that since $\overline{M}$ is disconnected by any of the above surgery $\overline{M}$ has to be the connected sum of the connected components of M' and hence diffeomorphic to a sphere.

The converse is proved (I hope!) by the figure 4.

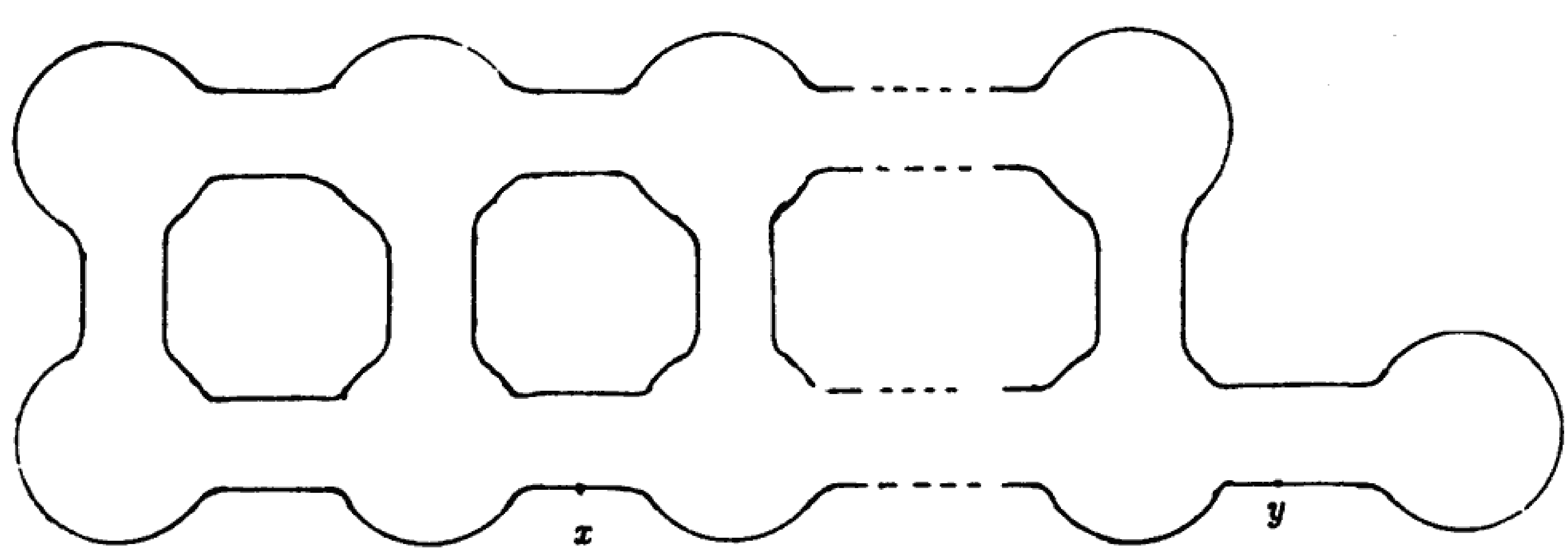

Figure 4. An immersion of a sphere with b_1 handles attached.
Conformal surgery at x does not disconnected, conformal surgery at y disconnects.

The above theorem looks exactly like the classical classification theorem for compact surfaces (which are conformally flat!).

The following natural question is still open:

Problem. Give a description of compact C.F. hypersurfaces in $\mathbf{R}^4$.

We observe explicitly that if an hypersurfaces of $\mathbf{R}^4$ has a principal curvature with multiplicity at least two then all the above arguments still hold true and we get the same conclusons. For such an hypersurface it is easy to see that, with the above notations, $\gamma = \lambda \left[A - \frac{1}{2}\lambda I\right]$, so that $R(X,Y) = \gamma(X) \wedge Y + X \wedge \gamma(Y)$ and, using the Codazzi-Mainardi equation for $A, (\nabla_X \gamma)(Y) = (\nabla_Y \gamma)(X)$ so that the manifold is C.F.

4. CONFORMALLY FLAT SUBMANIFOLDS

In the section we will study some of the local and global geometry of isometric immersions of a C.F. manifold M^n into $\mathbf{R}^{n+p}$. The first things to look for are analogues of the Cartan Lemma to characterize conformal flatness or, at least, give necessary conditions for it.

Let $f : M^n \to \mathbf{R}^{n+p}$ be an isometric immersion. We recall that a subspace $V \subseteq T_x M$ is said to be an umbilic subspace if these exists a normal vector $\xi \in v_x M$ such that

$$\alpha(X,Y) = \langle X,Y\rangle\xi \quad \forall X \in T_X M, Y \in V$$

A first partial analogue of Cartan's Lemma is the following result due to J.D. Moore (see [11] for proofs):

Theorem 4.1. *Let $f : M^n \to \mathbf{R}^{n+p}$ be an isometric immersion, $p \leq n-3$. If M^n is C.F., $\forall x \in M$ there exists an $(n-p)$-dimensional umbilic subspace $V \subseteq T_x M$.*

If M^n is compact the above reuslt gives, via the Morse theory for the height functions, restrictions on the topology of the manifold.

