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TWO-NORM SPACES AND SUMMABILITY

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Abstract. *We determine the general form of continuous linear functionals over the two-norm space of bounded Toeplitz summable sequences. From this we deduce some inclusion and consistency theorems. It was recognized many years ago [3], [7] that the two-normspace technique is the proper tool in investigation of the bounded efficiency field (i.e. the set of bounded summable sequences) of Toeplitz methods. However these results were so far not presented in full extent in any paper, in particular the general form of* γ *-continuous functionals (for the definition see below) in those spaces was not given. We fill this gap, which enables us to obtain some inclusion theorems which lead in turn, to the well known consistency theorems of Mazur and Orlicz [6] in a sharpened form. We adopt the following notation and conventions. The terms of all sequences will be indexed starting with zero, the* k *-th element of the sequence* x *will be denoted by* $\pi_k(x)$. *The symbols* l^∞, c *and* c_0 *will stand for the space of bounded, convergent and to null convergent sequences, respectively;* δ_{ij} *will denote the Kronecker symbol.*

1. THE UNDERLYING SPACES

First we recall some facts about two-norm spaces [1], [2], [5].

A two-norm space is a triplet $(X, \|\ \|, \|\ \|^0)$ in which X is vector space over the field of real and complex scalars, $\|\ \|$ and $\|\ \|^0$ are norms, the first being finer than the second one. Besides the convergence corresponding to these norms we shall consider as intermediate convergence, γ, defined as follows. The sequence (x_n) is called γ-convergent to x_0 (written $x_n \xrightarrow{\gamma} x_0$) if $\sup_n \|x_n\| < \infty$ and $\lim_{n\to\infty} \|x_n - x_0\|^0 = 0$. The two-norm space is called γ-complete if every sequence (x_n) satisfying $(x_{p(n)} - x_{q(n)}) \xrightarrow{\gamma} 0$ whenever $p(n) \to \infty$ and $q(n) \to \infty$ is γ-convergent. It is called γ-normal if $\lim_{n\to\infty} \|x_n - x_0\|^0 = 0$ implies that $\|x_0\| \leq \lim_{n\to\infty} \|x_n\|$. The space is both γ-complete and γ-normal if and only if the unit ball $S := \{x \in X : \|x\| \leq 1\}$ is complete for the metric $\rho(x; y) := \|x - y\|^0$. A set $A \subset X$ is called γ-closed if it contains all γ-limits of γ- convergent sequences of its elements, it is called γ-dense if every element of the space X is the γ-limit of a sequence of elements of this set.

A functional f over the space $(X, \|\ \|, \|\ \|^0)$ is called γ-linear if it is linear and sequentially continuous for the convergence γ. A two-norm space is said to possess the Banach property if the limit of every pointwise convergent sequence of γ-linear functionals is γ-linear. H. Steinhaus [9] seems to be the first in dealing with γ-continuous functionals in concrete cases. Not every γ-complete two-norm space has this property, however, there are known some sufficient conditions for this. The one we shall use reads

(σ) given any element $x_0 \in S$ and $\varepsilon > 0$, there exists an integer K and a $\delta > 0$ such that every element $x \in S$ satisfying $||x||^0 < \delta$ is of form:

$$x = \sigma_1 x_1 + \sigma_2 x_2 + \ldots + \sigma_n x_n,$$

where $x_1, \ldots, x_n \in S$, $\sigma_1 + \ldots + \sigma_n = 0$, $|\sigma_1| + \ldots + |\sigma_n| \leq K$ and $||x_0 - x_i||^0 < \varepsilon$ for $i = 1, 2, \ldots n$.

Theorem 1.1. *Let the space $(X, \|\ \|, \|\ \|^0)$ be γ-complete, γ-normal, and let is possess the property (σ), then it has the Banach property.*

