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SUPERCONVEX SETS AND σ-CONVEX SETS, AND THE EMBEDDING OF CONVEX AND SUPERCONVEX SPACES

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Dedicated to the memory of Professor Gottfried Köthe

0. INTRODUCTION

The theory of superconvex spaces has been developed over the past fifteen years in the circle of the first-named author in Saarbrücken; in particular fundamental contributions are due to Gerd Rodé [7] [8] and Norbert Kuhn. It is an abstract theory on the formation of countable convex combinations, with the primary purpose to extend and to reinforce the power of a technique due to Stephen Simons, the roots of which date back to R.C. James and J.D. Pryce. We refer to its systematic presentation in [1], with the subsequent new versions of the main theorems in [2]. We adopt the basic notions and facts on convex and superconvex spaces from [1] Section 1, which for the sake of completeness will be summarized in §1 below.

There are certain classes of natural examples of convex and superconvex spaces. The present paper is devoted to the problem whether and when a convex or superconvex space (X, I) can be identified with some member of such a class, and thus also to explore the relations between these classes.

The relevant kinds of morphisms are quite obvious: For convex spaces (X, I) and (Y, J) a map $\vartheta : X \to Y$ is called a c-*map* iff

$$\vartheta\left(I\begin{pmatrix}1-t & t\\ u & v\end{pmatrix}\right) = J\begin{pmatrix}1-t & t\\ \vartheta(u) & \vartheta(v)\end{pmatrix} \qquad \forall u, v \in X \text{ and } 0 \leq t \leq 1;$$

it is equivalent to require that

$$\vartheta\left(\overset{\infty}{\underset{l=1}{I}}(t_l, x_l)\right) = \overset{\infty}{\underset{l=1}{J}}\left(t_l, \vartheta(x_l)\right) \quad \forall t = (t_l)_l \in P \text{ and } x = (x_l)_l \in X^\infty.$$

Then $\vartheta(X) \subset Y$ is a convex subset of Y. If in addition $\vartheta : X \to Y$ is bijective then the inverse map $\vartheta^{-1} : X \leftarrow Y$ is a c-map as well; in this case ϑ will be called a c-*isomorphism.* Likewise for superconvex spaces (X, I) and (Y, J) a map $\vartheta : X \to Y$ is called an sc-*map* iff

$$\vartheta\left(\overset{\infty}{\underset{l=1}{I}}(t_l, x_l)\right) = \overset{\infty}{\underset{l=1}{J}}\left(t_l, \vartheta(x_l)\right) \quad \forall t = (t_l)_l \in Q \text{ and } x = (x_l)_l \in X^\infty.$$

Then $\vartheta(X) \subset Y$ is a superconvex subset of Y. If in addition $\vartheta : X \to Y$ is bijective then $\vartheta^{-1} : X \leftarrow Y$ is an sc-map as well; in this case ϑ will be called an sc-*isomorphism.* We mention one more obvious fact: If $\vartheta : X \to Y$ is a c-isomorphism between the convex spaces (X, I) and (Y, J), then the superconvex extensions $S : Q \times X^\infty \to X$ of I and $T : Q \times Y^\infty \to Y$ of J are in one-to-one correspondence via the condition that ϑ be an sc-isomorphism between (X, S) and (Y, T).

We shall start with the convex spaces. The standard example of a convex space is a nonvoid convex subset $Y \subset E$ of a real vector space E with its natural convex structure

$$C : C(t, x) = \sum_{l=1}^{\infty} t_l x_l \quad \forall t = (t_l)_l \in P \text{ and } x = (x_l)_l \in Y^\infty.$$

Thus a convex space (X, I) is c-isomorphic to some such standard example (Y, C) iff there exists an injective c-map $\vartheta : X \to E$ into a real vector space E. We shall see in §2 that this holds true iff (X, I) satisfies the *cancellation law* introduced in [1] Section 3, defined to mean that

$$I\begin{pmatrix} 1-t & t \\ a & u \end{pmatrix} = I\begin{pmatrix} 1-t & t \\ a & v \end{pmatrix} \Rightarrow u = v \quad \forall a, u, v \in X \text{ and } 0 < t < 1.$$

An equivalent condition is that the set Aff (X, I) of the affine functions $f : X \to \mathbb{R}$ separates the points of X. The proof will be based on an algebraic standard procedure.

We pass to the superconvex spaces. Let us define a nonvoid convex subset $Y \subset E$ of a real vector space E to be *superconvex* iff its natural convex structure C can be extended to some superconvex structure $SC : Q \times Y^\infty \to Y$ on Y. It is quite clear that such a superconvex extension does not exist for all nonvoid convex $Y \subset E$; in particular in case dim $E < \infty$ it will exist iff Y is bounded (in the pairwise equivalent norms on E). It follows that a superconvex space (X, I) is sc-isomorphic to some such standard example (Y, SC) iff its convex restriction $(X, I|P \times X^\infty)$ is c-isomorphic to some convex standard example (Y, C) as above, that is iff $X, I|P \times X^\infty)$ satisfies the cancellation law.

Now a basic uniqueness theorem due to Rodé [8] asserts that a superconvex subset $Y \subset E$ of a real vector space E carries a unique superconvex structure SC which extends its natural convex structure C. It is because of this uniqueness that the superconvex subsets $Y \subset E$ form a basic class of superconvex standard examples (Y, SC). In view of §2 below an equivalent statement is that a convex space (X, I) which satisfies the cancellation law admits at most one superconvex structure which extends I. This is the uniqueness result [1] Theorem 3.1 which appeared to be more comprehensive. However, the benefit of [1] Section 3 is that it isolates [1] Lemma 3.4 as the main step in the proof of the uniqueness theorem. In the present paper this lemma will be a basic component in the first proof of our main result of §4 below, which in its turn will be decisive for the rest of the paper. It asserts that, on a superconvex space (X, I) with the cancellation law, the set BAff (X, I) of the bounded $f \in$ Aff (X, I), which happen to be the superaffine functions $f : X \to \mathbb{R}$, separates the points of X. Besides the first proof, which leans on some basic points of the convex and superconvex theories, we shall present a second proof of this result which can be done with bare hands. Let us note that the main result of §4, and hence in particular its direct second proof, permits to reprove the basic uniqueness theorem. In §3 we isolate certain simple preliminaries on convex subsets of real vector spaces.

Besides the notion of a superconvex set one considers another kind of standard example of a superconvex space, which appears to be of a more concrete character. This is, in a real vector space E equipped with a Hausdorff vector-space topology $\mathscr{T}$, a nonvoid subset $Y \subset E$ which is σ-*convex* in $\mathscr{T}$, defined to mean that

$$\sum_{l=1}^{\infty} t_l x_l \text{ converges in } \mathscr{T} \text{ to some member } =: I(t, x) \in Y$$

$$\forall t = (t_l)_l \in Q \text{ and } x = (x_l)_l \in Y^{\infty}.$$

Then Y is convex, and the map $I : Q \times Y^{\infty} \to Y$ thus defined extends the natural convex structure C of Y. We shall see in §1 that a nonvoid subset $Y \subset E$ is σ-convex in $\mathscr{T}$ iff it is superconvex and bounded in $\mathscr{T}$, and that then I is the natural superconvex structure SC of Y. The problem arises to characterize those superconvex spaces (X, I) which are sc-isomorphic to some standard example of the new kind. In view of the former results, this amounts to characterize those superconvex subsets $Y \subset E$ of a real vector space E which are σ-convex in some Hausdorff vector-space topology $\mathscr{T}$ on E. Thus our natural problem is, for a superconvex subset $Y \subset E$ of a real vector space E, to characterize those Hausdorff vector-space topologies $\mathscr{T}$ on E in which Y is σ-convex.

