Note di Matematica Vol. IX - n. 2, 217-220(1989)

A REMARK ABOUT THE EMBEDDING $(H(E/F),\tau) \to (H(E),\tau)$, WITH $\tau = \tau_0, \tau_\omega$, IN FRECHET SPACES

S. PONTE [(1)]

Abstract. *In a recent paper by Aron-Moraes-Ryan [2], it is proved that when E is a complex Banach space, F is a closed subspace of E and U is a balanced open subset of E, then the mapping*

$$f \in H(\pi(U)) \to f \circ \pi \in H(U),$$

where π is the canonical mapping from E onto E/F, is a topological isomorphism from $(H(\pi(U)),\tau)$ onto a closed subspace of $(H(U),\tau)$, where $\tau = \tau_0, \tau_\omega$.

The aim of this remark is to show that the same result is true, with τ_0 for Fréchet spaces, and with τ_ω for Fréchet-Schwartz spaces. Also we prove that this result is not true with τ_ω for some Fréchet-Montel spaces and with τ_δ for some nuclear Fréchet spaces.

For a complex locally convex space E and an open subset U of E, $H(U)$ will denote the space of all holomorphic functions on U. On $H(U)$ we will consider the usual topologies τ_0, τ_ω and τ_δ : τ_0 is the compact open topology, τ_ω is the Nachbin ported topology, it is the locally convex topology generated in $H(U)$ by the seminorms p on $H(U)$ which are ported by some compact subset K of U; p is ported by K if for every open subset V of U, $K \subset V$, there exists $C > 0$ such that

$$p(f) \leq C\sup\{|f(x)| : x \in V\} \quad for\ all \quad f \in H(U).$$

τ_δ is the locally convex topology generated in $H(U)$ by the seminorms p on $H(U)$ such that for every increasing countable open cover of U, (U_n), there exist $C > 0$ and $k \in \mathbb{N}$ such that

$$p(f) \leq C\sup\{|f(x)| : x \in U_k\} \quad for\ all \quad f \in H(U).$$

To prove that the mapping $f \to f \circ \pi$ is an embedding of $(H(\pi(U)),\tau_0)$ in $(H(U),\tau_0)$, when E is a Fréchet space, we need the following result:

Lemma. *Let E be a Fréchet space and U an open subset of E. Then for every compact subset J of $\pi(U)$ there is a compact subset K in U such that $\pi(K) = J$.*

The proof of this Lemma is analogous to that of Proposition 18, Section 2, Chapter IX in [3] using the fact that every open subset of a Fréchet space is homeomorphic to a complete metric space ([7], Th. 2.76). See also [6], p. 57.

---

(1) Supported by the D.G.I.C.Y.T. of Spain. PB87-1031.

Theorem 1. *If E is a Fréchet space, then the mapping $f \to f \circ \pi$ is an isomorphism of $(H(\pi(U)), \tau_0)$ onto a closed subspace of $(H(U), \tau_0)$.*

Proof. It is clear that this mapping is linear, injective and continuous. It is also open onto its image as a direct consequence of the Lemma. Since $(H(\pi(U)), \tau_0)$ is complete ([4], p. 129), this image is closed in $(H(U), \tau_0)$.

Let us consider now a Fréchet-Schwartz space E and a closed subspace F of E. It is known ([5]) that E/F is also a Fréchet-Schwartz space.

By a result of Mujica [8], the topologies τ_0 and τ_ω agree on $H(U)$ (U being a balanced open subset of a Fréchet-Schwartz space), then from Theorem 1 we obtain

Theorem 2. *If E is a Fréchet-Schwartz space and F is a closed subspace of E, then the mapping $f \to f \circ \pi$ is a topological isomorphism of $(H(\pi(U)), \tau_\omega)$ onto a closed subspace of $(H(U), \tau_\omega)$ for every balanced open subset U of E.*

When one considers Fréchet-Montel spaces which are not Schwartz spaces the situation can change as the following shows.

Let $\lambda(A)$ be a Fréchet-Montel-Köthe echelon space. By a resault of Ansemil-Ponte [1], for this kind of spaces we have that

$$(H(U), \tau_0) = (H(U), \tau_\omega)$$

for every balanced open subset U of $\lambda(A)$.

On the other hand, if $\lambda(A)$ is a Fréchet-Montel, non Schwartz, Köthe echelon space, then it has a quotient which is isomorphic to ℓ^1 ([9], see also [10]). Then this quotient is not a Montel space.

Since for E a Fréchet space, the equality $(H(U), \tau_0) = (H(U), \tau_\omega)$ for some open subset U of E implies that E is a Montel space ([4]), we have the following

Theorem 3. *For every Fréchet-Montel, non Schwartz, Köthe echelon space $\lambda(A)$ there is a closed subspace F of $\lambda(A)$ such that the mapping*

$$(H(\pi(U)), \tau_\omega) \to (H(U), \tau_\omega)$$

$$f \to f \circ \pi$$

is not embedding for every balanced open subset U of $\lambda(A)$.

Proof. Let us consider a closed subspace F of $\lambda(A)$ such that $\lambda(A)/F$ is not a Montel space. Then $\tau_0 \neq \tau_\omega$ on $H(\pi(U))$. If the mapping $f \to f \circ \pi$ is an isomorphism from $(H(\pi(U)), \tau_\omega)$ onto a closed subspace of $(H(U), \tau_\omega)$, then, since $\tau_0 = \tau_\omega$ on $H(U)$, we get a contradiction by Theorem 1.

Note. For the τ_δ topology there is an analogous to Theorem 3 when E is a nuclear Fréchet space which has property (DN). Indeed, it is known that when E is a nuclear Fréchet space $\tau_0 = \tau_\delta$ on $H(E)$ if and only if E has (DN) ([4]). Since there are nuclear Fréchet spaces with (DN) (for exemple $H(\mathbb{C})$) with quotients which do not have (DN) ($\mathbb{C}^{\mathbb{N}}$ is a quotient of $H(\mathbb{C})$ which does not have (DN)), then a similar proof to one in Theorem 3 shows the note.

This last note is due to the referee whom I would like to thank.

REFERENCES

[1] J.M. Ansemil, S. Ponte, *The compact open and the Nachbin ported topologies on spaces of holomorphic functions*, Arch. Math.**51** (1988) 65-70.

[2] R.M. Aron, L.A. Moraes, R.A. Ryan, *Factorization of holomorphic mappings in infinite dimension*, Math. Ann. **277** (1987) 617-628.

[3] N. Bourbaki, *Elements of Mathematics: General Topology*, Part 2, Addison Wesley, 1966.

[4] S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Stud. **57**, 1981.

[5] A. Grothendieck, *Sur les espaces (F) and (DF)*, Summa Brasil. Math. **3**, 1954.

[6] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955)

[7] J.G. Hocking and G.S. Young, *Topology*, Addison-Wesley, 1961.

[8] J. Mujica, *A Banach-Dieudonné theorem for germs of holomorphic functions* , J. Funct. Anal. **27** (1984) 31-48.

[9] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland Math. Stud. **67**, 1982.

[10] M. Valdivia, *(F)-Spaces and strict (LF)-spaces*, Math. Z. **195** (1987) 345-364.

Received December 2, 1988.
S. Ponte
Departamento de Analisi Matemático
Facultad de Matemática
Universidad Complutense
28040 Madrid, Spain