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A CHARACTER-THEORY-FREE CHARACTERIZATION OF THE SIMPLE GROUPS M_{11} AND $L_3(3)$

DIETER HELD, JÖRG HRABĚ DE ANGELIS

Dedicated to the memory of Professor Gottfried Köthe

Abstract. *The existing proofs of the characterization of the Mathieu group M_{11} by the centralizer of one of its involutions make heavy use of the theory of group characters. There was a strong feeling that in small cases like this character theory was absolutely indispensable to make up for the poor local structure faced with in such situations. Up to now, the characterization of M_{11} has served as an illustration of the power of the theory of exceptional characters. Here, in the course of the post-classification effort to simplify proofs, we show that M_{11} can be treated in a completely elementary and group theoretical way while carrying out each step of the argument in detail.*

INTRODUCTION

The objective of this paper is to present a character-theory-free proof of the following result:

Theorem. *Let G be a finite nonabelian simple group which has the following two properties:*

(a) The center Z of a Sylow 2-subgroup T of G is cyclic;

(b) if z is the involution in Z, then the centralizer C of z in G is an extension of $\langle z \rangle$ by the symmetric group of degree four.

Then G is isomorphic to M_{11} or $L_3(3)$.

This theorem was stated in [6] as an easy consequence of a special case of a result of Richard Brauer [2] proved by W. J. Wong [20; theorem 6]. Both authors made use of mainly character-theoretical methods. For a typical proof of theorems of that genre in the character-theoretic mode, the reader is invited to study [13; § 5, pp. 341-366]. In this paper we employ a method introduced by H. Bender [1], an abstract definition for the Mathieu group M_{11} discovered by J. A. Todd [18], and a presentation of the projective special linear group $L_3(3)$ which can be found in § 3. Along similar lines, an elementary and completely character-theory-free proof of a characterization of the Mathieu-group M_{12} by the centralizer of a 2-central involution has been obtained in [9]. As for such characterizations of M_{22} and M_{23} done by Z. Janko [14], it has been shown in [10] that one has not to recourse to the theory of exceptional characters; as for an analogous characterization of M_{24} [7; § 5] it has been proved in [8] that one can do without invoking a block-theoretic result of R. G. Stanton [15] the proof of which has never been published in detail.

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1. PREPARATORY LEMMAS

Lemma 1.1. *The group C is isomorphic to $GL_2(3)$.*

Proof. Assume that C splits over $\langle z\rangle$. Then, a Sylow 2-subgroup of C, which obviously is a Sylow 2-subgroup of G, would not have a cyclic center. This contradicts (a). There are only two possibilities for a central non-splitting extension of a group of order 2 by Σ_4. These are $GL_2(3)$ and a group with a generalized quaternion group of order 16 as a Sylow 2-subgroup.

Assume by way of contradiction that T was a generalized quaternion group of order 16. Since each element of $T^{\#}$ is a root of the involution z, two elements of T are conjugate in G if and only if they are conjugate in $\mathbb{C}(z) = C$. Applying [5; 7.3.4] we compute

$$T = T \cap G' = \langle x^{-1}y | x, y \in T, x \text{ and } y \text{ conjugate in } G\rangle =$$
$$= \langle x^{-1}y | x, y \in T, x \text{ and } y \text{ conjugate in } C\rangle \subseteq C'.$$

The fact that $C/\langle z\rangle$ is isomorphic to Σ_4 yields $|C'| = 2^3 \cdot 3$, contrary to $T \subseteq C'$. Thus $C \cong GL_2(3)$, and the lemma is proved.

In what follows we shall identify C with $GL_2(3)$ using the following correspondences with entries from $GF(3)$:

$$z \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, t \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, a \rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, d \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We are able to compute the table I of conjugacy classes of C.

Table I

Element x	$o(x)$	x^2	$\lvert\mathbb{C}_C(x)\rvert$	$\lvert ccl_C(x)\rvert$	$\mathbb{C}_C(x)$
1	1		$2^4 \cdot 3$	1	C
z	2		$2^4 \cdot 3$	1	C
t	2		2^2	12	$\langle z\rangle \times \langle t\rangle$
d	3		$2 \cdot 3$	8	$\langle d\rangle \times \langle z\rangle$
a^2	4	z	2^3	6	$\langle a\rangle$
dz	6		$2 \cdot 3$	8	$\langle d\rangle \times \langle z\rangle$
a	8		2^3	6	$\langle a\rangle$
a^{-1}	8		2^3	6	$\langle a\rangle$
				48	

Lemma 1.2. *The group G contains precisely one class of involutions, precisely one class of elements of order 4, precisely one class of elements of order 6, and precisely two classes of elements of order 8.*

Proof. We know that T is a Sylow 2-subgroup of G which contains a cyclic group T_1 of order 8 as a maximal subgroup. A lemma of Thompson [16; 5.1.8] yields that each involution in G is conjugate to an involution in T_1. But the only involution in T_1 is z. Hence, G contains precisely one class of involutions.

Obviously, elements of order 4, 6 and 8 lie in the centralizer of a suitable involution. Therefore, the group G contains precisely one class of elements of order 4 and precisely one class of elements of order 6. Assume by way of contradiction that all elements of order 8 were conjugate in G. Then, a and a^{-1} would fuse in G. Since $a^4 = (a^{-1})^4 = z$, we see that this conjugation should take place in C, which - however - is not the case. The lemma is proved.

Lemma 1.3. *The normalizer of $\langle d\rangle$ in G is a splitting extension of a 3-group K of order at most 27 by the four-group $\langle z,t\rangle$. Furthermore, we have $K = \langle C_K(x) \mid x \in \langle z,t\rangle^{\#}\rangle$, and K is of exponent 3.*

Proof. We have $\mathbb{C}_C(d) = \langle d\rangle \times \langle z\rangle$ and $\mathbb{N}_C(\langle d\rangle) = \langle d\rangle\langle t\rangle \times \langle z\rangle$. Obviously, $\langle z\rangle$ is a Sylow 2-subgroup of $\mathbb{C}(d)$. Otherwise, a subgroup of order 4 in C would centralize d, which is not the case. Therefore, we get $\mathbb{C}(d) = K\langle z\rangle$, where K is a normal 2-complement of $\mathbb{C}(d)$ and of $\mathbb{N}(\langle d\rangle)$. This is a direct consequence of a transfer lemma of Burnside [5; 7.4.3]. The four-group $\langle z,t\rangle$ acts on K. Hence, the Brauer-formula [19; 1.1] - the proof of which can be obtained as an easy consequence of Thompson's order formula [11] - yields

$$|K| \cdot |\mathbb{C}_K(\langle z,t\rangle)|^2 = |\mathbb{C}_K(z)| \cdot |\mathbb{C}_K(t)| \cdot |\mathbb{C}_K(tz)|.$$

Obviously, we have $|\mathbb{C}_K(\langle z,t\rangle)| = 1$. Since the centralizer of an involution in G is of order 48, we obtain $|K| \in \{3, 3^2, 3^3\}$.

From the order of C we get that K is generated by elements of order 3. Since the order of K is at most 27, we obtain from [5; 5.3.9] that K is of exponent 3.

Lemma 1.4. *The normalizer of $\langle z,t\rangle$ in G is isomorphic to Σ_4. In particular, $\mathbb{C}(\langle z,t\rangle) = \langle z,t\rangle$. All four-groups in G are conjugate.*

Proof. We have $\mathbb{C}(\langle z,t\rangle) = \langle z,t\rangle$, and $\mathbb{N}_C(\langle z,t\rangle)$ is isomorphic to D_8 with center $\langle z\rangle$. Similarly, $\mathbb{N}(\langle z,t\rangle) \cap \mathbb{C}(t) \cong D_8$. It follows $\mathbb{N}(\langle z,t\rangle) \cong \Sigma_4$. Since all four-groups which contain the involution z are conjugate in C, and all involutions are conjugate in G, the last assertion follows as well.

In what follows K denotes the maximal odd-order normal subgroup of $\mathbb{N}(\langle d \rangle)$. Application of 1.3 yields that the order of K is equal to 3, 9, or 27. In the following sections we shall make use of the notation introduced so far.

2. THE CASE $|K| = 9$

In this section we argue under the assumption $|K| = 9$.

Lemma 2.1. *The normalizer of K in G is a splitting extension of an elementary abelian group of order 9 by a semidihedral group of order 16. Further, $\mathbb{C}(K) = K$ and K is a Sylow 3-subgroup of G. All elements of order 3 in G are conjugate in G.*

Proof. Obviously, the 3-group K is abelian. Application of lemma 1.3 gives that K is elementary abelian. Denote by q an element of $K^{\#}$ which is centralized by an involution of $\langle z, t \rangle \setminus \langle z \rangle$. Then, we have $\langle d, q \rangle = K$. Since K lies in $\mathbb{C}(d) = K\langle z \rangle$, we see that K is selfcentralizing.