The latter problem has a comprehensive and pleasant solution. Let $Y \subset E$ be a superconvex subset of a real vector space E. First note that if Y is σ-convex in a Hausdorff vector-space topology $\mathscr{T}$ on E then the same is true in each weaker $\mathscr{S} \subset \mathscr{T}$ of equal sort. Now assume that $\operatorname{Lin}(Y) = E$. Then our main theorem in §5 states that there exist topologies $\mathscr{T}$ as required, and that there is an (of course unique) maximum such one $\mathscr{M}$, which is a Banach-space topology on E and is obtained as follows: The circled convex hull $Z := \operatorname{conv}(Y \cup (-Y)) \subset E$ of Y is absorbent, and thus its Minkowski functional $||\cdot||$ is a seminorm on E. It turns out that $||\cdot||$ is a complete norm on E, and that its topology $\mathscr{M}$ is the desired maximum Hausdorff vector-space topology on E in which Y is σ-convex. The hardest point in the proof is that $||\cdot||$ is in fact a norm; this requires our main result of §4 quoted above.

In the above situation the open mapping theorem implies that the topology $\mathscr{M}$ of the complete norm $||\cdot||$ is the unique Banach-space topology on E in which Y is σ-convex. Also note that on a real vector space E the existence of a superconvex set $Y \subset E$ with $\operatorname{Lin}(Y) = E$ is equivalent to the existence of a complete norm $||\cdot||$ on E.

In conclusion we fix a real vector space E and ask for superconvex sets $Y \subset E$ with $\mathrm{Lin}(Y) = E$ and complete norms $|||\cdot|||$ on E such that Y is not σ-convex in $|||\cdot|||$. In case $\dim E < \infty$ all superconvex subsets $Y \subset E$ are σ-convex in the unique Hausdorff vector-space topology on E. Thus we assume E to be infinite-dimensional. Now one observes that on each infinite-dimensional Banach space $(E, ||\cdot||)$ there exists an abundance of complete norms $|||\cdot|||$ which are non-equivalent to $||\cdot||$ and to each other. In fact, fix a Hamel basis B of E, and form for each function $\varphi : B \to]0,\infty[$ the unique linear map $T_\varphi : E \to E$ with $T_\varphi u = \varphi(u)u \ \forall u \in B$, so that T_φ is bijective and T_1 is the identity map. Then $||\cdot||_\varphi : ||x||_\varphi = ||T_\varphi x|| \ \forall x \in E$ is a complete norm on E; and for $\varphi, \psi : B \to]0,\infty[$ the estimation $||\cdot||_\psi \leq c||\cdot||_\varphi$ with $c > 0$ implies that $\psi \leq c\varphi$. We refer to Laugwitz [4] [5] where this idea has been carried further in order to obtain the cardinality of the set of all Banach-space topologies on E.

Thus if $Y \subset E$ is superconvex with $\mathrm{Lin}(Y) = E$ and $||\cdot||$ is its natural complete norm, then Y is not σ-convex in any of the complete norms $|||\cdot|||$ on E which are non-equivalent to $||\cdot||$. On the other hand we obtain, in any infinite-dimensional real Banach space $(E, ||\cdot||)$, in form of the open and closed balls with respect to the non-equivalent complete norms $|||\cdot|||$ on E, myriads of superconvex sets $Y \subset E$ with $\mathrm{Lin}(Y) = E$ which are not σ-convex in $||\cdot||$.

1. CONVEX UND SUPERCONVEX SPACES

We summarize the basic notions and facts from [1] Section 1, with a few additions. Let Q consist of the sequences $t = (t_l)_l$ of real numbers $t_l \geq 0 \ \forall l \in \mathbf{N}$ with $\sum_{l=1}^{\infty} t_l = 1$, and P of the $t = (t_l)_l$ in Q with $t_l = 0$ for almost all $l \in \mathbf{N}$. For a nonvoid set X let X^∞ consist of the sequences $x = (x_l)_l$ of elements $x_l \in X \ \forall l \in \mathbf{N}$. A *superconvex structure* on X is defined to be a map

$$I : Q \times X^\infty \to X, \text{ written } I(t,x) = \mathop{I}_{l=1}^{\infty}(t_l, x_l) = I\begin{pmatrix} t_1 \ldots t_n \ldots \\ x_1 \ldots x_n \ldots \end{pmatrix},$$

with the properties

1) $\mathop{I}_{l=1}^{\infty}(\delta_l^p, x_l) = x_p \ \forall p \in \mathbf{N}$,

2) $\mathop{I}_{p=1}^{\infty}\left(t_p, \mathop{I}_{l=1}^{\infty}(t_l^p, x_l)\right) = \mathop{I}_{l=1}^{\infty}\left(\sum_{p=1}^{\infty} t_p t_l^p, x_l\right) \ \forall t \in Q$ and $t^p \in Q \ \forall p \in \mathbf{N}$;

note that this makes sense because $\left(\sum_{p=1}^{\infty} t_p t_l^p,\right)_l$ is in Q. A *convex structure* on X is defined to be a map $I : P \times X^\infty \to \infty$ with the respective properties 1) 2). Then (X, I) is called a *superconvex* resp. *convex space.* The axioms 1) and 2) permit to a wide extent to handle the operation I like conventional convex combinations. In particular one deduces that $t_l = 0$ for an $l \in \mathbf{N}$ implies that $I(t, x)$ is independent of $x_l \in X$; thus we can write

$$I\begin{pmatrix} t_1 \dots t_n \\ x_1 \dots x_n \end{pmatrix} := I\begin{pmatrix} t_1 \dots t_n & 0 & 0 & \dots \\ x_1 \dots x_n & * & * & \dots \end{pmatrix}$$

for $t_1 \dots, t_n \geq 0$ with $\sum_{l=1}^{n} t_l = 1$ and $x_1, \dots, x_n \in X$, where $* * \dots$ denote arbitrary elements of X.

In a convex space (X, I) a subset $A \subset X$ is called *convex* iff $I(t, x) \in A$ for all $t \in P$ and $x \in A^\infty$. In a superconvex space (X, I) a subset $A \subset X$ is called *superconvex* iff $I(t, x) \in A$ for all $t \in Q$ and $x \in A^\infty$.