Corollary 4.2. *Let M^n be a compact C.F. manifold and $f : M^n \to \mathbf{R}^{n+p}$ an isometric immersion, $p \leq (n-3)$. then M^n has the homotopy type of a cell complex with no cells in dimension $k, p < k < (n-p)$. In particular, for k in this range, the homology groups $H_k(M^n, \mathbf{Z})$ vanish.*

Proof. The existence of an umbilic subspace $V \subseteq T_x M$ of dimension at least $(n-p)$ implies that $\forall \xi \in v_x M, A_\xi$ has an eigenvalue of multiplicity at least $(n-p)$. If this eigenvalue is negative the index of h_ξ at x is at least $(n-p)$, if it is positive the index can not be bigger than p. The conclusion then follows from 3.12.

Definition 4.3. A normal vector $\xi \in v_x M$ is called quasi-umbilical if A_ξ has an eigenvalue of multiplicity at least $(n-1)$. The immersion $f : M^n \to \mathbf{R}^{n+p}$ is called quasi-umbilical if, $\forall x \in M$, there exists an orthonormal basis $\xi_1, \ldots, \xi_p$ in $v_x M$, with ξ_i quasi-umbilical, $i = 1, \ldots, p$.

The following characterization of conformal flatness for low codimensional submanifold was obtained by J.D. Moore and J.M. Morvan along the lines of Cartan's ideas (see [12]).

Theorem 4.4. *Let $f : M^n \to \mathbf{R}^{n+p}$ be an isometric immersion with $p \leq \min\{4, (n-3)\}$. Then M^n is C.F. if and only if f is quasi-umbilical.*

Remark. In [13] Morvan and Zafindratafa construct an immersion of an open neighbordhood of $0 \in \mathbf{R}^4$ into $\mathbf{R}^{10}$ with 0 as a flat point (with the induced metric) but not quasi-umbilical. This shows that some conditions on the codimension are essential for a theorem of the type of 4.4.

Theorems 4.1 and 4.4 have been used by M.H. Noronha to study the local geometry of a conformally flat n-dimensional submanifold of $\mathbf{R}^{n+2}$. We will describe now some of her results.

Definition 4.5. A submanifold $\Sigma \subseteq M$ is a geometric sphere of type $\varepsilon, \varepsilon = 0, 1$, if Σ is an umbilic submanifold of M with parallel mean curvature vector (in $v\Sigma \subseteq TM$) and the sectional curvatures of M along planes tangent to σ are zero ($\varepsilon = 0$) or a positive constant ($\varepsilon = 1$).

Definition 4.6. A manifold M^n is of type (ε, l) if it is locally foliated by codimensional l geometric spheres of type ε. From 4.1 using the Codazzi-Mainardi equation we get:

Corollary 4.7. *Let $f : M^n \to \mathbf{R}^{n+p}$ be as in 4.1. Then there exists an open and dense set $N \subseteq M$ such that each connected component of N is of type $(\varepsilon, l), l \leq p$.*

For a more detailed analysis of the local geometry of C.F. manifolds which can be isometrically immersed in euclidean space with low codimension we need some facts about normal curvature. We recall first that the vanishing of the normal curvature is a conformal invariant of the immersion. It is not true, in general, that C.F. submanifolds of codimension two have $R^\perp \equiv 0$. There are, in fact, flat surfaces in $\mathbf{R}^4$ with $R^\perp \neq 0$, if we want higher dimensional example is enough to take the product of such an immersion with the identity map $1{:}\mathbf{R}^k \to \mathbf{R}^k$. However to study the local intrinsic geometry of a C.F. manifold immersed in codimension two, we can suppose $R^\perp \equiv 0$, based on the following result (see [15] for a proof).

Theorem 4.8. *Let $f : M^n \to \mathbf{R}^{n+2}$ be an isometric immersion, M C.F. and $n \geq 5$. Then there exists a local isometric immersion $g : M^n \to \mathbf{R}^{n+2}$ with $R^\perp \equiv 0$.*

Let M^n be a C.F. manifold of type $(\varepsilon, 1), n \geq 5$. Combining results of J.D. Moore (see [14]) with the arguments of §3, we get:

Theorem 4.9. *In the above hypothesis there exists a local isometric immersion of M^n into $\mathbf{R}^{n+1}$ near each point of an open dense subset $N \subseteq M^n$. Moreover, given an isometric immersion $f : M^n \to \mathbf{R}^{n+2}, \forall x \in N$ there exists a neighborhood U_x where $f \setminus U_x$ is the composition of an isometric immersion $f_1 : U_x \to \mathbf{R}^{n+1}$ and a local isometric immersion $j : \mathbf{R}^{n+1} \to \mathbf{R}^{n+2}$.*

Codimension two C.F. submanifolds of type $(\varepsilon, 1)$ look like hypersurfaces. We will discuss now the case $(\varepsilon, 2)$.

