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STABILIZATION BY FREE PRODUCTS GIVING RISE TO ANDREWS-CURTIS EQUIVALENCES*

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION AND RESULT

Two compact, connected polyhedra [(1)] K^2 and L^2 are called *Andrews-Curtis equivalent* if there exists a sequence of expansions and collapses (in the sense of simple-homotopy theory) transforming K^2 into L^2, during which all cells have dimensions not exceeding 3 $(K^2 \overset{3}{\diagup\!\!\searrow} L^2)$. It is an open problem, whether simple-homotopy equivalence of K^2 and L^2 $(K^2 \diagup\!\!\searrow L^2)$ always implies $K^2 \overset{3}{\diagup\!\!\searrow} L^2$. The expectation «yes» as an answer to this question is the *generalized Andrews-Curtis conjecture* (AC'), the naming referring to a paper [1] of J. Andrews and M.L. Curtis in which the authors drew attention to the case of contractible K^2, L^2 *(Andrews-Curtis conjecture* (AC)). But there exist several notorious examples (s. [17]), for which the implication $K^2 \simeq * \Rightarrow K^2 \overset{3}{\diagup\!\!\searrow} *$ seems debatable. Moreover, we are convinced that counterexamples to (AC') for nontrivial fundamental groups are even more likely to exist, s. §3, A).

The main contribution of this paper is that, on the other hand, (unexpected) Andrews-Curtis equivalences can be constructed in a systematic way. They reveal that the search for $AC^{(\prime)}$-invariants is a delicate matter.

We shall prove the

Theorem. *Let K_0^2, L_0^2 be compact, connected polyhedra [(2)] which are simple-homotopy equivalent. Then by forming the one-point unions $K^2 = K_0^2 \vee K_1^2 \vee \ldots \vee K_n^2$ and $L^2 = L_0^2 \vee K_1^2 \vee \ldots \vee K_n^2$, where the $K_\nu^2, \nu = 1, \ldots, n$ are standard complexes of the presentation $\langle \alpha, \beta | \alpha^2 = [\alpha, \beta] = \beta^4 = 1 \rangle$ of $\mathbb{Z}_2 \times \mathbb{Z}_4$, and if n is big enough (depending on K_0^2, L_0^2), K^2 and L^2 are Andrews-Curtis equivalent.*

It is well known that, if the $K_\nu^2, \nu \geq 1$ of this theorem were replaced by an appropriate number of 2-spheres, Andrews-Curtis equivalences could already be obtained under the

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(1) Throughout this paper we could as well consider 2-dimensional CW-spaces (without a specified cell structure) s. [16], footnote 2 and [6], Thm. 5. But there is no real advantage in admitting nontriangulable spaces, compare the remark in §2.

(2) Compare footnote 1.

weaker assumptions: $\pi_1(K_0^2) \approx \pi_1(L_0^2)$ and $\chi(K_0^2) = \chi(L_0^2)$ instead of $K_0^2 \nearrow\searrow L_0^2$. The essential point of our theorem is that the $K_\nu^2, \nu \geq 1$ are complexes with *minimal Euler characteristic* for their fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_4$.

One-point unions with these complexes have already been studied in Metzler [18] so as to realize torsion values of $Wh(\pi_1(K_0^2))$ by maps between 2-complexes [(3)] of minimal Euler characteristic, thereby distinguishing homotopy type and simple-homotopy type in dimension 2. The $\mathbb{Z}_2 \times \mathbb{Z}_4$-factors allow to bypass the commutator problem (s. §2, (6) and §3; C)) which marks the main difference between complexes of dimension 2 and the case $n \neq 2$, where $K^n \nearrow\searrow L^n$ always implies $K^n \overset{n+1}{\nearrow\searrow} L^n$, s. Wall [24].

In [18], Thm. 3 examples of homotopy equivalent but simple-homotopy inequivalent 2-complexes are given which remain simple-homotopy distinct even after forming the one-point union with arbitrarily (finitely) many standard complexes of $\langle \alpha, \beta | \alpha^2 = [\alpha, \beta] = \beta^4 = 1 \rangle$. Hence the assumption $K_0^2 \nearrow\searrow L_0^2$ in the theorem of the present paper cannot generally be dismissed with.