Let (X, I) be a convex space. A function $f : X \to \mathbf{R}$ is called *convex* iff

$$f\left(I\begin{pmatrix} 1-t & t \\ u & v \end{pmatrix}\right) \leq (1-t)f(u) + tf(v) \quad \forall u, v \in X \text{ and } 0 \leq t \leq 1;$$

it is equivalent to require that

$$f(I(t, x)) \leq \sum_{l=1}^{\infty} t_l f(x_l) \quad \forall t = (t_l)_l \in P \text{ and } x = (x_l)_l \in X^\infty.$$

And $f : X \to \mathbf{R}$ is called *concave* resp. *affine* iff the above holds true with $\geq$ resp. $=$. Note that in the present paper we shall not need functions with infinite values. Let $\mathrm{Aff}(X, I)$ consist of the affine functions $f : X \to \mathbf{R}$, and $\mathrm{BAff}\ (X, I)$ of the bounded functions $f \in \mathrm{Aff}(x, I)$.

As an important example we transfer the notion of the Minkowski functional to the context of convex spaces. We shall not need but the particular case of the Minkowski functional $\phi_a : X \to [0, 1]$ relative to the entire space X and to a base point $a \in X$; it is somewhat simpler than its counterpart relative to a nonvoid convex subset $A \subset X$ and to a base point $a \in A$ (in the usual real vector space situation the base point will be $a = 0$). The definition is

$$\phi_a(x) := \mathrm{Inf}\left\{0 < t \leq 1 : x \in I\begin{pmatrix} 1-t & t \\ a & X \end{pmatrix}\right\} \quad \forall x \in X;$$

note that with $0 < t \leq 1$ each number in $[t, 1]$ is as required. This function appeared in [1] Section 3 under the notation $\phi_a(x) = \varepsilon(a, x)\ \forall a, x \in X$.

Proposition 1.1. *For $a \in X$ the function $\phi_a : X \to [0,1]$ is convex.*

Proof. It suffices to prove that

$$\phi_a(x) \le (1-\lambda)\phi_a(u) + \lambda\phi_a(v) \text{ for } u, v \in X \text{ and } 0 \le \lambda \le 1 \text{ with } x := I\begin{pmatrix} 1-\lambda & \lambda \\ u & v \end{pmatrix}.$$

Fix $0 < s, t \le 1$ such that

$$u = I\begin{pmatrix} 1-s & s \\ a & p \end{pmatrix} \text{ and } v = I\begin{pmatrix} 1-t & t \\ a & q \end{pmatrix} \text{ with } p, q \in X,$$

and put $\rho := (1-\lambda)s + \lambda t$; thus $0 < \rho \le 1$. Then

$$x = I\begin{pmatrix} 1-\lambda & \lambda \\ I\begin{pmatrix} 1-s & s \\ a & p \end{pmatrix} & I\begin{pmatrix} 1-t & t \\ a & q \end{pmatrix} \end{pmatrix} =$$

$$= I\begin{pmatrix} (1-\lambda)(1-s) + \lambda(1-t) & (1-\lambda)s & \lambda t \\ a & p & q \end{pmatrix} =$$

$$= I\begin{pmatrix} 1-\rho & \rho \\ a & z \end{pmatrix} \text{ with } z := I\begin{pmatrix} \frac{1}{\rho}(1-\lambda)s & \frac{1}{\rho}\lambda t \\ p & q \end{pmatrix} \in X.$$

It follows that $\phi_a(x) \le \rho$ and hence the assertion.

Now let (X, I) be a superconvex space. A function $f : X \to \mathbb{R}$ is called *superconvex* iff

$$f(I(t,x)) \le \sum_{l=1}^{\infty} t_l f(x_l) \quad \forall t = (t_l)_l \in Q \text{ and } x = (x_l)_l \in X^\infty$$

such that $\big(f(x_l)\big)_l$ is bounded above; note that then the infinite series involved converges in $[-\infty, \infty[$ and hence in $\mathbb{R}$. We call $f : X \to \mathbb{R}$ *superconcave* iff $-f$ is superconvex, and *superaffine* iff both f and $-f$ are superconvex. Let SAff (X, I) consist of the superaffine functions $f : X \to \mathbb{R}$.

We recall from [1] Remark 1.9 that

i) a convex function $f : X \to \mathbb{R}$ which is bounded above is superconvex, and ii) a superconvex function $f : X \to \mathbb{R}$ is bounded below; hence

i) a concave function $f : X \to \mathbb{R}$ which is bounded below is superconcave, and ii) a superconcave function $f : X \to \mathbb{R}$ is bounded above.

Therefore a function $f : X \to \mathbb{R}$ is superaffine iff it is affine and bounded; in other words SAff $(X, I) =$ BAff (X, I).

In conclusion we recall [1] Example 1.6 to take a first look at the σ-convex sets.

Proposition 1.2. *Let E be a real vector space and $\mathscr{T}$ be a Hausdorff vector-space topology on E. For a nonvoid subset $Y \subset E$ the following are equivalent.*

1) Y is σ-convex in $\mathscr{T}$.

2) Y is superconvex and bounded in $\mathscr{T}$.

In this case

$$I : I(t,x) = \sum_{l=1}^{\infty} t_l x_l := \mathscr{T} \lim_{n\to\infty} \sum_{l=1}^{n} t_l x_l \quad \forall t = (t_l)_l \in Q \text{ and } x = (x_l)_l \in Y^{\infty}$$

is the unique superconvex structure on Y which extends its natural convex structure C.

Proof of 1) $\Rightarrow$ 2). It has been shown in [1] Example 1.6 that the above $I : Q \times Y^{\infty} \to Y$ is a superconvex structure on Y; we remark that this can be done in a more natural manner without the use of the Banach-Steinhaus theorem.

Proof of 2) $\Rightarrow$ 1). Assume that $T : Q \times Y^{\infty} \to Y$ is a superconvex structure on Y which extends its natural convex structure C. Let $t = (t_l)_l \in Q \setminus P$ and $x = (x_l)_l \in Y^{\infty}$. For $n \in \mathbf{N}$ we put

$$\tau_n := \sum_{l=n+1}^{\infty} t_l > 0 \text{ and } z_n := \underset{l=n+1}{\overset{\infty}{T}} \left(\frac{1}{\tau_n} t_l, x_l\right) \in Y.$$

Then

$$T(t,x) = T\begin{pmatrix} t_1 \dots t_n \ \tau_n \\ x_1 \dots x_n \ z_n \end{pmatrix} = \sum_{l=1}^{n} t_l x_l + \tau_n z_n .$$

Now $\tau_n z_n \to 0$ for $n \to \infty$ in $\mathscr{T}$ since Y is bounded in $\mathscr{T}$. The assertion follows.

Note that this time we did not use the uniqueness theorem of Rodé.

Furthermore we recall from [1] Example 1.6 the particular classes of σ-convex sets which follow.

Remark 1.3. 1) Let E be a finite-dimensional real vector space and $\mathscr{T}$ be the unique Hausdorff vector-space topology on E. Then each nonvoid convex subset $Y \subset E$ which is bounded in $\mathscr{T}$ is σ-convex in $\mathscr{T}$. 2) Let $(E, ||\cdot||)$ be a real Banach space. Then each open or closed nonvoid bounded convex subset $Y \subset E$ is σ-convex in $||\cdot||$.

2. THE EMBEDDING THEOREM

The aim of the present section is the theorem which follows.

Theorem 2.1. *For a convex space* (X, I) *the following are equivalent.*

1) There exists an injective c-map $\vartheta : X \to E$ *into a real vector space* E.