Assume by way of contradiction that 3^3 divided the order of $\mathbb{N}(K)$. Then, K lies in a subgroup K_1 of order 27 of G, and there are at least six elements in K conjugate to d under $\mathbb{N}(K)$. Let x be an element in $\mathbb{Z}(K_1) \cap K^{\#}$. Clearly, x is not centralized by an involution in G. Since $\langle z, t \rangle$ acts on K, we see that x is conjugate to at least four elements in K. Hence, x and d would be conjugate in G. This contradicts the fact that 3^3 does not divide the order of $\mathbb{N}(\langle d \rangle)$. Therefore, the 3-group K is a Sylow 3-subgroup of G.

From a lemma of Burnside follows that d and q are conjugate in $\mathbb{N}(K)$, since they are conjugate in G. Therefore, we have $\mathbb{N}(K) \supset K\langle z, t \rangle$. Note that $\mathbb{N}(K)$ must be a $\{2, 3\}$-group by the order of $GL_2(3)$. Hence, $\mathbb{N}(K) = KS$, where S is either a dihedral group of order 8 or a Sylow 2-subgroup of G.

By way of contradiction we assume that S had order 8. Then, the element d has precisely four conjugates lying in K. These are d, d^{-1}, q, and q^{-1}. Hence, the element dq of K is not centralized by any involution in G, and we obtain $\mathbb{C}(dq) \cap \mathbb{N}(K) = K$. Therefore, each element in $K^{\#}$ would be conjugate to dq in $\mathbb{N}(K)$, which, however, is not possible. The assertion follows.

Lemma 2.2. *Let x be an element of order at least 3 in $\mathbb{N}(K)$. If $o(x) = 4$, assume that $\mathbb{C}(x) \subseteq \mathbb{N}(K)$. Then, we have $\mathbb{N}_G(\langle x \rangle) \subseteq \mathbb{N}(K)$. The only elements of $\mathbb{N}(K)^{\#}$ which are centralized by involutions of $G \setminus \mathbb{N}(K)$ are involutions.*

Proof. Let S be a Sylow 2-subgroup of $\mathbb{N}(K)$ containing z, and let r be an element of order 8 in S. Then, the involution r^4 acts fixed-point-freely on K, and $r^z = r^3$ holds.

We are able to compute the conjugacy classes of S and of $\mathbb{N}(K)$:

$$S : 1 \cdot 1 + 1 \cdot r^4 + 4 \cdot z + 2 \cdot r^2 + 4 \cdot zr + 2 \cdot r + 2 \cdot r^{-1};$$

$$\mathbb{N}(K) : 1 \cdot 1 + 9 \cdot r^4 + 12 \cdot z + 8 \cdot d + 18 \cdot r^2 + 36 \cdot zr + 24 \cdot dz + 18 \cdot r + 18 \cdot r^{-1}.$$

We have $o(r) = 8$, $o(z) = 2$, $o(d) = 3$, $o(zr) = 4$, $o(dz) = 6$.

Since $\mathbf{N}(\langle d\rangle) \subseteq \mathbf{N}(K)$ and all elements of order 3 of $\mathbf{N}(K)$ are conjugate in $\mathbf{N}(K)$, the assertion follows for elements of order 3.

The element dz represents the unique class of elements of order 6 in $\mathbf{N}(K)$. One easily sees that $\mathbf{N}(\langle dz\rangle) = \langle dz\rangle\langle r^4\rangle$ holds. This yields $\mathbf{N}(\langle dz\rangle) \subseteq \mathbf{N}(K)$.

Obviously, the centralizer of an element of order 4 is cyclic of order 8. Therefore, we only have to look at r^2. Since $\langle r^2\rangle$ is normalized by S, it follows $\mathbf{N}(\langle r^2\rangle) \subseteq \mathbf{N}(K)$.

Let x be an element of order 8 in $\mathbf{N}(K)$. We have $\mathbf{C}(x) = \langle x\rangle$. Together with the fact that x is not inverted in G we get $\mathbf{N}(\langle x\rangle) \subseteq \mathbf{N}(K)$.

Since the centralizer of an element of order 4 is cyclic of order 8, the last assertion follows immediately. The lemma is proved.

Lemma 2.3. *The order of G is equal to $2^4 \cdot 3^2 \cdot 5 \cdot 11$.*

Proof. Put

$$J = \text{ set of all involutions in } G,$$

$$H = \mathbf{N}(K),$$

$$b_n = \text{ number of cosets } Hg \neq H \text{ such that } |Hg \cap J| = n,$$

$$f = (|J|/|G : H|) - 1 = |H|/|C| - 1 = 2.$$

We want to determine the numbers b_n for $n \geq 2$. For this purpose we have to look only at the involutions in $G \setminus H$ which invert a nontrivial element in H. Note that 2.2 completely describes such elements.

Let x be an element of $H^\#$ which is inverted by an involution outside H. By 2.2 the order of x is equal to 2 or 4. First we assume $o(x) = 4$. Then, we have $H \cap \mathbf{C}(x) = \langle x\rangle$. The normalizer of $\langle x\rangle$ in G is a Sylow 2-subgroup of G, and $\mathbf{C}_G(x) \cong Z_8$. The normalizer of $\langle x\rangle$ in H is isomorphic to Q_8. Therefore, x is inverted by precisely four involutions none of which lies in H. Assume next that x is a 2-central involution in H. Then $\mathbf{C}(x) \cap H$ contains precisely five involutions. Hence, x is centralized by eight involutions of $G \setminus H$. Finally, we look at the case that x is an involution which does not lie in the center of a Sylow 2-subgroup of H. Then, $\mathbf{C}_H(x)$ is of order $2^2 \cdot 3$ and contains precisely seven involutions.It follows that x is centralized by six involutions of $G \setminus H$.

Let v and w be two different elements in $H^\#$ with $v \notin \langle w\rangle$ and $w \notin \langle v\rangle$. Assume that v and w were inverted by the same involution i of $G \setminus H$. We distinguish four cases suggested by 2.2.

Case 1. Here, we have $o(v) = o(w) = 2$. Then, i centralizes the element vw of H. Applying 2.2 this yields that vw is also an involution. We obtain $\langle v, w\rangle \cong E_4$, and therefore, it follows $\mathbf{C}(v) \cap \mathbf{C}(w) = \langle v, w\rangle$ by 1.4. This gives the contradiction $i \in \mathbf{C}(v) \cap \mathbf{C}(w) = \langle v, w\rangle \subseteq H$. Case 1 is not possible.

Case 2. Here, we assume $o(v) = 2$, and $o(w) = 4$. Then, i centralizes the involutions v and w^2. Case 1 yields $v = w^2$, i.e. $v \in \langle w \rangle$.

Case 3. We have $o(v) = 4$, and $o(w) = 2$. Thus, i centralizes the involutions v^2 and w. Case 1 forces $v^2 = w$, so $w \in \langle v \rangle$.

Case 4. Here, we have $o(v) = o(w) = 4$. Thus, $[v^2, i] = [w^2, i] = 1$, and case 1 implies $v^2 = w^2$. Then, we have $v, w \in O_2(\mathbb{C}(v^2))$. Remember that $O_2(\mathbb{C}(v^2)) \cong O_2(C)$ is a quaternion group. Compute $(vw)^i = v^i w^i = v^{-1}w^{-1} = vv^2w^2w = vw$. Obviously, vw is an element of order 4 in H. The structure of C yields $i = (vw)^2$. This is not possible, since i does not lie in H. Hence, case 4 does not occur.

Summarizing the above results we get that a coset of H different from H cannot contain precisely three involutions and that it cannot contain more than four involutions.

Now, we are able to compute the numbers b_n for $n \geq 2$. We have $b_3 = b_m = 0$ $(m \geq 5)$. The subgroup H contains precisely 36 elements of order 4 which are inverted by an involution of $G \setminus H$. Therefore, we have precisely 18 cosets of H in G which contain exactly four involutions. Since a 2-central involution of H is centralized by four of these 36 elements of order 4 and by precisely eight involutions of $G \setminus H$, we have that each involution of $G \setminus H$, which centralizes a 2-central involution of H, acts invertingly on a cyclic subgroup of order 4 of H which contains the 2-central involution. These involutions in $G \setminus H$ have been considered above.

Now, to compute b_2, we have only to look at the twelve non-2-central involutions in H. Each of them is centralized by six involutions of $G \setminus H$. Therefore, we count

$$b_2 = 12 \cdot 3 = 36.$$

A result in [1] forces

$$\begin{aligned} b_1 &< f^{-1}(|J \cap H| + b_2 + 2b_3 + 3b_4 + \ldots) - 1 - b_2 - b_3 - b_4 - \ldots = \\ &= \frac{1}{2}(9 + 12 + 36 + 0 + 3 \cdot 18 + 0) - 1 - 36 - 0 - 18 = \frac{1}{2}. \end{aligned}$$

Therefore $b_1 = 0$.