Proof. Let f_n be γ-linear functionals and let the sequence $(f_n(x))$ converge pointwise to $f(x)$, this functional is, of course, linear; to prove its γ-linearity it is enough to show that it is continuous at zero on the ball S endowed with the metric ρ. Since the functionals f_n are continuous on the ball S which is complete for the metric ρ, those functionals are, as well known, equicontinuous at a point x_0 in S, which means that, given any $\eta > 0$, there exists an $\varepsilon > 0$ such that $x \in S$ together with $||x - x_0||^0 < \varepsilon$ imply $|f_n(x) - f_n(x_0)| < \eta$ for every n. Choose δ and K according to the condition (σ) and let $x \in S, ||x - x_0||^0 < \delta$. Let $x_1, \ldots, x_N$, $\sigma_1, \ldots, \sigma_N$ have the same meaning as in (σ). Then $|f_n(x) - f_n(x_0)| < \eta$ for $j = 1, \ldots, N$ and

$$|f_n(x)| = \left|\sum_{j=1}^{N} \sigma_j f_n(x_j)\right| = \left|\sum_{j=1}^{N} \sigma_j [f_n(x_j) - f_n(x_0)]\right|$$
$$\leq \sum_{j=1}^{N} |\sigma_j| |f_n(x_j) - f_n(x_0)| \leq K\eta.$$

2. TOEPLITZ METHODS OF SUMMABILITY

All sequences we deal with are composed of complex numbers (the «real» case is simpler). Let $(A = (a_{ij}) : i, j = 0, 1, \ldots)$ be an infinite matrix of complex numbers. Given a sequence x, let

$$a_n(x) := \sum_{j=1}^{\infty} a_{nj} \pi_j(x);$$

if the above series converges for every n and if there exists

$$a(x) := \lim_{n \to \infty} a_n(x),$$

the sequence a is called A-summable to $a(x)$ we write also in this case $A\text{-lim}\ \pi_n(x) = a(x)$. The procedure assigning the A-limit to some sequences is called the Toeplitz method of summability.

In the sequel $\widetilde{A}(\widetilde{A}_0)$ will denote the set of all A-summable (to zero A-summable) sequences. Both are vector spaces under addition and multiplication, by scalars defined coordinatewise. The method A is called conservative (null-conservative) if every convergent sequence (to zero convergent sequence) is in $\widetilde{A}$.

Those methods are characterized by

Theorem 2.1. *(Toeplitz [10], Kojima [5], Schur [8]). The method A is null-conservative if and only if*

$$(i) \qquad ||A|| := \sup_n \sum_{j=0}^{\infty} |a_{nj}| < \infty,$$

(ii) for every j there exists

$$a_{.j} := \lim_{n\to\infty} a_{nj}.$$

The method is conservative if moreover

(iii) there exists

$$s_A := \lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj}.$$

Notice that for null-conservative methods $\sum_{j=0}^{\infty} |a_{.j}| \le ||A||$ and that $a_n(x)$ exists for every bounded sequence x. In the sequel all methods will be supposed to be al least null-conservative.

We shall be concerned with two subspaces $l^\infty \cap \widetilde{A}$ and $l^\infty \cap \widetilde{A}_0$ of $\widetilde{A}$ called the bounded efficiency and the bounded null-efficiency field, respectively. Both can be equipped with the two-norm space structure when setting

$$||x|| := \sup_n |\pi_n(x)|,$$

$$||x||^0 := \sup_n 2^{-n}|\pi_n(x)| + \sup_n |a_n(x)|.$$

The norm $||\ ||$ is finer than $||\ ||^0$. From the inequality $|a(x)| \le ||x||^0$ it follows that the functional a is γ-linear. A standard argument shows that the spaces $l^\infty \cap \widetilde{A}$ and $l^\infty \cap \widetilde{A}_0$ are γ-complete and γ-normal.

Lemma 2.3. *(Mazur and Orlicz [6]). Let the method $\widetilde{A}$ be null-conservative and let $x \in l^\infty \cap \widetilde{A}_0$. Then, given any $\varepsilon > 0$ and natural n, there exists a p and a sequence z such that*

$$\pi_k(z) = \begin{cases} \pi_k(x) & \text{for } k \leq n, \\ \sigma_k \pi_k(x) & \text{for } n < k \leq n+p, \\ 0 & \text{for } n+p < k, \end{cases}$$

where $1 \geq \sigma_{n+1} \geq \sigma_{n+2} \geq \dots \geq \sigma_{n+p} \geq 0$ and such that

$$|a_n(x) - a_n(z)| < \varepsilon$$

for every n.