2) Aff (X, I) *satisfies the cancellation law.*

3) (X, I) *satisfies the cancellation law.*

In this case it can be achieved that $E = \operatorname{Lin}(\vartheta(X)) \neq \operatorname{Lin}(\vartheta(X) - \vartheta(X))$.

Proof of 1) $\Rightarrow$ 2). For $u, v \in X$ with $u \neq v$ we have $\vartheta(u) \neq \vartheta(v)$. Thus there exists $\varphi \in E^*$ with $\varphi(\vartheta(u)) \neq \varphi(\vartheta(v))$. Then $f := \varphi \circ \vartheta$ is in $\operatorname{Aff}(X, I)$ and fulfills $f(u) \neq f(v)$.

Proof of 2) $\Rightarrow$ 3). Assume that $a, u, v \in X$ and $0 < t < 1$ are such that

$$I\begin{pmatrix} 1-t & t \\ a & u \end{pmatrix} = I\begin{pmatrix} 1-t & t \\ a & v \end{pmatrix}.$$

For $f \in \operatorname{Aff}(X, I)$ then

$$(1-t)f(a)+tf(u) = f\left(I\begin{pmatrix} 1-t & t \\ a & u \end{pmatrix}\right) = f\left(I\begin{pmatrix} 1-t & t \\ a & v \end{pmatrix}\right) = (1-t)f(a)+tf(v),$$

and hence $f(u) = f(v)$. From 2) we obtain $u = v$.

The proof of 3) $\Rightarrow$ 1) will be based on an algebraic standard procedure of folklore character. It will be formulated below without proof in the appropriate version.

Algebraic Lemma 2.2. *Assume that the nonvoid set* H *is equipped with*

1) an associative and commutative addition $+ : H \times H \to H$ *which fulfills the cancellation law* $(-)$: $a + u = a + v \Rightarrow u = v \ \forall a, u, v \in H$;

2) a scalar multiplication $]0, \infty[\times H \to H : (s, x) \mapsto sx$ *with the properties*

i) $1x = x \quad \forall x \in H$,

ii) $(s + t)x = sx + tx$ and $(st)x = s(tx)$ $\forall s, t > 0$ and $x \in H$,

iii) $s(u + v) = su + sv \ \forall s > 0$ *and* $u, v \in H$.

Then there exists an injective map $\Delta : H \to E$ *into a real vector space* E *such that*

$$\Delta(u + v) = \Delta(u) + \Delta(v) \text{ and } \Delta(su) = s\Delta(u) \ \forall s > 0 \text{ and } u, v \in H.$$

It can be achieved that

$$E = \{\Delta(u) - \Delta(v) : u, v \in H\}. \tag{0}$$

If (0) is fulfilled then for each map $\varphi : H \to F$ *into a real vector space* F *such that*

$$\varphi(u + v) = \varphi(u) + \varphi(v) \text{ and } \varphi(su) = s\varphi(u) \quad \forall s > 0 \text{ and } u, v \in H$$

there exists a unique linear map $\phi : E \to F$ *which satisfies* $\varphi = \phi \circ \Delta$.

From the last assertion one obtains as usual the canonical uniqueness of $\Delta : H \to E$ under the condition (0).

Proof of 3) $\Rightarrow$ 1) *and of the final assertion.* i) On the product set $H :]0,\infty[\times X$ we define an addition $+ : H \times H \to H$ to be

$$(s,u) + (t,v) := \left(s+t, I \begin{pmatrix} \frac{s}{s+t} & \frac{t}{s+t} \\ u & v \end{pmatrix} \right).$$

One verifies that + is associative and commutative, and that it satisfies the cancellation law (−) as a consequence of assumption 3). Then we define a scalar multiplication $]0,\infty[x H \to H$ to be $t(s,x) := (ts,x)$. The properties i) ii) iii) in 2.2.2) are immediate. Thus 2.2 furnishes an injective map $\Delta : H \to E$ into a real vector space E such that

$$\Delta((s,u)+(t,v)) = \Delta(s,u) + \Delta(t,v) \ \forall (s,u),(t,v) \in H,$$

$$\Delta(t(s,x)) = t\Delta(s,x) \quad \forall t > 0 \text{ and } (s,x) \in H,$$

and $E = \{\Delta(s,y) - \Delta(t,v) : (s,u),(t,v) \in H\}$. ii) We define an injective map $\theta : X \to H$ to be $\theta(x) := (1,x)$ $\forall x \in X$. Then the composition $\vartheta := \Delta \circ \theta$ is an injective map $\vartheta : X \to E$. We have $\vartheta(x) = \Delta(1,x)$ and hence

$$t\vartheta(x) = t\Delta(1,x) = \Delta(t(1,x)) = \Delta(t,x) \quad \forall t > 0 \text{ and } x \in X.$$

Thus ϑ is a c-map since for $u,v \in X$ and $0 < t < 1$ we have

$$(1-t)\vartheta(u) + t\vartheta(v) = \Delta(1-t,u) + \Delta(t,v) =$$

$$= \Delta((1-t,u)+(t,v)) = \Delta\left(1, I\begin{pmatrix} 1-t & t \\ u & v \end{pmatrix}\right) = \vartheta\left(I\begin{pmatrix} 1-t & t \\ u & v \end{pmatrix}\right).$$

iii) We define a map $\varphi : H \to \mathbf{R}$ to be $\varphi(s,x) = s$. By the definition of the operations in H it is as required in the final part of 2.2. Thus 2.2 furnishes a linear functional $\phi \in E^*$ which satisfies $\varphi = \phi \circ \Delta$, that means

$$s = \phi(\Delta(s,x)) \quad \forall (s,x) \in H =]0,\infty[\times X.$$

In particular $\phi(\vartheta(x)) = \phi(\Delta(1,x)) = 1$ $\forall x \in X$. Thus $\phi|\vartheta(X) = 1$ and hence $\phi|(\vartheta(X) - \vartheta(X)) = 0$. It follows that

$$E = \{s\vartheta(u) - t\vartheta(v) : s,t > 0 \text{ and } u,v \in X\} = \mathrm{Lin}(\vartheta(X)) \text{ is} \neq \mathrm{Lin}(\vartheta(X) - \vartheta(X)).$$

The proof is complete.

3. PRELIMINARIES ON CONVEX SETS IN REAL VECTOR SPACES

In the present section let $Y \subset E$ be a nonvoid convex subset of a real vector space E. Then $\mathrm{Lin}(Y-Y) \subset \mathrm{Lin}(Y)$ with

$$\mathrm{Lin}(Y) = \{su - tv : s,t > 0 \text{ and } u,v \in Y\},$$
$$\mathrm{Lin}(Y-Y) = \{s(u-v) : s > 0 \text{ and } u,v \in Y\}.$$

We have to distinguish between the two cases (=) $\mathrm{Lin}(Y-Y) = \mathrm{Lin}(Y)$ and ($\neq$) $\mathrm{Lin}(Y-Y) \neq \mathrm{Lin}(Y)$.