We are able to compute the order of G:

$$\begin{aligned} |G| = |J| \cdot |C| &= (|J \cap H| + \sum_{i \in \mathbb{N}} i \cdot b_i) \cdot |C| = \\ &= (|J \cap H| + 2 \cdot b_2 + 4 \cdot b_4) \cdot |C| = \\ &= (21 + 2 \cdot 36 + 4 \cdot 18) \cdot 48 = 2^4 \cdot 3^2 \cdot 5 \cdot 11 = |M_{11}|. \end{aligned}$$

The lemma is proved.

Lemma 2.4. *If ϵ is an element of order 11 in G, then the normalizer of $\langle\epsilon\rangle$ in G is a Frobenius-group of order $5 \cdot 11$. The group G contains precisely two classes of elements of order 11.*

Proof. Lemma 2.1 and the structure of C yield that $\mathbb{C}(\epsilon)$ is a $\{5,11\}$-group. Sylow's theorem gives that $\mathbb{N}(\langle\epsilon\rangle)$ has order $5 \cdot 11$. Now, the assertion directly follows from a transfer result of Burnside.

Lemma 2.5. *Let f be an element of order 5 in G. Then $\mathbb{C}(f) = \langle f\rangle$ and $\mathbb{N}(\langle f\rangle)$ is an extension of $\langle f\rangle$ by a cyclic subgroup of order 4. All elements of order 5 in G are conjugate in G.*

Proof. Application of lemma 2.1, 2.4, and the structure of C yield that $\mathbb{C}(f)$ is a 5-group. Therefore, $\mathbb{C}(f) = \langle f\rangle$ holds. The theorem of Sylow completes the proof.

Now, we are able to state the following lemma.

Lemma 2.6. *The conjugacy classes of G are uniquely determined, and we list them in table II. Here, f denotes an element of order 5 and ϵ an element of order 11.*

Table II

Element x	$o(x)$	$\lvert\mathbb{C}_G(x)\rvert$	$\lvert ccl_G(x)\rvert$	$\mathbb{C}_G(x)$
1	1	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	G
z	2	$2^4 \cdot 3$	165	C
d	3	$2 \cdot 3^2$	440	$K\langle z\rangle$
a^2	4	2^3	990	$\langle a\rangle$
f	5	5	1,584	$\langle f\rangle$
dz	6	$2 \cdot 3$	1,320	$\langle d\rangle \times \langle z\rangle$
a	8	2^3	990	$\langle a\rangle$
a^{-1}	8	2^3	990	$\langle a\rangle$
ϵ	11	11	720	$\langle\epsilon\rangle$
ϵ^{-1}	11	11	720	$\langle\epsilon\rangle$
			7,920	

Lemma 2.7. *Assume that G contains a subgroup of index 12. Then, G is isomorphic to M_{11}.*

Proof. The group G acts on the cosets of a suitable subgroup M of index 12 in G. Hence, we may and shall assume that G is a subgroup of A_{12} which acts transitively on the set of twelve symbols $\Omega = \{1,2,\dots,12\}$. Since a Sylow 11-normalizer of A_{12} is a Frobenius-group of order 55, we may further assume that

$$\alpha = (1,2,3,4,5,6,7,8,9,10,11) \text{ and}$$

$$\beta = (1,4,5,9,3)(2,8,10,7,6)$$

lie in G, and that $\langle\alpha,\beta\rangle$ is a Sylow 11-normalizer of G. We have

$$\alpha^\beta = \alpha^4, \alpha^{11} = \beta^5 = 1.$$

From lemma 2.5 we get the existence of an element γ of order 4 in G so that $\beta^\gamma = \beta^2$.

We have $G = \langle\alpha,\beta,\gamma\rangle$, namely: Assume false. Then $U = \langle\alpha,\beta,\gamma\rangle$ is a proper subgroup of G the order of which is divisible by $2^2 \cdot 5 \cdot 11$. The theorem of Sylow yields $|U| = 2^2 \cdot 3 \cdot 5 \cdot 11$. Hence, a Sylow 2-subgroup is cyclic of order 4. From a transfer lemma of Burnside [5; 7.4.3], we see that U has a normal 2-complement, say X. Then, a Sylow 11-subgroup of X is normal in X. It follows that G contains an element of order 33, contrary to 2.6. This proves $G = \langle\alpha,\beta,\gamma\rangle$.

Since $G = \langle\alpha,\beta,\gamma\rangle$ acts on Ω transitively, and since we may replace γ with $\gamma\beta^i (1 \le i \le 4)$, one easily sees that we have only the five possibilities γ_i for γ:

$$\gamma_1 = (1,2,4,10)(3,7,5,6)(8,9)(11,12);$$

$$\gamma_2 = (1,2,5,6)(3,7,9,8)(4,10)(11,12);$$

$$\gamma_3 = (1,2)(3,7,4,10)(5,6,9,8)(11,12);$$

$$\gamma_4 = (1,10,3,2)(4,6,9,7)(5,8)(11,12);$$

$$\gamma_5 = (1,8,5,2)(3,6,9,10)(4,7)(11,12).$$

We compute $o(\alpha^{-1}\beta\gamma_3) = 35$, $o(\alpha\beta\gamma_4) = o(\alpha\beta\gamma_5) = 10$. Since G does not contain elements of order 35 or 10, we have $\gamma \in \{\gamma_1,\gamma_2\}$. Obviously, the following relations hold:

(i) $\alpha^{11} = \beta^5 = \gamma_i^4 = 1$, $\alpha^\beta = \alpha^4$, $\beta^{\gamma_i} = \beta^2$ $(1 \le i \le 2)$;

(ii) $((\alpha^2)^4\gamma_1^2)^3 = (\beta\gamma_1^2)^2 = (\alpha^2\beta\gamma_1)^3 = 1$;

(iii) $(\alpha^4\gamma_2^2)^3 = (\beta\gamma_2^2)^2 = (\alpha\beta\gamma_2)^3 = 1$.

A result of Coxeter and Moser [4; p. 99] implies $G = \langle\alpha,\beta,\gamma\rangle \cong \langle\alpha,\beta,\gamma_1\rangle \cong \langle\alpha,\beta,\gamma_2\rangle \cong M_{11}$. This proves the lemma.

Theorem 2.8. *The group* G *is isomorphic to* M_{11}.

Proof. For the purpose of this proof we change our notation completely and use only structural information obtained so far. Let K be a Sylow 3-subgroup of G and denote by H its normalizer in G. Since H is an extension of $K \cong E_9$ by a semidihedral group of order 16, and since $\mathbb{C}(K) = K$, the group H is uniquely determined up to isomorphism. Studying Todd's presentation for M_{11} in [18] we see that we are able to find elements a, b, c, and e in H so that $H = \langle a, b, c, e\rangle$ and the following relations holds:

(i) $a^4 = 1$, $b^4 = 1$, $a^2 = b^2$, $a^b = a^{-1}$,

(ii) $c^2 = 1$, $c^a = (bc)^2$, $c^b = (ac)^2$,

(iii) $e^2 = 1$, $a^e = ab$, $b^e = b^{-1}$, $c^e = c$.

Note that $a^4 = b^4 = c^2 = e^2 = 1$ means here that $o(a) = o(b) = 4$ and that c and e are involutions. Direct computation shows that $K = \langle a^2c, aca\rangle$ holds. The centralizer of a^2 is isomorphic to $GL_2(3)$. Since $\mathbb{C}(a^2) \cap H = \langle a, b, e\rangle$ is a Sylow 2-subgroup of H, it is easy to see that there is an involution d in $G \setminus H$ so that

(iv) $d^2 = 1$, $a^d = a^{-1}$, $b^d = ab$, $(ed)^3 = 1$.

It follows $\mathbb{C}(a^2) = \langle a, b, d, e\rangle \cong GL_2(3)$.

It is our aim to determine the order of cd. Since cd is inverted by an involution, for example by c or d, we get that $o(cd)$ is equal to 1,2, 3, 4, 5, or 6 from 2.6. Since c lies in H, but d does not, it follows that the order of cd is not equal to 1.

We split our argument into five cases:

Case 1. Assume $o(cd) = 2$. By the proof of 2.3 the coset $Hd \neq H$ contains precisely four involutions, namely ad, a^2d, a^3d, and d. Since cd is an involution, we get $cd \in \langle a\rangle d$. Hence, the element c lies in $\langle a\rangle$ which is not possible, because a^2c is an element of order 3 in H. We have shown that $o(cd) \neq 2$.