The decisive role in this paper plays

Theorem 2.3. *The space $l^\infty \cap \widetilde{A}_0$ possesses the property (σ).*

Proof. Let $\sigma_1, \sigma_2, \sigma_3$ denote the three roots of degree 3 of 1. We will make use of the fact that for complex u, v such that $|u| \leq 1, |v| \leq 1$ we have $|u + \sigma_j v| \leq 1$ for some σ_j. Given any $x_0 \in S$ and $\varepsilon > 0$, choose $\eta > 0$ so that $(3 + 2\|A\|\eta) < \varepsilon/4$ and, then, natural n so that $2^{-n} \leq 3\delta/4$. By Lemma 2.1 there exists a sequence z and a p such that $\pi_k(z) = \pi_k(x_0)$ for $k \leq n, |\pi_k(z)| \leq |\pi_k(x_0)|$ for $n < k \leq n+p, \pi_k(z) = 0$ for $n+p < k$ and such that $|a_k(z) - a_k(x_0)| < \eta$ for every k. Therefore $z \in S$ and $\|x_0 - z\|^0 \leq 2^{-n} + \eta < \delta/4 + \eta$.

Let $x \in S, \|x\|^0 \leq \delta := \eta/2^{n+p}$. Then $|\pi_k(x)| \leq 2^k \delta$ for $k \leq n+p$. Since $|\pi_k(x)| \leq 1$ for every k, there exist $\varepsilon_k \in \{\sigma_1, \sigma_2, \sigma_3\}$ such that $|\pi_k(z) - \varepsilon_k \pi_k(x)| \leq 1$. Let the sequences x_1, x_2, x_3 be defined by

$$\pi_k(x_j) = \begin{cases} \pi_k(z) + \varepsilon_k \pi_k(x) & \text{for } k \leq n, \\ \pi_k(z) & \text{for } n < k \leq n+p, \\ \pi_k(x)/3\sigma_j & \text{for } n+p < k. \end{cases}$$

Then $x_1, x_2, x_3 \in S, x = \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3, |\sigma_1| + |\sigma_2| + |\sigma_3| = 3$ and $|\pi_k(z) - \pi_k(x_j)| \leq |\pi_k(x)|$ for $j = 1, 2, 3$ and every k. Therefore $|a_m(z - x_j)| \leq$

$$\left|\sum_{k=0}^{n+p} a_{mk}\pi_k(z - x_j)\right| + \left|\sum_{k>n+p} a_{mk}\pi_k(z - x_j)\right| \leq 2\sum_{k=0}^{n+p} |a_{mk}\pi_k(x)| + |a_m(x)| \leq$$

$\eta + 2\|A\|2^{n+p} \leq \eta + 2\|A\|2^{n+p}\delta$, for $|\pi_k(z - x_j)| \leq |\pi_k(x)|$ for $k \leq n+p$. Since $|\pi_k(z - x_j)| \leq 2$ and $\pi_k(z - x_j) = 0$ for $n < k \leq n+p$, we obtain

$$\|z - x_j\|^0 \leq 2^{-n} + 3\eta + 2\|A\|\eta \leq 3\eta/4$$

and finally

$$\|x_0 - x_j\|^0 \leq \|x_0 - z\|^0 + \|z - x_j\|^0 < \varepsilon.$$

From Theorem 1.1 we obtain

Theorem 2.4. *Let the method A be null-conservative, then every pointwise limit of a sequence of γ-linear functionals on $l^\infty \cap \widetilde{A}_0$ is γ-linear.*

A similar property shares the space $(l^\infty \cap \widetilde{A}, \|\ \|, \|\ \|^0)$.

Theorem 2.5. *Let the method A be conservative, then the conclusion of Theorem 2.4 is valid in the space $(\widetilde{A}, \|\ \|, \|\ \|^0)$.*

Proof. Let the sequence (f_n) of γ-linear functionals converge pointwise to the functional f, let $x_0 \in l^\infty \cap \widetilde{A} \setminus l^\infty \cap \widetilde{A}_0$ (if such element does not exist it is nothing to prove) and let $x_n \xrightarrow{\gamma} 0$. Then $x_n = u_n + [a(x_n)/a(x_0)]x_0$, where $u_n \in l^\infty \cap \widetilde{A}_0$ and $u_n \xrightarrow{\gamma} 0$. Therefore $f_n(x_n) = f_n(u_n) + a(x_n)[f_n(x_0)/a(x_0)] \to 0$, from which the assertion follows.

3. LINEAR FUNCTIONALS OVER THE SPACE $l^\infty \cap \widetilde{A}$

Now we shall determine the general form of γ-linear functionals over the spaces we deal with.