The one-point union of polyhedra corresponds to forming the free product of fundamental groups. That free products, free products with amalgamation, HNN-extensions and graphs of groups as fundamental groups of 2-complexes reveal (simple)-homotopy phenomena that are not merely a «sum» of those of the factors, has already turned out several times. For instance, the deficiency of groups is not additive under the forming of the free product (Hog-Lustig-Metzler [10], s. also Hog-Angeloni [8]). Our paper is part of a program to study these phenomena. For simple-homotopy purposes the free factors $\mathbb{Z}_2 \times \mathbb{Z}_4$ have the advantage of not contributing to $Wh(\pi_1)$ (s. [18], footnote 3). But for the stabilization process yielding Andrews-Curtis equivalences they can be replaced by bigger factors. Such possible extensions of the theorem and their connection with further ideas towards a solution of the Andrews-Curtis problem are discussed in the concluding §3.

2. PROOF OF THE THEOREM

It clearly suffices to establish the assertion for polyhedra which arise by varying K_0^2 and L_0^2 within their Andrews-Curtis classes. Hence, by starting with p.l. cell-decompositions and contracting spanning trees (which are modifications of type $\overset{3}{\nearrow\searrow}$), we may assume without loss of generality that

(1) K_0^2 and L_0^2 are standard complexes of finite presentations.

(3) The theorem of the present paper can be used to show that the complexes $L_{\tau_0}^2$ of the claim of [18], Thm. 1 are unique «up to Andrews-Curtis equivalence» after (further) stabilization with these standard $\mathbb{Z}_2 \times \mathbb{Z}_4$-complexes.

If $f : K_0^2 \to L_0^2$ is a simple-homotopy equivalence, we may moreover apply further $\overset{3}{\nearrow\!\!\searrow}$-modifications to K_0^2 and L_0^2 and deform f homotopically until we also get:

(2) (a) The 1-skeleta of K_0^2 and L_0^2 coincide;
(b) the restriction of f to the 1-skeleta is the identity map.

(Compare [15] for these normalizations.)
Thus K_0^2 may be given as the standard complex of a presentation $\mathscr{P}_0 = \langle a_1, \dots, a_g | R_1(a_i), \dots, R_h(a_i)\rangle$, L_0^2 as the standard complex of $\mathscr{Q}_0 = \langle a_1, \dots, a_g | S_1(a_i), \dots, S_h(a_i)\rangle$, where the normal closure of the R_j and that of the S_j in the free group $F(a_i)$ coincide (and will be denoted by N henceforth). (2) together with the fact that f is a simple-homotopy equivalence implies that with respect to cellular bases

(3) the induced map $C_2(\tilde{K}_0) \to C_2(\tilde{L}_0)$ determines an (invertible and) Whitehead-trivial $\mathbb{Z}(\pi_1)$-matrix F.

Here $\tilde{K}_0, \tilde{L}_0$ denote the universal covering complexes of K_0^2 resp. L_0^2. $\pi_1 = F(a_i)/N$ is isomorphic to the fundamental groups of K_0^2 and L_0^2. The R_j resp. S_j give rise to fundamental systems $\tilde{R}_j$ resp. $\tilde{S}_j$ of 2-cells for $\tilde{K}_0$ resp. $\tilde{L}_0$; and if $\tilde{R}_j$ is mapped to the $\mathbb{Z}(\pi_1)$-linear combination $\varphi_{j1}\tilde{S}_1 + \dots + \varphi_{jh}\tilde{S}_h$, then its coefficients $\varphi_{j1}, \dots, \varphi_{jh}$ constitute the j-th row of F.

The fact that F is trivial in $Wh(\pi_1)$ means that F can be transformed into the identity matrix by a sequence of the following elementary operations (s. [4], p. 31):

(4) (a) adding a row of F to another row;
(b) multiplying a row of F by plus or minus an element of π_1;
(c) passing from F to the prolonged matrix $\begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix}$.

These operations can be effected by $\overset{3}{\nearrow\!\!\searrow}$-modifications of K_0^2 which correspond to the following operations applied to the presentation $\mathscr{P}_0$:

(5) (a) multiplication of a defining relator to another one (from the right or left);
(b) inversion of a defining relator and/or conjugation of it by an element $w \in F(a_i)$;
(c) prolongation, i.e. introduction of a new generator and a new defining relator reading this generator.