Case 2. Here, we have $o0(cd) = 4$. Using the relations in (iii), and (iv) we get

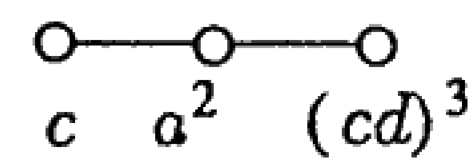
$$(cd)^{ede} = c^{ede}d^{ede} = c^{de}e = edcdee = edcd.$$

Hence, $edcd$ is an element of order 4 which is centralized by the involution c. It follows $e(dcd) \in O_2(\mathbb{C}(c))$ by the structure of the centralizer of an involution in G. Hence, we have $e(dcd) \in H$, consequently $(cd)^2 \in H$ and cdc is an involution in Hd. The fact that $Hd \neq H$ contains precisely four involutions forces $cdcd = a^2$. But $(cd)^2$ centralizes c and a^2 does not. This contradiction shows that the possibility $o(cd) = 4$ does not occur.

Case 3. Now, we consider the case that the order of cd is equal to 6. Then, the involution $(cd)^3$ does not lie in H, namely: if the assertion were false, then the involution $(cd)^3d$ would lie in Hd. But Hd contains precisely four involutions, and this implies $(cd)^3 = a^2$ which is against $o(ca^2) = 3$. Thus, $(cd)^3 \in \mathbb{C}(c) \setminus H$.

We have a^2, $(cd)^3 \in \mathbb{C}(d)$. Hence, the order of $a^2(cd)^3$ is 1,2,3,4, or 6. Remember that an element of order 8 is not inverted by an involution in G. If $(cd)^3$ centralized a^2, then $(cd)^3$ would lie in $\mathbb{C}(a^2c)$, against $(cd)^3 \notin H$, but $\mathbb{C}(a^2c) \subset H$. Hence, $o(a^2(cd)^3) \notin \{1,2\}$.

Assume next $o(a^2(cd)^3) = 3$. Put $A = \langle c, a^2, (cd)^3 \rangle$. We know that the relations of the following diagram hold in A:



Thus, $A \cong \Sigma_4$. Note that A contains $\langle a^2, c\rangle \cong \Sigma_3$ as a proper subgroup. From the relations in (i) to (iv) we obtain

$$(cd)^{a^2cd} = d(ca^2c)da^2cd = da^2c(a^2da^2)cd = da^2cdcd =$$
$$= a^2dcdcd = a^2c(cd)^3 \in A.$$

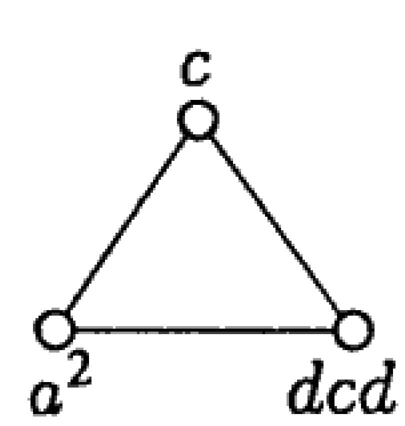
Hence, there is an element of order 6 lying in A, contrary to $A \cong \Sigma_4$. The order of $a^2(cd)^3$ is not equal to 3.

Assume now $o(a^2(cd)^3) = 4$. Since d centralizes $a^2(cd)^3$, we have $d = (a^2(cd)^3)^2 = a^2c(dcdcd)a^2(cd)^3$. It follows

$$d^{cd} = dcdcd = ca^2(d(dc)^3)a^2 = (ca^2c)dcdca^2 =$$
$$= a^2c(a^2d)cdca^2 = a^2cd(a^2c)dca^2 = (a^2c)^{dca^2}$$

which, however, is not possible, since d is an involution and $o(a^2c) = 3$. This forces that $a^2(cd)^3$ is not an element of order 4.

Finally, we have to treat the case that $o(a^2(cd)^3) = 6$. Since d centralizes $a^2(cd)^3$, the structure of the centralizer of an involution in G implies $d = (a^2(cd)^3)^3$. Hence, $(a^2cdcdc)^3 = 1$. We have obtained the following diagram

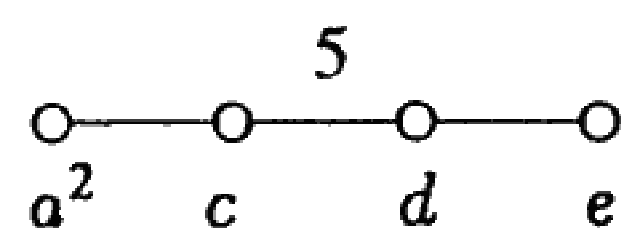


Note that $a^2dcd = d(a^2c)d$ holds. Put $B = \langle a^2, c, dcd\rangle$. Let B^* be the abstract group generated by a^2, c, and dcd so that the relations implied by the above diagram together with

$(a^2 \cdot c \cdot (dcd) \cdot c)^3 = 1$ hold. We make use of the Todd-Coxeter-enumeration-method described in [4]. It follows that the order of B^* is 54, and therefore $|B| \in \{2 \cdot 3, 2 \cdot 3^2\}$. In any case, B is 3-closed and lies in $\mathbb{N}(\langle a^2 c\rangle) \subseteq H$. Note that $\langle a^2, c\rangle \cong \Sigma_3$ is a subgroup of $\mathbb{N}(\langle a^2 c\rangle)$. This proves $(cd)^2 \in K$, since $(cd)^2$ is an element of order 3 of H. Consequently, $cd \in H$, since cd centralizes the element $(cd)^2$ of K, and we conclude $d \in H$ which, obviously, is a contradiction. We have shown that $o(a^2(cd)^3) = 6$ does not occur.

Therefore, we have obtained a contradiction to the assumption that the order of cd would be equal to 6.

Case 4. Consider the possibility $o(cd) = 5$. Here, the relations described by the following diagram are obviously satisfied:



Let B^* be the abstract group generated by a^2, c, and d together with the relations between these generators given in the above diagram. Then, we have $B^* \cong Z_2 \times A_5$. Since an involution does not centralize an element of order 5 in G, it follows $\langle a^2, c, d\rangle \cong A_5$, and consequently $(a^2 cd)^5 = 1$. It is easy to verify that the abstract group C^* - defined by the above diagram together with the additional relation $(a^2 cd)^5 = 1$ - is isomorphic to $L_2(11)$. We refer to [3] where these defining relations can also be found. This implies $\langle a^2, c, d, e\rangle \cong L_2(11)$, and G contains a subgroup of index 12. By lemma 2.7, the assertion follows in this case.

Case 5. Here, the order of cd is equal to 3. A result of J. A. Todd in [18] gives that $\langle a, b, c, d, e\rangle$ is isomorphic to M_{11}. Since $|G| = |M_{11}|$, we have $G \cong M_{11}$.

The theorem is proved.

3. THE CASE $|K| = 27$

Here, we argue under the hypothesis $|K| = 27$.

Lemma 3.1. *A Sylow 3-subgroup of G is extra-special of order 27 and exponent 3.*

Proof. By way of contradiction we assume that K was abelian. Application of 1.3 yields that K is elementary abelian. Obviously, we have $\mathbb{C}(K) = K$. Hence, the order of $\mathbb{N}(K)/K$ devides $2^4 \cdot 3^3 \cdot 13$. We split our argument into two cases.

Case 1. Here, K is a Sylow 3-subgroup of G. Note that $\langle z, t\rangle$ acts on K and that $\mathbb{C}(\langle d\rangle)$ lies in the normalizer of K. Therefore, the factor group $\mathbb{N}(K)/K$ is a $\{2, 13\}-$ group, and its order is divisible by 4. Furthermore, the element d has at most 26 conjugates under the action of $\mathbb{N}(K)$, and it has at least six conjugates under $\mathbb{N}(K)$, namely the elements in

$\mathbb{C}_K(z)^{\#} \cup \mathbb{C}_K(t)^{\#} \cup \mathbb{C}_K(zt)^{\#}$, since these are conjugate in G. This follows immediately from a lemma of Burnside. We obtained

$$|\mathbb{N}(K)/K| \in \{2^4, 2^2 \cdot 13\}.$$

Since $GL_3(3)$ does not contain a subgroup of order $2^2 \cdot 13$, it follows $|\mathbb{N}(K)/K| = 2^4$. Let T_1 be a subgroup of order 8 of $\mathbb{N}(K)$ containing $\langle z,t\rangle$, and let x be an involution in $\mathbb{Z}(T_1) \cap \langle z,t\rangle$. Then, T_1 acts on $\mathbb{C}_K(x)$ which is group of order 3. Therefore, an element of order 3 in $\mathbb{C}_K(x)$ is centralized by a group of order 4. This contradicts the structure of $\mathbb{C}(d)$. Note that $\langle d\rangle$ and $\mathbb{C}_K(x)$ are conjugate in G. Case 1 does not occur.