Lemma 3.1. *For each affine function $f \in \mathrm{Aff}(Y)$ there exists a unique $\varphi \in (\mathrm{Lin}(Y))^*$ such that*

$$f - \varphi|Y = \left\{ \begin{array}{ll} \text{const} = c \in \mathbb{R} & \text{in case } (=) \\ 0 & \text{in case } (\neq) \end{array} \right\}.$$

Proof. 0) The uniqueness assertion is clear in both cases. i) There exists a unique function $\phi : \mathrm{Lin}(Y-Y) \to \mathbb{R}$ such that

$$\phi(s(u-v)) = s(f(u) - f(v)) \quad \forall s > 0 \text{ and } u,v \in Y.$$

In fact, the uniqueness of ϕ is obvious. In order to see its existence let

$$s_1(u_1 - v_1) = s_2(u_2 - v_2) \text{ with } s_l > 0 \text{ and } u_l, v_l \in Y (l = 1,2);$$

then

$$\frac{1}{s}s_1 u_1 + \frac{1}{s}s_2 v_2 = \frac{1}{s}s_2 u_2 + \frac{1}{s}s_1 v_1 =: x \in Y \text{ with } s := s_1 + s_2 > 0,$$

$$f(x) = \frac{1}{s}s_1 f(u_1) + \frac{1}{s}s_2 f(v_2) = \frac{1}{s}s_2 f(u_2) + \frac{1}{s}s_1 f(v_1),$$

$$s_1 \left(f(u_1) - f(v_1)\right) = s_2 \left(f(u_2) - f(v_2)\right),$$

so that the assertion follows. ii) One verifies that $\phi \in (\mathrm{Lin}(Y-Y))^*$. iii) Assume (=). For $u,v \in Y$ then $\phi(u) - \phi(v) = \phi(u-v) = f(u) - f(v)$, or $f(u) - \phi(u) = f(v) - \phi(v)$. Thus $\varphi := \phi$ is as required. iv) Assume ($\neq$) and fix $a \in Y$. It is obvious that $\mathrm{Lin}(Y) = (\mathbb{R}a)) \oplus \mathrm{Lin}(Y-Y)$. Let $\varphi \in (\mathrm{Lin}(Y))^*$ be the extension of ϕ with $\varphi(a) = f(a)$. For $x \in Y$ then $\varphi(x) = \varphi(a) + \varphi(x-a) = f(a) + \phi(x-a) = f(a) + (f(x) - f(a)) = f(x)$. Thus φ is as required.

Consequence 3.2. *In case* $(\neq)$ *the restriction map* $(\mathrm{Lin}(Y))^* \to \mathrm{Aff}(Y) : \varphi \mapsto \varphi|Y$ *is a bijection. In particular there exists a unique* $\alpha \in (\mathrm{Lin}(Y))^*$ *such that* $\alpha|Y = 1$; *it satisfies*

$$\mathrm{Lin}(Y - Y) = \{x \in \mathrm{Lin}(Y) : \alpha(x) = 0\}.$$

Proof. In the last assertion we have $\subset$, and hence $=$ since $\mathrm{Lin}(Y - Y) \subset \mathrm{Lin}(Y)$ has codimension one.

Lemma 3.3. *The following are equivalent. 1)* $\mathrm{BAff}\,(Y)$ *separates the points of* Y. *2)* $\{\varphi \in (\mathrm{Lin}(Y))^* : \varphi|Y$ *bounded*$\}$ *separates the points of* $\mathrm{Lin}(Y)$.

Proof. 2) $\Rightarrow$ 1) is obvious. In order to prove 1) $\Rightarrow$ 2) fix a nonzero $x \in \mathrm{Lin}(Y)$. We have to find $\varphi \in (\mathrm{Lin}(Y))^*$ such that $\varphi|Y$ is bounded and $\varphi(x) \neq 0$. If $x \notin \mathrm{Lin}(Y - Y)$ then we have $(\neq)$, and $\varphi := \alpha$ as obtained in 3.2 does it. If $x \in \mathrm{Lin}(Y - Y)$ and thus $x = s(u - v)$ with $s > 0$ and $u, v \in Y$ with $u \neq v$, then by assumption there is a bounded $f \in \mathrm{Aff}\,(Y)$ with $f(u) \neq f(v)$. Thus $\varphi \in (\mathrm{Lin}(Y))^*$ as obtained in 3.1 has a bounded restriction $\varphi|Y$ and satisfies $\varphi(x) = s\,(\varphi(u) - \varphi(v)) = s\,(f(u) - f(v)) \neq 0$.

The next proposition introduces the candidate $||\cdot||$ for the rôle prophesied in the Introduction.

Proposition 3.4. *Assume that* $\mathrm{Lin}(Y) = E$. *Define* $||\cdot|| : E \to [0, \infty[$ *to be*

$$||x|| := \mathrm{Inf}\,\{s + t : x = su - tv \text{ with } s, t > 0 \text{ and } u, v \in Y\} \quad \forall x \in E.$$

Then 1) $||\cdot||$ *is a seminorm on* E. *2) For* $\varphi \in E^*$ *we have*

$$\mathrm{Sup}\{|\varphi(x)| : x \in E \text{ with } ||x|| \leq 1\} = \mathrm{Sup}\{|\varphi(x)| : x \in Y\}.$$

In particular $|\varphi| \leq ||\cdot||$ *iff* $|\varphi|Y| \leq 1$, *and hence by the Hahn-Banach theorem*

$$||x|| = \mathrm{Max}\,\{|\varphi(x)| : \varphi \in E^* \text{ with } |\varphi|Y| \leq 1\} \quad \forall x \in E.$$

3) The circled convex hull $Z := \mathrm{conv}(Y \cup (-Y))$ *of* Y *is absorbent. And* $||\cdot||$ *is the Minkowski functional for* Z; *that means*

$$||x|| = \mathrm{Inf}\,\{\lambda > 0 : x \in \lambda Z\} \quad \forall x \in E.$$

Proof. 1) is obvious. 2) For $c \geq 0$ the relation $|\varphi| \leq c||\cdot||$ or $\varphi \leq c||\cdot||$ means that

$$s\varphi(u) - t\varphi(v) = \varphi(su - tv) \leq c(s + t) \quad \forall s, t > 0 \text{ and } u, v \in Y.$$

But this is equivalent to $|\varphi(x)| \leq c\ \forall x \in Y$. 3) Let $x \in E$ and $x = su - tv$ with $s, t > 0$ and $u, v \in Y$. Then

$$x = (s+t)\left(\frac{s}{s+t}u - \frac{t}{s+t}v\right) \in (s+t)Z.$$

Thus Z is absorbent. Let $\vartheta : E \to [0, \infty[$ be the Minkowski functional for Z. Then the above shows that $\vartheta(x) \leq s + t$; hence $\vartheta(x) \leq ||x||\ \forall x \in E$. In order to see $||x|| \leq \vartheta(x)$ fix $x \in E$ and $\lambda > \vartheta(x)$. Then $x \in \lambda Z$ or $x = \lambda(su - tv)$ for some $s, t \geq 0$ with $s + t = 1$ and $u, v \in Y$. For $\varepsilon > 0$ we have

$$x = \lambda\left((s+\varepsilon)\frac{su + \varepsilon v}{s+\varepsilon} - (t+\varepsilon)v\right);$$

hence by definition $||x|| \leq \lambda(s+\varepsilon) + \lambda(t+\varepsilon) = \lambda(1+2\varepsilon)$. It follows that $||x|| \leq \vartheta(x)$.