In the case (c) the prolongation resp. the corresponding 2-dimensional expansion has also to be applied to $\mathscr{Q}_0$ resp. to its standard complex L_0^2.

Remark. *Together with the operation of (d) replacing the a_i by the result of a free transformation in all defining relators and (e) the inverse of (c) (if possible), the operations of type (5) generate all Q^{**}-transformations; they completely characterize Andrews-Curtis classes of 2-dimensional polyhedra (or CW-spaces) in terms of combinatorial group theory, s. Wright [25] and the references given in footnote 1.*

After this further normalization we may assume without loss of generality that the given simple-homotopy equivalence $f : K_0^2 \to L_0^2$ fulfills (1), (2) and:

(3') the induced map $C_2(\tilde{K}_0) \to C_2(\tilde{L}_0)$ sends $\tilde{R}_j$ to $\tilde{S}_j, j = 1, \ldots, h$.

This does not imply $R_j = S_j, j = 1, \ldots, h$; but the following weaker implication holds:

(6) The $R_j \cdot S_j^{-1}$ are contained in the commutator subgroup $[N, N]$ of the group N of relators [4].

A proof of this (well known) fact can be given by an argument mainly due to Reidemeister [22]. It will be supplied for the convenience of the reader at the end of this section. (6) can be related to handlebody considerations which are an important motivation for the Andrews-Curtis problem, s. Quinn [21]. If the commutator «obstructions» given by (6) could be trivialized by applying Q^{**}-transformations to $\langle a_i | R_j \rangle$, then (AC') would be proven. But although it is possible to impose further restrictions on the commutators $R_j S_j^{-1}$, at the moment they might just as well point towards (AC')-invariants, s. §3, C).

Nevertheless (6) gives rise to an interplay with additional free $\mathbb{Z}_2 \times \mathbb{Z}_4$-factors of π_1; it is the main idea of this proof (and similar to the one of [18], Thm. 1):

By (6) the $R_j S_j^{-1}$ can be expressed as products of elementary commutators $[T_{\rho_1}, T_{\rho_2}]$, altogether using only finitely many $T_\rho \in N$. For each ρ and each pair (ρ_1, ρ_2) that occurs we bijectively reserve one of the indices ν between 1 and n (thereby determining n).

For this number n we consider the complexes K^2 and L^2 of the assertion. They are standard complexes of the presentation $\mathcal{P} = \langle a_i, \alpha_\nu, \beta_\nu | R_j, \alpha_\nu^2, [\alpha_\nu, \beta_\nu], \beta_\nu^4 \rangle$ resp. $\mathcal{Q} = \langle a_i, \alpha_\nu, \beta_\nu | S_j, \alpha_\nu^2, [\alpha_\nu, \beta_\nu], \beta_\nu^4 \rangle$.

By multiplication of the β_ν^4 to $\alpha_\nu \beta_\nu \alpha_\nu^{-1} \beta_\nu^{-1}$ we get the Q^{**}-transformation

$$(7) \qquad \mathcal{P} \to \langle a_i, \alpha_\nu, \beta_\nu | R_j, \alpha_\nu^2, \alpha_\nu \beta_\nu \alpha_\nu^{-1} \beta_\nu^3, \beta_\nu^4 \rangle = \mathcal{P}_1.$$

As the T_ρ are consequences of the R_j, we may apply the Q^{**}-transition

$$(8) \quad \mathcal{P}_1 \to \langle a_i, \alpha_\nu, \beta_\nu | R_j, \alpha_\nu^2, \alpha_\nu \beta_\nu \alpha_\nu^{-1} \beta_\nu^3, \beta_{\nu(\rho)}^4 T_\rho^{-1} \text{ resp. } \beta_{\nu(\rho_1, \rho_2)}^4 (T_{\rho_1} T_{\rho_2})^{-1} \rangle = \mathcal{P}_2.$$

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(4) Conversely, if $\langle a_i | R_1, \ldots, R_h \rangle$ and $\langle a_i | S_1, \ldots, S_h \rangle$ are finite presentations with the same relator subgroup $N \subseteq F(a_i)$ and if (6) is fulfilled, then the corresponding standard complexes are simple-homotopy equivalent.