Theorem 4.10. *Let M^n be a C.F. manifold of type $(\varepsilon, 2), n \geq 5$. A necessary and sufficient condition for the existence of a local isometric immersion into $\mathbf{R}^{n+2}$ is that M^n admits two orthogonal codimension 1 foliations whose leaves are, in the induced metric, C.F. manifold of type $(\varepsilon, 1)$ and the intersectionof the leaves of the two foliations gives the codimension two foliation by geometric spheres.*

Idea of the proof. Let us suppose M^n is (locally) immersed in $\mathbf{R}^{n+2}$. By 4.8 we can suppose $R^{\perp} \equiv 0$ and therefore, by the Ricci equation for all $x \in M^n$ there exists an orthonormal basis $\{X_1, \ldots, X_n\}$ in $T_x M^n$ which diagonalizes simultaneously all the operators $A_\xi, \xi \in v_x M$. By 4.4 there are two orthogonal quasi-umbilic vectors $\xi_1, \xi_2 \in v_x M$. Up to reordering the basis $\{X_1, \ldots, X_n\}$ we have:

$$A_{\xi_1} = \begin{bmatrix} a_1 & & & 0 \\ & a & & \\ & & \ddots & \\ 0 & & & a \end{bmatrix} \quad A_{\xi_2} = \begin{bmatrix} b & & & & 0 \\ & b_1 & & & \\ & & b & & \\ & & & \ddots & \\ 0 & & & & b \end{bmatrix}$$

with $b_1 \neq b, a_1 \neq a$ otherwise M^n would be of type $(\varepsilon, 1)$.

It is not difficult to see that we can choose the $X_i' s$ and $\xi_i' s$ differentiably near x so that we get four differentiable distributions:

$$D_1 = span\{X_1, X_3, \ldots, X_n\}$$

$$D_2 = span\{X_2, X_3, \ldots, X_N\}$$

$$D = D_1 \cap . D_2$$

$$D^{\perp} = span\{X_1, X_2\}.$$

From the Codazzi-Mainardi equations it follows that each distribution is integrable. Let $\Sigma_i, i = 1, 2, \Sigma, \Sigma^{\perp}$ denote the leaves of the foliations. A little more playing with the basic equations show that the $\Sigma_i' s$ are C.F. manifolds of type $(\varepsilon, 1)$ and the $\Sigma' s$ foliated Σ_i and are geometric spheres.

For the converse we first observe that if $\varepsilon = 0$, the Σ_i's are of type (0.1). Then Σ_i is flat and with a few calculation is possible to see flat M is flat so we can locally embed it in $\mathbf{R}^{n+2}$. Let us suppose $\varepsilon = 1$. We want to construct two commuting symmetric operators A_1, A_2 : $T_xM \to T_xM$ and a connection in $M \times \mathbf{R}^2$ such that the basic equations 3.1, 3.2, 3.3, are satisfied. It is possible to show that there exists a local orthonormal frame $\{X_1, \dots, X_n\}$ that diagonalizes the operator γ (defined in 2.1) such that $\{X_1, X_3, \dots, X_n\}$ diagonalizes the operator γ_1 (relative to the C.F. manifold Σ_1) and $\{X_2, X_3, \dots, X_n\}$ diagonalizes γ_2. In this basis γ takes the form

$$\begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \mu^2 & & \\ & & & \ddots & \\ 0 & & & & \mu^2 \end{bmatrix}$$

We are looking for operators A_i which in the above basis look like

$$A_1 = \begin{bmatrix} a_1 & & & & 0 \\ & a_2 & & & \\ & & a & & \\ & & & \ddots & \\ 0 & & & & a \end{bmatrix} \quad A_2 = \begin{bmatrix} b & & & & 0 \\ & b_1 & & & \\ & & b & & \\ & & & \ddots & \\ 0 & & & & b \end{bmatrix}$$

The basic idea is that the Gauss equations give four relations between a, b, a_1, a_2, b_1 so we can determine them in function of one parameter. So we can play with this parameter and using the fact that γ verifies an equation of type Codazzi-Mainardi, define a suitable connection on $M \times \mathbf{R}^2$ such that our data verify also the equation of Codazzi-Mainardi and Ricci. The calculations however are non trivial and we refer to [14] for a complete proof.

The results described above give a reasonable description of the local geometry of C.F. submanifolds of $\mathbf{R}^N$ with codimension two. However, probably, they have to be refined to answer the following natural questions:

Problem. Give a global geometric description of compact C.F. submanifold in $\mathbf{R}^N$ with codimension two similar to the one given in 3.5 and 3.10 for hypersurfaces.

Probably the first and essential step is to understand haw the various pieces of type (ε, l) glue together.

Another interesting question is the following:

Problem. Given a C.F. manifold of type (ε, l), which are there geometric invariants of the immersion into $\mathbf{V}^{n+1}$ (cf. 2.4) which detect the type?

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Francesco Mercuri
IMECC - UNICAMP
Caixa Postal 6065
13.081 - CAMPINAS, S.P., BRAZIL
e-mail MERCURI @ BRUC. BITNET