3.5 Addendum to 3.4. *Assume that $Lin(Y) = E$. If* BAff (Y) *separates the points of Y then $||\cdot||$ is a norm on E.*

Proof. Combine 3.4.2) with 3.3.

4. THE SEPARATION THEOREM

The primary purpose of the present section is the basic theorem which follows. We shall present two proofs.

Theorem 4.1. *Let (X, I) be a superconvex space which satisfies the cancellation law. Then* SAff (X, I) = BAff(X, I) *separates the points of X.*

The first proof leans on some basic points of the abstract convex and superconvex theories. On the one hand, one specializes the abstract Hahn-Banach theorem due to Rodé [6] [3] to obtain the adequate version for convex spaces.

Hahn-Banach Theorem 4.2. *On the convex space (X, I) let $G : X \to \mathbf{R}$ be concave and $H : X \to \mathbf{R}$ be convex with $G \leq H$. Then there exists an $f \in$* Aff (X, I) *such that $G \leq f \leq H$.*

On the other hand, we adopt from [1] Lemma 3.4 the basic result below on the extended Minkowski functional as introduced in §1, the proof of which has been quite technical.

Lemma 4.3. *Let (X, I) be a superconvex space with the cancellation law. Assume that $u, v \in X$ are such that $\phi_u(a_l) \to 0$ and $\phi_v(a_l) \to 0$ for some sequence $(a_l)_l$ in X. Then $u = v$.*

First proof of 4.1: Fix $u, v \in X$ with $u \neq v$. By 4.3 then

$$d := \text{Inf}\,\{\phi_u(x) + \phi_v(x) : x \in X\} > 0.$$

By 1.1 the function $G := d - \phi_u$ is concave and the function $H := \phi_v$ is convex, and we have $G \leq H$. By 4.2 there exists an $f \in \mathrm{Aff}(X, I)$ such that $G \leq f \leq H$, and hence f is bounded. Now

$$f(u) \geq G(u) = d - \phi_u(u) = d > 0, \quad f(v) \leq H(v) = \phi_v(v) \leq 0,$$

and hence $f(u) \neq f(v)$. The proof is complete.

We pass to the second proof of 4.1. The basic observation is that part of the later results in §5 can be proved with bare hands, and at the same time via 2.1 suffice to obtain 4.1. This will be carried out in the two lemmata below; the first one will on purpose be done without reference to the uniqueness theorem of Rodé.

Lemma 4.4. *Let $Y \subset E$ be a nonvoid convex subset of a real vector space E such that* $\mathrm{Lin}(Y - Y) \neq \mathrm{Lin}(Y)$. *Then each superconvex structure $I : Q \times Y^\infty \to Y$ on Y which extends its natural convex structure C can be extended to some $J : Q \times Z^\infty \to Z$ of equal sort on the circled convex hull $Z :=$* $\mathrm{conv}(Y \cup (-Y))$ *of Y.*

Proof. 1) Fix $\lambda = (\lambda_l)_l \in Q$ and $x = (x_l)_l \in Z^\infty$. For a fixed sequence of representations

$$x_l = s_l u_l - t_l v_l \text{ where } s_l, t_l \geq 0 \text{ with } s_l + t_l = 1 \text{ and } u_l, v_l \in Y \; \forall l \in \mathbb{N}$$

we form

$$s := \sum_{l=1}^{\infty} \lambda_l s_l \text{ and } t := \sum_{l=1}^{\infty} \lambda_l t_l, \text{ so that } s, t \geq 0 \text{ with } s + t = 1,$$

$$u := \mathop{I}_{l=1}^{\infty} \left(\frac{1}{s}\lambda_l s_l, u_l\right) \in Y \text{ if } s > 0 \text{ and } u \in Y \text{ arbitrary if } s = 0,$$

$$v := \mathop{I}_{l=1}^{\infty} \left(\frac{1}{t}\lambda_l t_l, v_l\right) \in Y \text{ if } t > 0 \text{ and } v \in Y \text{ arbitrary if } t = 0.$$

Then $z := su - tv \in Z$. We assert that the resultant $z \in Z$ is independent of the sequence of representations it started from. In fact, if

$$x_l = s'_l u'_l - t'_l v'_l \text{ where } s'_l, t'_l \geq 0 \text{ with } s'_l + t'_l = 1 \text{ and } u'_l, v'_l \in Y \quad \forall l \in \mathbb{N}$$

is another such sequence, then application of the functional $\alpha \in (\mathrm{Lin}(Y))^*$ obtained in 3.2 leads to $s_l - t_l = s'_l - t'_l$ and hence to $s_l = s'_l$ and $t_l = t'_l$; thus

$$s_l u_l + t_l v'_l = s_l u'_l + t_l v_l =: w_l \in Y \quad \forall l \in \mathbb{N}.$$

By [1] Rule 1.4 for superconvex structures

$$\mathop{I}_{l=1}^{\infty}(\lambda_l, w_l) = \mathop{I}_{l=1}^{\infty}\left(\lambda_l, I\begin{pmatrix} s_l\ t_l \\ u_l\ v'_l \end{pmatrix}\right) = I\begin{pmatrix} \lambda_l s_l & \lambda_l t_l \\ u_l & v'_l \\ \scriptstyle l\geq 1 & \scriptstyle l \geq 1 \end{pmatrix} = I\begin{pmatrix} s\ t \\ u\ v' \end{pmatrix} = su + tv',$$

and likewise $= su' + tv$. Thus $su - tv = su' - tv' = s'u' - t'v'$, as claimed. Therefore we have a well-defined map $J : Q \times Z^{\infty} \to Z$. 2) In case $\lambda \in P$ we have

$$su = \sum_{l=1}^{\infty} \lambda_l s_l u_l \text{ and } tv = \sum_{l=1}^{\infty} \lambda_l t_l v_l, \text{ and hence } z = \sum_{l=1}^{\infty} \lambda_l x_l.$$

Thus J extends the natural convex structure C of Z. 3) In case $x = (x_l)_l \in Y^{\infty}$ we can take $s_l = 1$ and $t_l = 0$, $u_l = x_l$ and v_l arbitrary $\in Y$ $\forall l \in \mathbb{N}$. Then $z = u = I(\lambda, x)$. Thus J extends the superconvex structure I on Y. 4) It remain to show that J is indeed a superconvex structure on Z. This is another routine verification as before, and will not be carried out in detail.