Note that in $\mathscr{P}_2$ $\alpha_\nu^2 = 1$ and $\alpha_\nu\beta_\nu\alpha_\nu^{-1} = \beta_\nu^{-3}$ together (already) imply $\beta_\nu^8 = 1$. The relations $T_\rho = \beta^4_{\nu(\rho)}$ resp. $T_{\rho_1}T_{\rho_2} = \beta^4_{\nu(\rho_1,\rho_2)}$ hence imply $T_\rho^2 = 1, T_{\rho_1}T_{\rho_2}T_{\rho_1}T_{\rho_2} = 1$, from which $[T_{\rho_1}, T_{\rho_2}] = 1$ follows. In achieving these consequences, the original relators R_j of K_0^2 have not been used. Thus the representation of the $R_jS_j^{-1}$ as products of the $[T_{\rho_1}, T_{\rho_2}]$ gives rise to a Q^{**}-transformation

$$(9)\quad \mathscr{P}_2 \to \langle a_i, \alpha_\nu, \beta_\nu | S_j, \alpha_\nu^2, \alpha_\nu\beta_\nu\alpha_\nu^{-1}\beta_\nu^3, \beta^4_{\nu(\rho)}T_\rho^{-1} \text{ resp. } \beta^4_{\nu(\rho_1,\rho_2)}(T_{\rho_1}T_{\rho_2})^{-1}\rangle = \mathscr{Q}_2.$$

In total analogy to (7) and (8), $\mathscr{Q}$ Q^{**}-transforms to $\mathscr{Q}_2$. Reversing these last transitions, we have completed a chain of Q^{**}-transformations

$$\mathscr{P} \overset{(7)}{\to} \mathscr{P}_1 \overset{(8)}{\to} \mathscr{P}_2 \overset{(9)}{\to} \mathscr{Q}_2 \overset{(8)^{-1}}{\to} \mathscr{Q}_1 \overset{(7)^{-1}}{\to} \mathscr{Q}$$

which establish the desired Andrews-Curtis equivalence of K^2 and L^2, q.e.d.

We close this section with the postponed argument for (6): with the finite presentation $\mathscr{P}_0 = \langle a_1, \dots, a_g | R_1, \dots, R_h\rangle$ we associate a (bigger) free group $F(a_i, r_j), i = 1, \dots, g$, $j = 1, \dots, h$ and a projection $p : F(a_i, r_j) \to F(a_i)$ given by $a_i \to a_i, r_j \to R_j$. Let $\overline{F}(r_j)$ be the normal closure of the r_j in $F(a_i, r_j)$. $F(a_i)$ operates on $\overline{F}(r_j)$ by conjugation; p induces a surjection $p_* : \overline{F}(r_j) \to N$. The kernel of p_* is the *group of identities of* $\mathscr{P}_0$. *Peiffer identities*, i.e. identities of type

$$(10)\qquad (r, s) = r \cdot s \cdot r^{-1} \cdot p_*(r)s^{-1}p_*(r)^{-1}, r, s \in \overline{F}(r_j)$$

are of particular relevance for the Andrews-Curtis problem.

There exists a natural homomorphism

(11) $\Theta_{\mathscr{P}_0} : \overline{F}(r_j) \to C_2(\tilde{K}_0)$ which sends wr_jw^{-1} to $[w]\tilde{R}_j$, where $w \in F(a_i)$ and $[w]$ denotes the residual class of w in $\pi_1 = F(a_i)/N$.

Clearly commutators $[r, s]$ and Peiffer identities $(r, s), r, s \in \overline{F}(r_j)$ are contained in $ker\Theta_{\mathscr{P}_0}$. The result of Reidemeister [22] which we need, is

(12) $ker\Theta_{\mathscr{P}_0}$ is normally generated by commutators and Peiffer identities as an $\overline{F}(r_j)$-subgroup.