Case 2. Here, the order of G is divisible by 3^4. Note that by assumption K is still abelian. We get that K is normal in a suitable subgroup of order 3^4 of G. We have $\mathbb{C}(d) \subseteq \mathbb{N}(K)$ and the fact that d has at most 26 conjugates under the action of $\mathbb{N}(K)$. Therefore, we compute

$$|\mathbb{N}(K)/K| \in \{2^2 \cdot 3, 2^2 \cdot 3^2, 2^3 \cdot 3, 2^4 \cdot 3\}.$$

In what follows we shall rule out these possibilities.

Case 2.1. Here, we consider $|\mathbb{N}(K)/K| \in \{2^3 \cdot 3, 2^4 \cdot 3\}$. Let T_1 be a subgroup of order 8 of $\mathbb{N}(K)$ which contains $\langle z,t\rangle$, and let x be an involution in $\mathbb{Z}(T_1) \cap \langle z,t\rangle$. Then, a contradiction follows as in case 1.

Case 2.2. $|\mathbb{N}(K)/K| = 2^2 \cdot 3^2$. The factor group $\mathfrak{N} = \mathbb{N}(K)/K$ acts on K. Since $\langle z,t\rangle$ normalizes $\mathbf{0}(\mathbb{N}(K))$, it follows that $\mathbf{0}(\mathbb{N}(K)) = K$ and $\mathbf{0}(\mathfrak{N}) = \mathbf{0}_3(\mathfrak{N}) = \langle 1\rangle$ holds. Hence,we see that $\mathbf{0}_2(\mathfrak{N}) = \mathbf{0}_{3'}(\mathfrak{N})$ is a group of order 2 or 4. In any case, an element $\mathfrak{p}$ of order 3 in $\mathfrak{N}$ centralizes an involution $\mathfrak{r}$ in $\mathbf{0}_2(\mathfrak{N})$. Therefore, $\mathfrak{p}$ centralizes $\mathbb{C}_K(\mathfrak{r})$ which is a cyclic group of order 3 conjugate to $\langle d\rangle$. We see that $\langle d\rangle$ would be centralized by a group of order 3^4 in G which is not possible.

We have shown that the cases 2.1 and 2.2 do not occur. In what follows we have to deal with the case $|\mathbb{N}(K)/K| = 2^2 \cdot 3$.

Put $\mathfrak{N} = \mathbb{N}(K)/K$. Since $\mathbf{0}(\mathfrak{N}) = \langle 1\rangle$, we get that $\mathbf{0}_2(\mathfrak{N}) \neq \langle 1\rangle$. Let $\mathfrak{p}$ be an element of order 3 in $\mathfrak{N}$. Assume that $\mathfrak{p}$ centralized an element $\mathfrak{r}$ of order 2 in $\mathfrak{N}$. Then, we get a contradiction in the same way as in case 2.2. Hence, $\mathbf{0}_2(\mathfrak{N})\langle\mathfrak{p}\rangle$ is isomorphic to A_4, and so, the normalizer of K in G is an extension of K by A_4. Since $\mathbf{0}_{3,2}(\mathbb{N}(K)) = K\langle z,t\rangle$, the Frattini argument yields that this extension splits.

Denote by $\mathbf{P}$ an element of order 3 which acts on $\langle z,t\rangle$ so that $z^p = t$. Under p we have

$$\mathbb{C}_K(z) \to \mathbb{C}_K(t) \to \mathbb{C}_K(zt).$$

Hence, there exist elements q and r of order 3 in K, so that $\langle q\rangle = \mathbb{C}_K(t)$, $\langle r\rangle = \mathbb{C}_K(zt)$, and $p : d \to q \to r \to d$ hold. Note that $D = K\langle p\rangle$ is a Sylow 3-subgroup of

$\mathbb{N}(K)$. One easily sees that K is the only abelian subgroup of order 27 of D as $\mathbb{Z}(D) = \langle dqr \rangle$. This forces $\mathbb{N}(D) = D$, and consequently, D is a Sylow 3-subgroup of G.

By a lemma of Burnside the 3-central elements dqr and $(dqr)^{-1}$ are not conjugate in G. The element qr is not conjugate to d in G, since such a conjugation would be performed in $\mathbb{N}(K)$; but d has only six conjugates under the action of $\mathbb{N}(K)$ and qr is not among them. Further, we get that qr is not a 3-central element, since it is inverted by the involution z.

We have obtained the following result: there are precisely four G-classes of elements of order 3 passing through K with representatives d, qr, dqr, and $(dqr)^{-1}$; note that these elements are also representatives for the $\mathbb{N}(K)$-classes of $K^{\#}$. We have $\mathbb{C}(d) = K\langle z \rangle$, $\mathbb{C}(qr) \cap \mathbb{N}(K) = K$, and $\mathbb{C}(dqr) \cap \mathbb{N}(K) = \mathbb{C}((dqr)^{-1}) \cap \mathbb{N}(K) = D$.

We are now able to determine the structure of $\mathbb{C}(qr)$. Put $A = \mathbb{C}(qr)$, and $N = \mathbb{N}(\langle qr \rangle)$. Since z inverts qr, we have $N = A\langle z \rangle$. Obviously, the order of A is odd. We know that $\mathbb{N}(K) \cap A = K$. Note that K is a Sylow 3-subgroup of A. A transfer lemma of Burnside yields that A has a normal 3-complement, say X. Since $X = \mathbb{O}_{3'}(A)$, we see that X is a normal subgroup of N. Denote by R the subgroup $\langle qr, qr^{-1} \rangle X$ of A. Obviously, the involution z acts on R. Since $\mathbb{C}(z) \cap N = \langle z \rangle \langle d \rangle$ holds, it follows that the action of z on R is fixed-point-free. Therefore, the group R is abelian. The element q of order 3 is contained in R. This yields $R \subseteq \mathbb{C}(q) = K\langle t \rangle$. Consequently, we have $R \subseteq K$, since R is of odd order. By the definition of R we get $R = \langle qr, qr^{-1} \rangle$ and $X = \langle 1 \rangle$. Further, we have $\mathbb{N}(\langle qr \rangle) = K\langle z \rangle$.

Let H be the subgroup $K\langle z, t \rangle$ of G. The order of an element of H is always equal to 1,2,3, or 6. The above result yields that elements of order 3 or 6 in $H^{\#}$ are not inverted by an involution of $G \setminus H$; note that $K\langle z, t \rangle$ is normal in $\mathbb{N}(K)$. Furthermore, the involutions of H are the only elements in $H^{\#}$ which are centralized by an involution of $G \setminus H$. These facts are basic to the following computations.

Let x and y be two different involutions in H. Assume that there was an involution i in $G \setminus H$ which centralizes both x and y. Then, we have $i \in \mathbb{C}(xy)$. Therefore, the element xy of H is of order 2, and $\langle x, y \rangle$ is a four-group. Hence, it follows $i \in \mathbb{C}(\langle x, y \rangle) = \langle x, y \rangle \subseteq H$, contrary to $i \in G \setminus H$.

We are in a position which allows to apply the method of H. Bender [1] introduced in § 2 with respect to the subgroup H of G. We use the same notation as in 2.3.

From the above results we get $b_i = 0$ for $i \geq 3$. Each involution in H is centralized by a subgroup isomorphic to $\Sigma_3 \times Z_2$ in H. Furthermore, there are precisely 27 involutions lying in H. Thus, we compute $b_2 = 3 \cdot 27 = 81$. Hence

$$b_1 < \frac{4}{5}(|J \cap H| + b_2) - 1 - b_2 = \frac{4}{5}(27 + 3 \cdot 27) - 1 - 3 \cdot 27 = \frac{22}{5}.$$

This implies $b_1 \leq 4$. Obviously, the group H acts regulary on the set of all involutions x of $G \setminus H$ with $\mathbb{C}(x) \cap H = \langle 1 \rangle$. But the number of these involutions is b_1. Therefore, the order of H devides b_1, and so $b_1 = 0$. Now we are able to compute the order of G:

$$|G| = |C| \cdot (|J \cap H| + b_1 + 2 \cdot b_2) = 2^4 \cdot 3 \cdot (27 + 0 + 2 \cdot 3 \cdot 27) =$$
$$= 2^4 \cdot 3^4 \cdot 7.$$

Let S be a Sylow 7-subgroup of G. Then, S is not centralized by an element of order 2. The theorem of Sylow yields

$$|G : \mathbf{N}(S)| \in \{2^4 \cdot 3^4, 2^3\}.$$

Assume first that $|\mathbf{N}(S)| = 2 \cdot 3^4 \cdot 7$. Here, a Sylow 3-subgroup D_1 of G normalizes S. The theorem of Sylow forces $\mathbf{N}(D_1) \cap \mathbf{N}(S) \supset D_1$, contrary to the fact that $\mathbf{N}(D) = D$ holds. If $|N(S)| = 7$, then a transfer lemma of Burnside [5; 7.4.3] gives a contradiction to the simplicity of G.

We have arrived at a contradiction to the main assumption. It follows that K is not abelian.