Lemma 4.5. *Let the subset $Z \subset E$ of the real vector space E be circled convex and absorbent. If Z is superconvex then the Minkowski functional $\|\cdot\| : E \to [0, \infty[$ for Z is a norm on E.*

Proof. Let $x \in E$ with $\|x\| = 0$; we have to show that $x = 0$. By definition $x \in tZ$ $\forall t > 0$; thus $2^l x \in Z$ $\forall l \in \mathbb{N}$. Define $z := \mathop{I}_{l=1}^{\infty}(2^{-l}, 2^l x) \in Z$. Then on the one hand

$$\frac{1}{2}x + \frac{1}{2}z = I\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ x & z \end{pmatrix} = I\begin{pmatrix} \frac{1}{2} & \frac{1}{2}2^{-l} \\ x & 2^l x \\ & \scriptstyle l \geq 1 \end{pmatrix} = \mathop{I}_{l=1}^{\infty}(2^{-l}, 2^{l-1}x),$$

and on the other hand

$$\frac{1}{2}0 + \frac{1}{2}z = I\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & z \end{pmatrix} = I\begin{pmatrix} \frac{1}{2}2^{-l} & \frac{1}{2}2^{-l} \\ 0 & 2^l x \\ \scriptstyle l\geq 1 & \scriptstyle l \geq 1 \end{pmatrix} = \mathop{I}_{l=1}^{\infty}(2^{-l}, 2^{l-1}x).$$

It follows that $x = 0$, as claimed.

We come to our second proof of 4.1, based on 4.4 and 4.5 (and on 2.1) as announced in the Introduction.

Second proof of 4.1: Let (X,S) be a convex space which satisfies the cancellation law, and let $T : Q \times X^\infty \to X$ be a superconvex structure on X which extends S. By 2.1 the convex space (X,S) admits an injective c-map $\vartheta : X \to E$ into a real vector space E such that $Y := \vartheta(X) \subset E$ satisfies $E = \operatorname{Lin}(Y) \neq \operatorname{Lin}(Y - Y)$. Then $Y \subset E$ is a superconvex set, and carries a superconvex structure $I : Q \times Y^\infty \to Y$ which extends its natural convex structure C and is such that ϑ is an sc-map with respect to T and I. By 4.4 I can be extended to some superconvex structure $J : Q \times Z^\infty \to Z$ on $Z := \operatorname{conv}((Y \cup (-Y))$ which extends the natural convex structure C of Z. By 3.4.3) Z is absorbent, and by 4.5 the Minkowski functional $\|\cdot\|$ for Z is a norm on E. Now $Y \subset Z$ is contained in the closed unit ball of $(E, \|\cdot\|)$. Thus the functionals $\varphi \in E' \subset E^*$ produce bounded affine functions on Y, and by the conventional Hahn-Banach theorem their collection separates the points of Y. It follows that $\{\varphi \circ \vartheta : \varphi \in E'\} \subset \operatorname{BAff}(X,S)$ separates the points of X. This proves 4.1.

Remark 4.6. We assert that the basic uniqueness result is an immediate consequence of 4.1. In fact, assume that (X,I) is a convex space with the cancellation law, and that S and T are superconvex structures on X which extend I. For $t = (t_l)_l \in Q$ and $x = (x_l)_l \in X^\infty$ let $u := S(t,x)$ and $v := T(t,x)$. Then for $f \in \operatorname{SAff}(X,S) = \operatorname{BAff}(X,S) = \operatorname{BAff}(X,I) = \operatorname{BAff}(X,T) = \operatorname{SAff}(X,T)$ we have

$$f(u) = \sum_{l=1}^{\infty} t_l f(x_l) = f(v).$$

Thus $u = v$ by 4.1. It follows that $S = T$.

5. SUPERCONVEX SETS AND σ-CONVEX SETS

Theorem 5.1. *Let $Y \subset E$ be a superconvex subset of a real vector space E such that* $\operatorname{Lin}(Y) = E$. *Then the Minkowski functional $\|\cdot\|$ for $Z :=$* $\operatorname{conv}(Y \cup (-Y))$ *is a complete norm on E.*

Proof. 1) We combine 4.1 applied to (Y, SC) with 3.5 to conclude that $\|\cdot\|$ is a norm on E. 2) In order to prove that $\|\cdot\|$ is complete consider a sequence $(x_l)_l$ in E such that $\sum_{l=1}^{\infty} \|x_l\| < \infty$. We have to show that $\sum_{l=1}^{\infty} x_l$ is convergent in $(E, \|\cdot\|)$. We fix representations

$$x_l = s_l u_l - t_l v_l \text{ with } s_l, t_l > 0 \text{ and } u_l, v_l \in Y \text{ such that } \sum_{l=1}^{\infty} (s_l + t_l) < \infty.$$

Then with

$$s := \sum_{l=1}^{\infty} s_l > 0 \text{ and } t := \sum_{l=1}^{\infty} t_l > 0, \ \sigma_n := \sum_{l=n+1}^{\infty} s_l > 0 \text{ and } \tau_n := \sum_{l=n+1}^{\infty} t_l > 0$$

we form

$$\begin{aligned} u : &= \underset{l=1}{\overset{\infty}{SC}} \left(\tfrac{1}{s} s_l, u_l\right) \in Y \quad \text{and} \quad v : = \underset{l=1}{\overset{\infty}{SC}} \left(\tfrac{1}{t} t_l, v_l\right) \in Y, \\ a_n : &= \underset{l=n+1}{\overset{\infty}{SC}} \left(\tfrac{1}{\sigma_n} s_l, u_l\right) \in Y \quad \text{and} \quad b_n : = \underset{l=n+1}{\overset{\infty}{SC}} \left(\tfrac{1}{\tau_n} t_l, v_l\right) \in Y \quad \forall n \in \mathbf{N}. \end{aligned}$$

It follows that

$$u = SC \begin{pmatrix} \frac{1}{s} s_1 & \dots & \frac{1}{s} s_n & \frac{1}{s} \sigma_n \\ u_1 & \dots & u_n & a_n \end{pmatrix} = \sum_{l=1}^{n} \frac{1}{s} s_l u_l + \frac{1}{s} \sigma_n a_n,$$

$$v = SC \begin{pmatrix} \frac{1}{t} t_1 & \dots & \frac{1}{t} t_n & \frac{1}{t} \tau_n \\ v_1 & \dots & v_n & b_n \end{pmatrix} = \sum_{l=1}^{n} \frac{1}{t} t_l v_l + \frac{1}{t} \tau_n b_n,$$

and hence

$$su - tv = \sum_{l=1}^{n} x_l + \sigma_n a_n - \tau_n b_n,$$

$$\left\| \sum_{l=1}^{n} x_l - (su - tv) \right\| = \|\sigma_n a_n - \tau_n b_n\| \leq \sigma_n + \tau_n,$$

since $Y \subset Z$ is in the closed unit ball of $(E, \|\cdot\|)$. The assertion follows.

Before we proceed further we insert the next consequence which is natural after 4.4.