(This can be seen as follows: Choose representatives for the residual classes of $F(a_i)$ mod N. Let $w = vR$, where $R \in N$ and $R = p_*(r)$. Commutators and Peiffer identities generate equivalence classes in $\overline{F}(r_j)$ such that $wr_jw^{-1} = vRv^{-1}(vr_jv^{-1})vR^{-1}v^{-1} \sim vrv^{-1} \cdot vr_jv^{-1} \cdot vr^{-1}v^{-1} \sim vr_jv^{-1}$. Hence every element $s \in \overline{F}(r_j)$ is equivalent to a product of factors $\overline{g}r_j^{n(r_j,g)}\overline{g}^{-1}$ (with $n(r_j,g) \in \mathbb{Z}$, $\overline{g}$ the representative of $g \in \pi_1$), each pair r_j, g occurring at most once. A product of this form is mapped to zero under $\Theta_{\mathscr{P}_0}$ (if and) only if all exponents $n(r_j,g)$ vanish; hence $\Theta_{\mathscr{P}_0}(s) = 0$ implies $s \sim 1$).

By (1) and (2) the given map $C_2(\tilde{K}_0) \to C_2(\tilde{L}_0)$ lifts to an $F(a_i)$-equivariant homomorphism

$$\begin{array}{ccc} \overline{F}(r_j) & \xrightarrow{\varphi} & \overline{F}(s_j) \\ p_* \searrow & & \swarrow q_* \\ & N & \end{array} \tag{13}$$

such that $p_* = q_*\varphi$, where the data for $\mathscr{Q}_0$ are constructed and denoted in analogy to those for $\mathscr{P}_0$. By (3') φ fulfills

$$\varphi(r_j)s_j^{-1} \in ker\Theta_{\mathscr{Q}_0}; \tag{14}$$

hence – because of (12) – $\varphi(r_j)s_j^{-1}$ is a product of commutators of $\overline{F}(s_j)$ and conjugates of Peiffer identities of $\mathscr{Q}_0$. As the latter are trivialized under the projection q_*, $q_*(\varphi(r_j)s_j^{-1}) = R_jS_j^{-1}$ is a product of commutators of N, q.e.d.

3. DISCUSSION AND ADDITIONS

In this section we (almost) confine ourselves to the algebraic notation of the Andrews-Curtis problem via Q^{**}-classes of finite presentations, as this is more adequate in treating the examples below.

Presentation classes ϕ and ψ may be added by taking «disjoint» representatives $\mathscr{P} = \langle a_i|R_j\rangle, \mathscr{Q} = \langle b_k|S_\ell\rangle$ and passing to the presentation $\mathscr{P} + \mathscr{Q} = \langle a_i, b_k|R_j, S_\ell\rangle$. The sum $\phi + \psi$ corresponds to the free product of the groups which are presented and to the one-point union of standard 2-complexes. Thus an abelian *semigroup of presentation classes* arises; the class ϕ_0 which consists of the presentations $\mathscr{K} \sim \langle - | - \rangle$ (i.e. $K^2 \overset{3}{\searrow} *$) is the neutral element of this semigroup (compare [16] and [17]).

A) If $\mathscr{K}$ is a balanced presentation of the trivial group (i.e. $K^2 \searrow *$) and if $\mathscr{P}$ is a finite presentation, then $\mathscr{P}$ and $\mathscr{P} + \mathscr{K}$ determine standard complexes of the same simple-homotopy type. But it is at least as hard to disprove (AC') via $\mathscr{P}, \mathscr{P} + \mathscr{K}$ as to establish $\mathscr{K}$ as a counterexample to (AC). A striking reason is that $\mathscr{K} \sim \langle -|- \rangle$ implies $\mathscr{P} \sim \mathscr{P} + \mathscr{K}$. Moreover, even if (AC) is false with $\mathscr{K} \not\sim \langle -|- \rangle$ a counterexample, then for some $\ast_{i=1}^{n} (\mathbb{Z}_2 \times \mathbb{Z}_4)_i$ the theorem of §1 provides a $\mathscr{P}$ of maximal deficiency [5] such that $\mathscr{P} \sim \mathscr{P} + \mathscr{K}$. (The falsity of (AC) thus would yield that a «cancellation law» $\mathscr{P} + \mathscr{Q} \sim \mathscr{P} + \mathscr{Q}' \Rightarrow \mathscr{Q} \sim \mathscr{Q}'$ needs strong assumptions on the summands, compare Sieradski [23]).