Obviously, the center of K is equal to $\langle d \rangle$, since K lies in $\mathbf{C}(d)$. As K is nonabelian, it follows $K' = \mathbf{Z}(K) = \Phi(K)$. Further, K is of exponent 3 by 1.3.

Assume by way of contradiction that K was not a Sylow 3-subgroup of G. Then, K is a normal subgroup of a suitable subgroup K_1 of order 3^4 of G. Since $\langle d \rangle = \mathbf{Z}(K)$, it follows that K_1 lies in $\mathbf{C}(d)$, contrary to $K \in Syl_3(\mathbf{C}(d))$. The lemma is proved.

Lemma 3.2. *Let $\mathfrak{G} \neq \langle 1 \rangle$ be a group generated by elements $\mathfrak{d}$, $\mathfrak{r}$, $\mathfrak{q}$, $\mathfrak{z}$, $\mathfrak{t}$, $\mathfrak{u}$ and $\mathfrak{s}$ with the following relations:*

(i) $\mathfrak{d}^3 = \mathfrak{r}^3 = \mathfrak{q}^3 = 1,\ \mathfrak{r}^{-1}\mathfrak{d}\mathfrak{r} = \mathfrak{d},\ \mathfrak{q}^{-1}\mathfrak{r}\mathfrak{q} = \mathfrak{r}\mathfrak{d}^{-1},\ \mathfrak{q}^{-1}\mathfrak{d}\mathfrak{q} = \mathfrak{d}$,

(ii) $\mathfrak{z}^2 = 1,\ (\mathfrak{z}\mathfrak{r})^2 = 1,\ \mathfrak{z}\mathfrak{d}\mathfrak{z} = \mathfrak{d},\ (\mathfrak{z}\mathfrak{q})^2 = 1$,

(iii) $\mathfrak{t}^2 = 1,\ (\mathfrak{z}\mathfrak{t})^2 = 1,\ (\mathfrak{t}\mathfrak{r})^2 = 1,\ (\mathfrak{t}\mathfrak{d})^2 = 1,\ \mathfrak{t}\mathfrak{q}\mathfrak{t} = \mathfrak{q}$,

(iv) $\mathfrak{u}^2 = 1,\ \mathfrak{u}\mathfrak{z}\mathfrak{u} = \mathfrak{z}\mathfrak{t},\ \mathfrak{u}\mathfrak{r}\mathfrak{u} = \mathfrak{d}^{-1}$,

(v) $\mathfrak{s}^2 = 1,\ \mathfrak{s}\mathfrak{z}\mathfrak{s} = \mathfrak{t},\ \mathfrak{s}\mathfrak{q}\mathfrak{s} = \mathfrak{d}^{-1},\ (\mathfrak{u}\mathfrak{s})^3 = 1,\ (\mathfrak{u}\mathfrak{z}\mathfrak{q})^3 = 1$.

Then $\mathfrak{G}$ is isomorphic to $L_3(3)$.

Proof. Coset enumeration yields that the order of the abstract group $\mathfrak{G}^*$ defined by the above relations is equal to $2^4 \cdot 3^3 \cdot 13$ which is the order of $L_3(3)$.

Note that $L_3(3) \cong SL_3(3)$ holds. With the identification

$$\mathfrak{d} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \ \mathfrak{r} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathfrak{q} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ \mathfrak{z} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$\mathfrak{t} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ \mathfrak{u} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \text{ and } \mathfrak{s} \to \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

with entries from $GF(3)$, the above relations are obviously satisfied. Hence, we get $\mathfrak{G} \cong \mathfrak{G}^* \cong L_3(3)$. This completes the proof.

Lemma 3.3. *The group G contains a subgroup isomorphic to $L_3(3)$.*

Proof. Let R_1 be the elementary abelian subgroup $\langle \mathbf{C}_K(z), \mathbf{C}_K(t) \rangle$ of order 9. Denote by P_1 the normalizer of R_1 in G. Obviously $\mathbf{N}(\langle d \rangle) \subseteq P_1$ and $\mathbf{C}(R_1) = R_1$. Consider the group $\mathbf{N}(\mathbf{C}_K(t))$. Since $\langle d \rangle$ and $\mathbf{C}_K(t)$ are conjugate in G, there exists an extra-special subgroup L of order 27 such that $\mathbf{N}(\mathbf{C}_K(t)) = L\langle z,t \rangle$. Clearly, $L\langle z,t \rangle \neq K\langle z,t \rangle$, $R_1 = K \cap L$, and R_1 is normal in $\langle K, L \rangle$. Thus, P_1/R_1 is not 3-closed and contains a four-subgroup. This forces $P_1/R_1 \cong GL_2(3)$. Since zt acts invertingly on R_1, the subgroup $R_1\langle z,t \rangle$ lies normal in P_1, and the Frattini argument yields that P_1 splits over R_1.

Similarly, if $P_2 = \mathbf{N}(R_2)$ with $R_2 = \langle \mathbf{C}_K(z), \mathbf{C}_K(zt) \rangle$ then $\mathbf{C}(R_2) = R_2$ and P_2 is a splitting extension of R_2 by $GL_2(3)$. We remark that t acts fixed-point-freely on R_2.

In what follows we shall show that $\langle P_1, P_2 \rangle$ is isomorphic to $L_3(3)$. Since P_1 is a splitting extension of R_1 by a group isomorphic to $GL_2(3)$, and since $\mathbf{C}(R_1) = R_1$, we can choose elements q, r, and s in P_1 so that $R_1 = \langle d, q \rangle$, $P_1 = \langle d, q \rangle \langle r, z, t, s \rangle$, and the following relations are satisfied:

(I) $d^3 = r^3 = q^3 = 1$, $d^r = d^q = d$, $q^r = dq$,

(II) $z^2 = 1$, $(zr)^2 = (zq)^2 = 1$, $d^z = d$,

(III) $t^2 = 1$, $(zt)^2 = (tr)^2 = (td)^2 = 1$, $q^t = q$,

(IV) $s^2 = 1$, $z^s = t$, $q^s = d^{-1}$.

The relations in (I) imply $r^q = rd^{-1}$. Obviously, we have $\mathbf{N}(\langle d \rangle) = \langle d, q, r \rangle \langle z, t \rangle$, hence $K = \langle d, q, r \rangle$. The group R_2 is equal to $\langle d, r \rangle$. Let X be a complement of R_2 in P_2 with $\langle z, t \rangle \subseteq X$. Then X is isomorphic to $GL_2(3)$, and consequently X contains an involution u which acts on $\langle z, t \rangle$. Since t acts fixed-point-freely on R_2, the element u interchanges z and zt. Because z and zt are not conjugate in P_1, the involution u cannot lie in P_1. Note that $ztR_1 \in \mathbf{Z}(P_1/R_1)$. We have

$$(\mathbf{C}_{R_2}(z))^u = \mathbf{C}_{R_2}(z^u) = \mathbf{C}_{R_2}(zt),$$

hence $d^u \in \{r, r^{-1}\}$. Interchanging u and tu if necessary we may and shall assume that $d^u = r^{-1}$ holds. We remark that $P_2 = \langle d, q, r, z, t, u\rangle$, since $\langle d, q, r, z, t\rangle / R_2 \cong \Sigma_3 \times Z_2$ is a maximal subgroup of $P_2/R_2 \cong GL_2(3)$.

Now we are able to describe the action of u, z, and q on R_2. Compute

$$u : d \to r^{-1} \to d,$$

$$z : d \to d, r \to r^{-1},$$

$$q : d \to d, r \to d^{-1}r \to dr \to r.$$

One easily sees that $(uzq)^3 \in \mathbb{C}(R_2) = R_2$. Hence, the order of uzq is equal to 3. This follows from the fact that K is a Sylow 3-subgroup of G which does not contain an element of order 9.

Both u and s lie in $\mathbb{N}(\langle z, t\rangle)$ which is isomorphic to Σ_4. Their action on $\langle z, t\rangle$ is described by

$$u : t \to t, z \to zt,$$

$$s : t \to z, zt \to zt.$$

Thus $(us)^3 \in \mathbb{C}(\langle z, t\rangle)$, and the order of us is equal to 3 or 6. But a group isomorphic to Σ_4 does not contain an element of order 6. So, we have obtained $o(us) = 3$.

Summarizing the above results we get

(V) $u^2 = 1$, $z^u = zt$, $r^u = d^{-1}$, $(uzq)^3 = 1$, $(us)^3 = 1$.

Lemma 3.3 yields - together with (I) to (IV) - that $\langle P_1, P_2\rangle = \langle d, q, r, z, t, s, u\rangle$ is isomorphic to $L_3(3)$. The lemma is proved.

Theorem 3.4. *The group G is isomorphic to $L_3(3)$.*

Proof. By the preceding lemma, the group G contains a subgroup M isomorphic to $L_3(3)$. We have $|\mathbb{C}_M(i)| = 48$ for any involution i in M. Hence $\mathbb{C}_M(i) = \mathbb{C}_G(i)$ for all these involutions.