Let X be a vector space, $(Y, \|\ \|_Y)$ - a normed space and L - a linear operator from X to Y. The functional defined as $\|x\|^* = \|Lx\|_Y$ is a seminorm on X. In this situation we have the

Lemma 3.1. *A linear functional f is continuous over the space $(X, \|\ \|^*)$ if and only if there exists a continuous functional g over $(Y, \|\ \|_Y)$ such that $f(x) = g(Lx)$.*

Proof. Necessity. Let f be continuous, then $|f(x)| \le \|f\|\ \|x\|^* = \|f\|\ \|Ax\|$. We define a linear functional h over the subspace $RngA$ of Y by setting

$$h(z) = f(Lx) \quad for \quad z = Lx.$$

This functional is uniquely determined, for if $Lx_1 = Lx_2$, then $f(Lx_1) = f(Lx_2)$ it is also bounded since

$$|h(z)| = |f(Lx)| \le \|f\|\ \|Lx\| = \|f\|\ \|x\|^*.$$

By the Hahn-Banach theorem there exists a bounded linear functional g over the space $(Y.\|\ \|_Y)$ suth that $g(z) = h(z)$ for $z \in RngA$, whence $f(x) = g(Lx)$ for every $x \in X$.

The sufficiency of the condition is obvious.

Theorem 3.2. *Let the method A be null-conservative, then the functional f is linear over the space $l^\infty \cap \widetilde{A}_0$ if and only if there exists a sequence (α_n) such that $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ and*

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \pi_n(x).$$

Proof. For $x \in l^\infty \cap \widetilde{A}_0$ let $||x||_1^0 = \sup_n 2^{-n}|\pi_n(x)|, ||x||_2^0 = \sup_n |a_n(x)|$ then $|| \, ||_1^0$ is a norm and $|| \, ||_2^0$ - a seminorm and $||\chi||^0 = ||w||_1^0 + ||x||_2^0$.

First we establish the general form of a continuous linear functional f over the space $(l^\infty \cap \widetilde{A}_0, || \, ||_1^0)$. By a theorem of Zeller ([11], p. 468), $f(x) = f_1(x) + f_2(x)$ where f_1 and f_2 are linear functionals continuous over the space $(l^\infty \cap \widetilde{A}_0, || \, ||_1^0)$ and $(l^\infty \cap \widetilde{A}_0, || \, ||_2^0)$, respectively. Let $L_1 x = (2^{-n}\pi_n(x)), L_2(x) = (a_n(x))$, then L_1, L_2 act from the space $l^\infty \cap \widetilde{A}_0$ to $c_0, ||x||_1^0 = ||L_1 x||, ||x||_2^0 = ||L_2 x||$, so by 3.1

$$f(x) = \sum_{n=0}^{\infty} b_n 2^{-n} \pi_n(x) + \sum_{n=0}^{\infty} c_n a_n(x).$$

where $\sum_{n=0}^{\infty} (||b_n|| + ||c_n||) < \infty$. Now

$$\sum_{n=0}^{\infty} c_n a_n(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n a_{nk} \pi_k(x) = \sum_{k=0}^{\infty} \pi_k(x) \sum_{n=0}^{\infty} c_n a_{nk},$$

since $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |c_n||a_{nk}| \leq ||A|| \sum_{n=0}^{\infty} |c_n|$. Let $d_n = 2^{-n} b_n + \sum_{k=0}^{\infty} c_k a_{nk}$, then $\sum_{n=0}^{\infty} |d_n| < \infty$ and

$$(*) \qquad f(x) = \sum_{n=0}^{\infty} d_n \pi_n(x).$$

Thus every functional linear and continuous over the space $(l^\infty \cap \widetilde{A}_0, || \, ||^0)$ is of form (*) where $\sum_{n=0}^{\infty} |d_n| < \infty$ (the converse is not true).

Now let f be a γ-linear functional over the space $(l^\infty \cap \widetilde{A}_0, || \, ||, || \, ||^0)$. By a theorem of Alexiewicz and Semadeni ([4], p. 132) there exist continuous linear functionals g_n over $l^\infty \cap \widetilde{A}_0$ such that

$$f(x) = \sum_{n=0}^{\infty} g_n(x)$$

and $\sum_{n=0}^{\infty} ||g_n|| < \infty$, where $||g_n|| = \sup\{||g_n(x)|| : ||x|| \leq 1\}$. Thus $g_n(x) = \sum_{k=0}^{\infty} \alpha_{nk} \pi_k(x)$

and $g_n = \sum_{k=0}^{\infty} |\alpha_{nk}| < \infty$. Therefore $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |\alpha_{nk}| < \infty$ and

$$f(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \alpha_{nk} \pi_k(x) = \sum_{n=0}^{\infty} \alpha_n \pi_n(x),$$

where $\alpha_n = \sum_{k=0}^{\infty} \alpha_{kn}$ and $\sum_{n=0}^{\infty} |\alpha_n| < \infty$.