The preceding discussion gives rise to the following definition: Suppose that $\mathscr{P}_1, \mathscr{P}_2$ constitute a counterexample to (AC') such that $\mathscr{P}_1$ and $\mathscr{P}_2$ remain inequivalent (even) if we pass to the coarser equivalence classes for which we allow Q^{**}-transitions *and* the addition or deletion (if possible) of a balanced presentation of the trivial group (s. [17]); then we call $\mathscr{P}_1, \mathscr{P}_2$ a counterexample to (AC') which is *specific for the group* presented by $\mathscr{P}_1$, and $\mathscr{P}_2$.

In our opinion it is worth while to focus on such group specific counterexamples to (AC') for $\pi \neq \{1\}$. They might be detected by invariants which by definition are unable to distinguish between $\mathscr{P}$ and $\mathscr{P} + \mathscr{K}$, $\mathscr{K}$ a balanced presentation of the trivial group; however the structure of π may play an essential role.

Remark. *Our expectation is also based on work in progress on concrete «exotic» presentations. Their construction involves «typical» nontrivial elements of π and of $N \subseteq F(a_i)$ (such as: elements of finite order, commutators, stable letters of HNN-extensions). Exotic presentations of $\pi = G * H$ were first considered by us in [10], s. also [13]. Originally we confined to nonsplittable homotopy types with respect to the given factorization of π. In the meantime we have obtained various examples, the corresponding 2-complexes of which are all of the same simple-homotopy type (s. [3], [12], compare also [7], [8], [9], [15], [23]); but only in some cases we have achieved Q^{**}-equivalences. Moreover we are fairly convinced that establishing such counterexamples to (AC') would naturally be done by proving at once that they are specific for the group in question.*

B) The presentations $\langle \alpha, \beta | \alpha^2 = [\alpha, \beta] = \beta^4 = 1 \rangle$ provided *universal* factors of minimal Euler characteristic which turn simple-homotopy equivalences into Andrews-Curtis equivalences. For the stabilization process one might as well use the presentations $\langle \alpha, \beta | \alpha^{2n} = [\alpha^n, \beta^n] = \beta^{4n} = 1 \rangle$; in the defining relators of (7), (8) and (9) one merely has to substitute α, β by α^n, β^n throughout. The presentation $\langle \alpha, \beta | \alpha^3 = \beta^5, \alpha^3 = (\alpha\beta)^2, \alpha^3 = 1 \rangle$ of the icosahedral group is also a candidate, as already the first two of the defining relators together imply $\alpha^6 = 1$. But in certain concrete situations we even know a bigger variety of trivializing factors. Consider the presentation $\mathscr{P}_0 = \langle a, b | b^3 = ab^2a^{-1}, a^3 = ba^2b^{-1} \rangle$ of $\pi = \{1\}$, for which an AC-trivialization has not yet been achieved (s. Crowell-Fox [5], Osborne [20],

(5) In particular: no trivial relator can be isolated by applying Q^{**}-transformations to $\mathscr{P}$.

Metzler [17]). C.F. Miller and P.E. Schupp observed that, in fact, the second relator of $\mathscr{P}_0$ can be replaced by any R with exponent sum 1 in a, still yielding a presentation $\mathscr{P}_{M-S}$ of $\pi = \{1\}$, s. [19]. For these presentations we may extend our theorem to the following

Proposition [(6)]. *Let* $(m,n) \neq 1$ *and* $\mathscr{Q} = \langle \alpha, \beta | \alpha^m = 1, [\alpha, \beta] = 1, \beta^n = 1 \rangle$ *be the «standard» presentation of* $\mathbb{Z}_m \times \mathbb{Z}_n$. *Then* $\mathscr{P}_{M-S} + \mathscr{Q} \sim \mathscr{Q}$.

Proof. By Miller and Schupp's result a given presentation $\mathscr{P}_{M-S} + \mathscr{Q} = \langle a, b, \alpha, \beta | \alpha^m = 1, [\alpha, \beta] = 1, \beta^n = 1, b^3 = ab^2a^{-1}, R = 1 \rangle Q^{**}$-transforms into

$$\langle a, b, \alpha, \beta | \alpha^m = 1, \alpha\beta\alpha^{-1} = \beta^{1+n}, \beta^n = b, b^3 = ab^2a^{-1}, R = 1 \rangle. \tag{15}$$

The first two relations of (15) imply that β is of an order which divides $(1+n)^m - 1$ (a multiple of n); hence already by the first 3 relations b has finite order too. Now the fourth relation shows that b^3 and b^2 have the same order which in turn then has to be the order of b. Therefore b^2 and b^3 are primitive elements in the cyclic group generated by b. $b^3 = ab^2 = a^{-1}$ thus implies

$$aba^{-1} = b^k, \quad a^{-1}ba = b^\ell \quad \text{for some } k, \ell \in \mathbf{N}; \tag{16}$$

and in achieving these consequences, the last relator R has not yet been used.