Assume by way of contradiction that M was a proper subgroup of G. Clearly, $|M|$ is even, $|\bigcap_{g\in G} M^g|$ is odd, and $\mathbb{N}(T^*) = T^*$ holds for a Sylow 2-subgroup T^* of M. By a lemma of Thompson [17; 5.35], there exists a subgroup M_0 of M so that the order of M_0 is odd and $M = M_0\,\mathbb{C}(j)$ holds for a suitable involution j of M. Hence, the order of M_0 is divisible by $3^2 \cdot 13$. But $L_3(3)$ does not contain such a subgroup. This contradiction completes the proof.

4. THE CASE $|K| = 3$

Finally, we have to handle the case $|K| = 3$. Here, K is a Sylow 3-subgroup of G, and all elements of order 3 are conjugate in G. By 1.4 the normalizer of $\langle z, t\rangle$ in G is isomorphic to Σ_4. Denote by r an element of order 3 in $\mathbf{N}(\langle z,t\rangle)$ with $z^r = t$. Further, let g be an involution in $\mathbf{N}(\langle z,t\rangle)$ which inverts r and centralizes z. Since all elements of order 3 are conjugate in G, there exists an element l of order 2 in $\mathbb{C}(\langle r\rangle)$. Then, l centralizes g. Note that $\mathbf{N}(\langle d\rangle)$ is isomorphic to $\Sigma_3 \times Z_2$. Denote by U the subgroup of G generated by z, t, r, g and l.

Lemma 4.1. *The subgroup U of G is isomorphic to Σ_5.*

Proof. We have $z, l \in \mathbb{C}(g)$. Since all involutions in G are conjugate, it follows $o(zl) \in \{1,2,3,4,6\}$. Obviously, the order of zl is not equal to 1. Put $S = \langle z,t\rangle\langle r,g\rangle$. Then, $S \cong \Sigma_4$.

Assume by way of contradiction that $(zl)^2 = 1$ held. Then, we see that l lies in $\mathbb{C}(\langle z, g\rangle)$. But $\langle z,g\rangle$ is selfcentralizing. Therefore, l is contained in $\langle z,t,r,g\rangle = S$. But no involution centralizes an element of order 3 in $S \cong \Sigma_4$, contrary to $l \in S \cap \mathbb{C}(r)$.

Now, we assume $o(zl) = 3$. Since r acts fixed-point-freely on $\langle z,t\rangle$ and centralizes l, we compute $3 = o(zl) = o(tl) = o(ztl)$. Hence, the relations $z^2 = l^2 = t^2 = (zl)^3 = (tl)^3 = 1$ force $\langle z,t,l\rangle \cong \Sigma_4$. By the structure of Σ_4 it follows $o(ztl) = 4$, contrary to $(ztl)^3 = 1$.

Next we assume $o(zl) = 4$. We know that g lies in $\mathbb{C}(zl)$. Therefore, we have $g = (zl)^2$. Compute

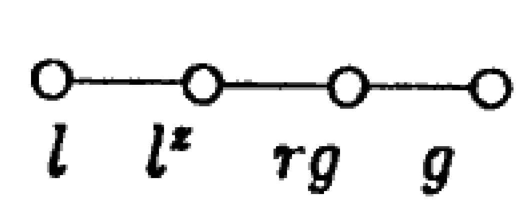
$$S^l = \langle z,t,r,g\rangle^l = \langle lzl, ltl, r, g\rangle = \langle lzl, (lzl)^r, r, g\rangle =$$
$$= \langle zg, (zg)^r, r, g\rangle = S.$$

Thus, the element l normalizes S; hence it normalizes the characteristic subgroup $\langle z,t\rangle$ of S as well. But S is the normalizer of $\langle z,t\rangle$ in G. Hence, l lies in S, which, however, is not possible.

We have shown that the order of zl must be equal to 6. Since g lies in $\mathbb{C}(\langle zl\rangle)$, it follows $g = (zl)^3$. Taking into account the structure of $\mathbb{C}(g)$, we compute

$$(l^z(rg))^3 = zlzr(gzlzrg)zlzrg = zlz(rzlzr^{-1})zlzrg = zlzztlztzlzrg =$$
$$= zltltlzrg = zt(tl)^3zrg = zt((zl)^3)^r zrg = ztr^{-1}(grzrg) =$$
$$= ztr^{-1}r^{-1}zr^{-1} = zt(rzr^{-1}) = ztzt = 1.$$

Here, we have used the equations $z^g = z^t = z$, $z^r = t$, $t^r = zt$, $(zt)^r = z$, $l^g = l$, $r^g = r^{-1}$, $z^2 = r^3 = 1$, and $(zl)^3 = g$. We have obtained the following diagram:



Hence, the subgroup $W = \langle l, l^z, rg, g \rangle$ is isomorphic to Σ_5 as $S \subseteq W$. Obviously, W is a subgroup of U, and $\langle l, r, g \rangle$ lies in W. Further, we have

$$l^z l l^z g = zlzlzlzg = (zl)^3 zg = gzg = z,$$

consequently $z, z^r = t \in W$. This forces $U = W$, and the assertion follows.

Lemma 4.2. *Let w be an element of order 5 in U. Then the normalizer of $\langle w \rangle$ in G is an extension of $\mathbf{C}(w)$ by a cyclic group of order 4. For any $u \in \mathbf{C}(w)^{\#}$, we have $\mathbf{C}(u) = \mathbf{C}(w)$. The order of $\mathbf{C}(w)$ is equal to 25 and the order of G is equal to $2^4 \cdot 3 \cdot 5^2 \cdot 13$.*

Proof. The normalizer of $\langle w \rangle$ in U is equal to $\langle w \rangle \langle y \rangle$ with a suitable element y of order 4 in U, and $\mathbf{C}_U(w) = \langle w \rangle$. Denote by F the centralizer of w in G. Since $|F|$ is not divisible by 2 or 3, we see that the involution y^2 acts fixed-point-freely on F. Hence, F is abelian and y^2 acts invertingly on F. It follows $\mathbf{C}(x) = F$ for all elements x in $F^{\#}$. Put $\varphi = |F|$.

As in § 2, we apply the method of H. Bender and put $H = \mathbf{N}(\langle w \rangle) = F\langle y \rangle$. We use the same notation as in 2.3. The fact that $\mathbf{C}(x) = F$ holds for any element x in $F^{\#}$ yields that an involution which inverts an element of $F^{\#}$ lies in H, and so, only elements of order 2 and 4 in $H^{\#}$ are inverted by involutions of $G \setminus H$. From the structure of the centralizer of an involution in G we get that the only elements of $H^{\#}$ centralized by involutions of $G \setminus H$ are involutions. Compute - arguing similarly as in the proof of 2.3

$$b_4 = \varphi,\ b_3 = 0,\ b_2 = 4 \cdot \varphi, \text{ and } b_j = 0 \text{ for } j > 4.$$

Assume $\varphi \geq 25$. Then, we have $f = (4 \cdot \varphi)/48 - 1 = (\varphi - 12)/12$. Hence,

$$b_1 < \frac{12}{\varphi - 12}(\varphi + 4 \cdot \varphi + 3 \cdot \varphi) - 1 - 4 \cdot \varphi - \varphi = \left(\frac{12}{\varphi - 12} \cdot 8 - 5\right) \cdot \varphi - 1.$$

If $\varphi > 25$, then clearly $\varphi \geq 35$, and

$$b_1 < \left(\frac{12}{23} \cdot 8 - 5\right) \cdot \varphi - 1 < 0,$$

contrary to $b_1 \geq 0$. The case $\varphi > 25$ does not occur. Hence, in this case, $\varphi = 25$. Compute

$$b_1 < 762/13 < 59.$$

Since b_1 is the number of all involutions i in $G \setminus H$ with $\mathbf{C}(i) \cap H = \langle 1 \rangle$, it follows $b_1 = 0$. Hence,

$$|G| = |C| \cdot (|J \cap H| + b_1 + 2 \cdot b_2 + 4 \cdot b_4) =$$
$$= 48 \cdot (25 + 0 + 2 \cdot 4 \cdot 25 + 4 \cdot 25) = 15{,}600.$$

If we are able to rule out the case $\varphi = 5$, then the assertion will follow.

Assume by way of contradiction that $\varphi = 5$ held. Consider the subgroup U of G. Note that $U \cong \Sigma_5$. The fact that $\varphi = 5$ and that all elements of order 5 in U are conjugate in U yields that involutions are the only elements of $U^\#$ which are centralized or inverted by an involution of $G \setminus U$.