Consequence 5.2. *Let $Y \subset E$ be a superconvex subset of a real vector space E. Then its circled convex hull $Z := \operatorname{conv}(Y \cup (-Y)) \subset E$ is superconvex as well.*

Proof. We can assume that $\operatorname{Lin}(Y) = E$. Fix $\lambda = (\lambda_l)_l \in Q$ and $x = (x_l)_l \in Z^\infty$, and let

$$x_l = s_l u_l - t_l v_l \text{ where } s_l, t_l \geq 0 \text{ with } s_l + t_l = 1 \text{ and } u_l, v_l \in Y \ \forall l \in \mathbf{N}.$$

As earlier we form

$$s := \sum_{l=1}^{\infty} \lambda_l s_l \text{ and } t := \sum_{l=1}^{\infty} \lambda_l t_l, \text{ so that } s, t \geq 0 \text{ with } s + t = 1,$$

$$u := \underset{l=1}{\overset{\infty}{SC}} \left(\frac{1}{s} \lambda_l s_l, u_l\right) \in Y \text{ if } s > 0 \text{ and } u \in Y \text{ arbitrary if } s = 0,$$

$$v := \underset{l=1}{\overset{\infty}{SC}} \left(\frac{1}{t} \lambda_l t_l, v_l\right) \in Y \text{ if } t > 0 \text{ and } v \in Y \text{ arbitrary if } t = 0.$$

By 5.1 and 1.2 the series $\sum_{l=1}^{\infty}\lambda_l s_l u_l$ and $\sum_{l=1}^{\infty}\lambda_l t_l v_l$ converge in $||\cdot||$ with sums su and tv; hence $\sum_{l=1}^{\infty}\lambda_l x_l$ converges in $||\cdot||$ with sum $su - tv \in Z$. Therefore Z is σ-convex in $||\cdot||$ and thus superconvex.

Theorem 5.3. *Let $Y \subset E$ be a superconvex subset of a real vector space E such that* $\mathrm{Lin}(Y) = E$. *Let $||\cdot||$ be the Minkowski functional for $Z := \mathrm{conv}(Y \cup (-Y))$ and $\mathscr{M}$ be its norm topology on E. Then for any Hausdorff vector-space topology $\mathscr{T}$ on E we have*

$$Y \text{ is } \sigma\text{-convex in } \mathscr{T} \Leftrightarrow \mathscr{T} \subset \mathscr{M}.$$

Thus by the open mapping theorem $\mathscr{M}$ is the unique Banach-space topology on E in which Y is σ-convex.

Proof. $\Leftarrow$) By 5.1 and 1.2 Y is σ-convex in $\mathscr{M}$, and hence σ-convex in any $\mathscr{T} \subset \mathscr{M}$. $\Rightarrow$) Let $U \subset E$ be a 0-neghbourhood in $\mathscr{T}$, and take another one $V \subset E$ such that $V - V \subset U$ and $tV \subset V$ for $0 < t \leq 1$. By 1.2 there exists $\lambda > 0$ such that $Y \subset \lambda V$. Now consider a point $x \in E$ with $||x|| < 1$. Then $x = su - tv$ with $0 < s, t < 1$ and $u, v \in Y$. It follows that $su \in s\lambda V \subset \lambda V$ and $tv \in t\lambda V \subset \lambda V$, and hence $x \in \lambda V - \lambda V \subset \lambda U$. Thus the open unit ball of $(E, ||\cdot||)$ is contained in λU. It follows that $\mathscr{T} \subset \mathscr{M}$.

Special case 5.4. *Let E be a finite-dimensional real vector space and $\mathscr{T}$ be the unique Hausdorff vector-space topology on E. For a nonvoid $Y \subset E$ the following are equivalent.*

1) Y is superconvex.

2) Y is convex and bounded in $\mathscr{T}$.

3) Y is σ-convex in $\mathscr{T}$.

Proof. We have 1) $\Rightarrow$ 2) by 5.1 applied to $\mathrm{Lin}(Y) \subset E$, and 2) $\Rightarrow$ 3) by 1.3.1).

At this point we have obtained the results announced in the Introduction. We add one more consequence; it looks innocent but does not seem to be accessible without the results developed so far.

Remark 5.5. Let $A, B \subset E$ be superconvex subsets of a real vector space E such that $A \subset B$. Then the natural superconvex structure $SC : Q \times A^{\infty} \to A$ of A is equal to the natural superconvex structure $SC : Q \times B^{\infty} \to B$ of B restricted to $Q \times A^{\infty}$.

Proof. We can assume that $\mathrm{Lin}(B) = E$. Let $||\cdot||$ be the Minkowski functional for $Z := \mathrm{conv}(B \cup (-B))$. By 5.1 and 1.2 then A and B are σ-convex in the topology of $||\cdot||$; thus the assertion follows from 1.2.

In conclusion we return to the problem of the identification of a superconvex space (X, I) with members of our two classes of standard examples. We can summarize our results from 1.2, 2.1, 3.2 and 3.4, 4.1 and 5.1 as follows.

Theorem 5.6. *For a superconvex space (X, I) the following are equivalent.*

1) (X, I) *satisfies the cancellation law.*

2) $\mathrm{Aff}\,(X, I)$ *separates the points of* X.

3) $\mathrm{SAff}\,(X, I) = \mathrm{BAff}\,(X, I)$ *separates the points of* X.

4) (X, I) *is sc-isomorphic to some superconvex subset* $Y \subset E$ *of a real vector space* E.

5) (X, I) *is sc-isomorphic to some* σ*-convex subset* $Y \subset E$ *of a real vector space* E *equipped with a Hausdorff vector-space topology* $\mathscr{T}$.

6) (X, I) *is sc-isomorphic to some* σ*-convex subset* $Y \subset E$ *of a real Banach space* $(E, \|\cdot\|)$ *such that* $E = \mathrm{Lin}(Y) \neq \mathrm{Lin}(Y - Y)$.

In this case let $\vartheta : X \to E$ be an injective c-map of (X, I) into a real vector space E such that $Y := \vartheta(X) \subset E$ fulfills $E = \mathrm{Lin}(Y) \neq \mathrm{Lin}(Y - Y)$, and let $\|\cdot\|$ be the complete norm on E which is the Minkowski functional for $Z := \mathrm{conv}(Y \cup (-Y))$. Then the transposed map

$$E^* \to \mathrm{Aff}\,(X, I) : \varphi \mapsto f := \varphi \circ \vartheta$$

is a bijection, and induces a bijection

$$E' \to \mathrm{SAff}\,(X, I) = \mathrm{BAff}\,(X, I) : \varphi \mapsto f := \varphi \circ \vartheta$$

which is isometric in the sense that $\|\varphi\| = \|f\| := \mathrm{Sup}\{|f(x)| : x \in X\}$.

Let us also note the characterization consequence below which, however, does not require our full results.

Proposition 5.7. *For a convex space (X, I) the following are equivalent.*

1) $\mathit{BAff}\,(X, I)$ *separates the points of* X.

2) (X, I) *is c-isomorphic to some convex subset of a superconvex space which satisfies the cancellation law.*

3) (X, I) *is c-isomorphic to some bounded convex subset* $Y \subset E$ *of a real Banach space* $(E, \|\cdot\|)$.

Proof. In order to obtain 1) $\Rightarrow$ 3) it suffices to combine 2.1 with 3.5. The rest is obvious.

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We want to thank Jürgen Kindler for his hint to the paper H. Kneser, *Konvexe Räume*, Arch. Math. 3, 198-206, 1952, where the abstract notion of a convex space (with cancellation law) has been defined (even over an ordered skewfield instead of the real number field) and an

embedding theorem equivalent to our theorem 2.1 has been proved. König has to confess that somewhen in the fifties in Oberwolfach he had talked to the late Hellmuth Kneser about that paper and its context, but that over the decades this had fallen into oblivion.

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