Hence because of (16), this relator in (15) can be changed into $a = b^i$ for some $i \in \mathbf{N}$ by a Q^{**}-transformation. But now a final chain of Q^{**}-transitions is apparent:

$$\langle a, b, \alpha, \beta | \alpha^m = 1, \alpha\beta\alpha^{-1} = \beta^{1+n}, \beta^n = b, b^3 = ab^2a^{-1}, a = b^i \rangle \to$$
$$\langle a, b, \alpha, \beta | \alpha^m = 1, \alpha\beta\alpha^{-1} = \beta^{1+n}, \beta^n = b, b = 1, a = b^i \rangle \to$$
$$\langle a, b, \alpha, \beta | \alpha^m = 1, \alpha\beta\alpha^{-1} = \beta^{1+n}, \beta^n = 1, b = 1, a = 1 \rangle \to \mathscr{Q}, \quad \text{q.e.d.}$$

Note that $(m,n) \neq 1$ didn't enter the computation. This sassumption solely has been made to ensure that $\mathscr{Q}$ is of maximal deficiency.

C) In a forthcoming paper [11] it will be shown that the commutator property (6) of §2 can be sharpened as follows:

(6) The argument of Miller and Schupp is given in the even more general situation in which the exponents 2, 3 in the first relator of $\mathscr{P}_0$ and $\mathscr{P}_{M-S}$ are substituted by arbitrary adjacent natural numbers. This generalization also holds for the proposition.

(6') If $\mathcal{P} = \langle a_1, \ldots, a_g | R_1, \ldots, R_h \rangle$ and $\mathcal{Q} = \langle q_1, \ldots, a_g | S_1, \ldots, S_h \rangle$ are finite presentations with the same relator subgroup $N \subseteq F(a_i)$ and if (6) holds, then by applying appropriate transformations of type (5) (a), (b) (*Q-transformations*) to (one of) the presentations $\mathcal{P}, \mathcal{Q}$ we can achieve

$$R_j S_j^{-1} \in N^{(n)} \quad \text{for any } n \geq 1, \quad \text{where}$$

$N^{(n)}$ denotes the $n-th$ derived group of N (i.e. $N^{(1)} = [N, N], N^{(n)} = [N^{(n-1)}, N^{(n-1)}]$).

This is an extension to simple-homotopy of an (unpublished) result of W. Browning [2], who was mainly interested in the (AC)-case. It prevents us from attempts to disprove (AC') by projecting the R_j and S_j to a quotient of $F(a_i)$ mod F_n (lower central series), $F^{(n)}$, N_n or $N^{(n)}$ and trying to achieve Q-invariants for the projections of the «vectors» $(R_1, \ldots, R_h)$ and $(S_1, \ldots, S_h)$ as a first step towards Q^{**}-invariants.

On the other hand, a study of the terms $R_j S_j^{-1} \in N^{(n)}$ for increasing values of n may be worth while – in particular for $\pi \neq \{1\}$ – in order to obtain Q^{**}-invariants, s. [14].

(6') moreover is a motivation to look for «refined» factors instead of S^2 (serving for N), $\mathbb{Z}_2 \times \mathbb{Z}_4$ (serving for $[N, N]$), which are appropriate for a modification of the commutator-trick of §2 on the level of a preassigned $n \geq 1$.

Finally we want to point out that even via their consequences of providing universal stabilizing factors (6) and (6') prevent from concentrating on futile invariants: A Q^{**}-invariant must «die» after wedging on sufficiently many copies of a universal stabilizing factor, and for many suggested invariants this requirement implies that they are unable to distinguish between different Q^{**}-classes at all.

These considerations are disillusions for quick attempts to disprove (AC'). But they also contain «positive» aspects which – in connection with concrete potential counterexamples as mentioned in [17] and the remark at the end of A) above – leave the Andrews-Curtis problem as a challenging topic for further research.

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