Conversely, let the functional f be of the above form. Then the functionals $f_k := \alpha_k \pi_k(x)$ are continuous over $l^\infty \cap \widetilde{A}_0 \|f_k\| = \|\alpha_k\|, f(x) = \sum_{k=0}^{\infty} f_k(x)$ so, again, by [1] p. 132 the functional f is γ-linear.

Theorem 3.3. *Let the method A be conservative, then the functional f is γ-linear over the space $(l^\infty \cap \widetilde{A}_0, \| \|, \| \|^0)$ if and only if there exist scalars α and $\alpha_n (n = 0, 1, \ldots)$ such that $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ and*

$$f(x) = \alpha a(x) + \sum_{n=0}^{\infty} \alpha_n \pi_n(x).$$

Proof. We need only to consider the case where there exists an $w_0 \in l^\infty \cap \widetilde{A}$ such that $a(x_0) \neq 0$. Then as shown in Section 2, there exists a projection map $L : l^\infty \cap \widetilde{A} \to l^\infty \cap \widetilde{A}_0$ such that $x = Lx + [f(x_0)/a(x_0)]x_0$. Therefore $f(x) = (f \circ L)(x) + [f(x)/a(x_0)]a(x)$. Let us denote by $\overline{f}$ the restriction of the functional f to the space $l^\infty \cap \widetilde{A}_0$, then $f \circ L = \overline{f} \circ L$ and by Theorem 3.2 $f(Lx) = \sum_{n=0}^{\infty} \alpha_n \pi_n(Lx)$, where $\sum_{n=0}^{\infty} |\alpha_n| < \infty$. Since $Lx = x - [a(x)/a(x_0)]x_0$, we see that $\pi_n(Lx) = \pi_n(x) - [a(x)/a(x_0)]\pi_n(x_0)$, whence

$$f(x) = \alpha a(x) + \sum_{n=0}^{\infty} \alpha_n \pi_n(x)$$

with $\alpha = \left[f(x_0) - \sum_{n=0}^{\infty} \alpha_n \pi_n(x_0) \right] / a(x_0)$.

Conversely, each functional of this type is obviously γ-linear.

4. INCLUSION THEOREMS

Now we shall deal with two Toeplitz methods A and B. Here b_{nk}, $b_{.n}$, b will have the obvious meaning.

We are interested with the case when the bounded efficiency field of the method A is contained in the efficiency field of the method B.

In this case, if the method A is null-conservative, so is the method B.

Theorem 4.1. *Let the method A be null-conservative and let $l^\infty \cap \widetilde{A}_0 \subset \widetilde{B}$, then for every $x \in l^\infty \cap \widetilde{A}_0$*

$$b(x) = \sum_{n=0}^{\infty} \alpha_n \pi_n(x),$$

where $\sum_{n=0}^{\infty} |\alpha_n| < \infty$.

Proof. By Theorem 2.4 the functionals $b_{.n}$ are γ-linear as pointwise limits of the functionals $x \mapsto \sum_{j=0}^{m} b_{nj}\pi_j(x)$ so, again, the functional b is γ-linear. Apply Theorem 3.2.

Theorem 4.2. *Let the method A be conservative and let $l^\infty \cap \widetilde{A} \subset \widetilde{B}$, then there are scalars α, α_n such that $\sum_{n=0}^{\infty} |\alpha_n| < \infty$ and for every $x \in l^\infty \cap \widetilde{A}$*

$$b(x) = \alpha a(x) + \sum_{j=0}^{\infty} \alpha_j \pi_j(x).$$

Proof. The same as for Theorem 4.1.