Let x and x^* be two different involutions in U. Assume that i is an involution in $G \setminus U$ which centralizes both x and x^*. Then, $i \in C(xx^*)$. The above result yields $\langle x, x^* \rangle \cong E_4$. Hence, $i \in C(x, x^*) = \langle x, x^* \rangle$, contrary to $i \in G \setminus U$.

We are able to apply the method of Bender again using the same notation as before. Here, we consider the subgroup U in the role of H. Obviously $b_i = 0$ for $i \geq 3$. Note that U contains precisely two classes of involutions with centralizers in U isomorphic to D_8 or $Z_2 \times \Sigma_3$. Hence, we have

$$b_2 = 15 \cdot 4 + 10 \cdot 3 = 90.$$

Further, we compute $f = |U|/48 - 1 = \frac{3}{2}$. This yields

$$b_1 < \frac{2}{3}(15 + 10 + 90) - 1 - 90 < 0,$$

contrary to $b_1 \geq 0$. We have shown that $\varphi = 5$ does not occur. The lemma is proved.

Lemma 4.3. *A Sylow 13-normalizer of G is a Frobenius-group of order $2^2 \cdot 13$.*

Proof. Obviously, a Sylow 13-subgroup is selfcentralizing. Hence, the assertion follows by the theorem of Sylow.

Lemma 4.4. *Let F be a Sylow 5-subgroup of G. Then, the normalizer of F in G is a splitting extension of the elementary abelian subgroup $\mathbb{C}(F) = F$ of order 25 by $SL_2(3)$. All elements of order 5 of G are conjugate. The group G contains only elements of order 1, 2, 3, 4, 5, 6, 8, and 13.*

Proof. The fact that $\mathbb{C}(F) = F$ and $|F| = 25$ is a direct consequence of 4.2. Note that F is either cyclic or elementary abelian, and that $2^2 \cdot 5^2$ divides $|\mathbb{N}(F)|$, but 13 does not. The theorem of Sylow yields $|\mathbb{N}(F)| \in \{2^3 \cdot 3 \cdot 5^2, 2^2 \cdot 5^2\}$.

Consider the case $|\mathbb{N}(F)| = 2^2 \cdot 5^2$. Then, G has precisely six classes of nontrivial 5-elements. This forces that G contains precisely 18,720 elements, contrary to 4.2.

We have shown that $\mathbb{N}(F)$ is of order $2^3 \cdot 3 \cdot 5^2$. Clearly, F is elementary abelian, since $\mathbb{C}(x) = F$ holds for any $x \in F^\#$. Moreover, all elements of order 5 are conjugate in G. Let i be an involution in $\mathbb{N}(F)$. Then, i acts invertingly on F, and $F\langle i \rangle$ is a normal subgroup of $\mathbb{N}(F)$. The Frattini argument yields $\mathbb{N}(F) = F\langle i \rangle(\mathbb{C}(i) \cap \mathbb{N}(F))$, and consequently $\mathbb{N}(F) = \mathbb{C}_{\mathbb{N}(F)}(i)F$. Clearly $\mathbb{C}(i) \cap F = \langle 1 \rangle$, and $\mathbb{C}_{\mathbb{N}(F)}(i)$ is a subgroup of index 2 in $\mathbb{C}(i)$. Hence, we have $\mathbb{C}_{\mathbb{N}(F)}(i) \cong SL_2(3)$.

The last part of the assertion is an easy consequence of § 1 and the preceding lemmas.

Theorem 4.5. *The group G does not exist.*

Proof. Let F be a Sylow 5-subgroup of G. Denote by B the normalizer of F in G. Then, B is a splitting extension of F by a subgroup isomorphic to $SL_2(3)$. Up to isomorphism, such an extension is uniquely determined. We may regard G as a permutation group on the set $\Omega = \{0, 1, 2, \ldots, 25\}$. The elements of Ω stand for the 26 conjugates of B in G. We identify $0 \in \Omega$ with B. Note that B is a maximal subgroup of G with $|G : B| = 26$. An element of order 5 in G is contained in precisely one conjugate of B. Hence, F acts on the set $\Omega \setminus \{0\}$ transitively. Thus, we may and shall assume that $F = \langle v, w \rangle$ with

$$v = (1,2,3,4,5)(6,7,8,9,10)(11,12,13,14,15)(16,17,18,19,20)$$
$$(21,22,23,24,25)$$

and

$$w = (1,6,11,16,21)(2,7,12,17,22)(3,8,13,18,23)(4,9,14,19,24)$$
$$(5,10,15,20,25).$$

Denote by Σ the symmetric group on Ω. Since F acts on $\Omega \setminus \{0\}$ transitively, we have $\mathbb{C}_\Sigma(F) = F$ by [12; II.3.1]. Furthermore, we see that $\mathbb{N}_\Sigma(F)$ is a splitting extension of F by $GL_2(5)$ as follows. In Σ we compute that $\mathbb{N}_\Sigma(F) \cap \mathbb{C}_\Sigma(v)$ is a group of order $2^2 \cdot 5^3$. Note that 5^6 divides the order of $\mathbb{C}_\Sigma(v)$, and that

$$u = (6,11,21,16)(7,12,22,17)(8,13,23,18)(9,14,24,19)$$
$$(10,15,25,20).$$

lies in $\mathbb{N}_\Sigma(F) \cap \mathbb{C}_\Sigma(v)$. An analogous conclusion is admissible in $\mathbb{C}_\Sigma(w)$. It follows that all elements in $F^\#$ are conjugate under $\mathbb{N}_\Sigma(F)$. The fact that $\mathbb{N}_\Sigma(F) \cap \mathbb{C}_\Sigma(v)$ is a group of order $2^2 \cdot 5^3$ forces that $\mathbb{N}_\Sigma(F)$ is an extension of $F = \mathbb{C}_\Sigma(F)$ by $GL_2(5)$ which obviously splits over F. Since $GL_2(5)$ contains precisely one class of subgroups isomorphic to $SL_2(3)$, we may and shall assume that $B = \langle v, w \rangle \langle h, k, c \rangle$ with

$$h = (2,4,5,3)(6,11,21,16)(7,14,25,18)(8,12,24,20)(9,15,23,17)$$
$$(10,13,22,19),$$

$$k = (2,11,5,16)(3,21,4,6)(7,13,25,19)(8,23,24,9)(10,18,22,14)$$
$$(12,15,20,17),$$

and

$$c = (2,9,22)(3,12,18)(4,20,14)(5,23,10)(6,8,19)(7,11,15)(13,21,24)(16,17,25).$$

Note that $\langle h, k\rangle$ is a quaternion group and that $S = \langle h, k, c\rangle$ is isomorphic to $SL_2(3)$. Put $N = \mathbf{N}_G(S)$. Then, N is isomorphic to $GL_2(3)$. This forces the existence of an involution i in N so that the relations

$$i^2 = 1,\ h^i = h^{-1},\ k^i = hk^{-1},\ c^i = c^{-1}$$

are satisfied. Since $G = \langle B, i\rangle$, the element i interchanges the symbols 0 and 1 of Ω. Note that both 0 and 1 are fixed under S, and that 0 is stabilized by B. Since S acts on the set $\{2,3,\ldots,25\}$ transitively, it follows by the proof of [12; II.3.1] that i is uniquely determined by the image of $2 \in \Omega$ under i which then has to lie in $\{2,3,\ldots,25\}$. Only twelve of the 24 possibilities for the choice of i respect the above relations. Since we may replace i with ih^2, we have only the following 6 possibilities i_j for i:

$$i_1 = (0,1)(3,4)(6,16)(7,24)(8,25)(9,22)\cdot (10,23)(11,21)(12,14)(13,15)(17,19)(18,20),$$

$$i_2 = (0,1)(2,3)(4,5)(7,17)(8,19)(9,18)\cdot (10,20)(11,16)(12,22)(13,24)(14,23)(15,25),$$

$$i_3 = (0,1)(2,7)(3,14)(4,18)(5,25)(6,19)\cdot (9,15)(10,16)(11,22)(12,20)(13,21)(17,23),$$

$$i_4 = (0,1)(2,8)(3,12)(4,20)(5,24)(6,9)\cdot (7,25)(10,13)(11,17)(15,16)(19,22)(21,23),$$

$$i_5 = (0,1)(2,9)(3,15)(4,17)(5,23)(6,24)\cdot (7,18)(8,21)(11,12)(13,19)(14,25)(16,20), \text{ and}$$

$$i_6 = (0,1)(2,10)(3,13)(4,19)(5,22)(6,14)\cdot (7,11)(8,20)(9,23)(12,24)(16,25)(18,21).$$

Compute $o(vi_1) = 10$, $o(vi_2) = 16$, $o(vi_3) = 60$, $o(vi_4) = 133$, $o(vi_5) = 12$, and $o(vi_6) = 105$. This contradicts 4.4.

The theorem is proved.

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Dieter Held, Jörg Hrabě de Angelis
Fachbereich Mathematik (17)
Universität Mainz
Germany