Now we introduce the following notation. Let $e = (\delta_{kk} : k = 0, 1, \ldots), e_n = (\delta_{nk} : k = 0, 1, \ldots)$, then $a_{.n} = a(e_n), b_{.n} = b(e_n)$. We shall denote also $k_A(x) = a(x) - \sum_{j=0}^{\infty} a_{.j}\pi_j(x)$. For conservative methods

$$k_A(e) = \lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj} - \sum_{j=0}^{\infty} a_{.j}.$$

Let now $l^\infty \cap \widetilde{A} \subset \widetilde{B}$, then $b_{.n} = \alpha a_{.n} + \alpha_n$, therefore

$$b(x) = \alpha a(x) + \sum_{j=0}^{\infty} (b_{.j} - \alpha a_{.j})\pi_j(x)$$

for every $x \in l^\infty \cap \widetilde{A}$.

Suppose now that $k_A(x) = 0$ for every x in $l^\infty \cap \widetilde{A}$, then $a(x) = \sum_{j=0}^{\infty} a_{.j}\pi_j(x)$, whence

$$b(x) = \sum_{j=0}^{\infty} b_{.j}\pi_j(x)$$

in $l^\infty \cap \widetilde{A}$.

Consider now the case when there exists an $x_0 \in l^\infty \cap \widetilde{A}$ such that $k_A(x_0) \neq 0$. Then

$$b(x_0) = \alpha a(x_0) + \sum_{j=0}^{\infty} (b_{.j} - \alpha a_{.j})\pi_j(x_0).$$

which gives

$$b(x) = (k_B(x_0)/k_A(x_0)) \left[a(x) - \sum_{j=0}^{\infty} a_{.j}\pi_j(x) \right] + \sum_{j=0}^{\infty} b_{.j}\pi_j(x).$$

So we obtain

Theorem 4.3. *Let the method A be conservative and let $l^\infty \cap \widetilde{A} \subset \widetilde{B}$, then for every $x \in l^\infty \cap \widetilde{A}$*

$$b(x) = k_A(x)k_B(x_0)/k_A(x_0) + \sum_{j=0}^{\infty} b_{.j}\pi_j(x)$$

if $k_A(x_0) \neq 0$ for some x_0, or else

$$b(x) = \sum_{j=0}^{\infty} b_{.j}\pi_j(x).$$

Now we are able to state some consistency theorems. Two Toeplitz mehtods A and B are said to consistent for sequences of a set E if $a(x) = b(x)$ for every $x \in \widetilde{A} \cap \widetilde{B} \cap E$.

Theorem 4.4. *Let the method A be null-conservative, let $k_A(x) = 0$ for every $x \in l^\infty \cap \widetilde{A}$ and let $l^\infty \cap \widetilde{A} \subset \widetilde{B}$. If those methods are consistent for null-convergent sequences, they are consistent for bounded sequences.*

Proof. By Theorem 3.2 $a(x) = \sum_{n=0}^{\infty} a_{.n}\pi_n(x)$. Now $a_{.n} = b_{.n}$, apply Theorem 4.3.

Our main theorem reads

Theorem 4.5. *Let the method A be null-conservative, let $l^\infty \cap \widetilde{A} \subset \widetilde{B}$ and let those methods be consistent for null-convergent sequences. Then those methods are consistent for bounded sequences if and only if either $k_A(x) = 0$ for every x in $l^\infty \cap \widetilde{A}$, or else if there exists a sequence x_0 in $l^\infty \cap \widetilde{A}$ such that $a(x_0) = b(x_0)$ and $k_A(x_0) \neq 0$.*

Proof. Necessity of the condition is obvious. Suppose it to be satisfied. The first case is covered by Theorem 4.4. In the second case $a_{.j} = b_{.j}$ for every j, whence $k_A(x_0) = k_B(x_0) \neq 0$ and, by Theorem 4.3, $a(x) = b(x)$ in $l^\infty \cap \widetilde{A}$.

As application one obtains

Corollary 4.6. *Let the method A be null conservative, let $l^\infty \cap \widetilde{A} \subset \widetilde{B}$, and let those methods be consistent for null-convergent sequences. If $k_A(x_0) \neq 0$ for some null-convergent sequence x_0, then those methods are consistent for bounded sequences.*

The case when $x_0 = e$ is the Mazur-Orlicz consistency Theorem ([4], p. 140).

In Corollary 4.6 one cannot get rid of the condition $k_A(x_0) \neq 0$. Indeed let the methods A, B be such that $a_n(x) = \pi_{2n}(x) - \pi_{2n+1}(x), b_n(x) = a_{2n}(x)$. Both are conservative, $a(x) = b(x)$ for null-convergent sequences, however for $x_0 = (1, -1, 1, \ldots) : a(x_0) = 2, b(x_0) = 1$.

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