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SMOOTH POINTS OF THE POSITIVE PART OF THE UNIT BALL OF $C(K, E)$

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *Let K be a compact Hausdorff space and let E be a Banach lattice. By $C(K,E)$ we denote the Banach lattice of E-valued continuous functions on K. The aim of the paper (*) is to characterize explicitly the smooth points and quasi interior points of the positive part of the unit ball of $C(K,E)$. Then we translate this result to the space of compact operators $\underline{K}(E, C(K))$. We describe also extreme points of the positive part of $C(K,E)$. The description of smooth points as well as extreme and exposed points gives us additional knowledge about the geometry of an important class of convex sets in functional analysis. In this paper we would like to use this tool for the study of the geometry of the positive part of the unit ball of various well-known Banach spaces. In Section 1 we collect general facts about supporting points, smooth points and quasi interior points of the positive part of the unit ball of an arbitrary Banach lattice. Singer [6] has described the extreme points of the unit ball of the space dual to the space $C(K,E)$ of continuous functions from a compact set K into a Banach space E. In Section 2 we adapt his proof to get an analogous characterization of the extreme points of the positive part of the unit ball of the $C(K,E)$ (*), where E is a Banach lattice. Sundaresan [8] has described smooth points of the unit ball of $C(K,E)$. In Section 3 we continue this for the positive part of the unit ball of $C(K,E)$. Moreover we give a characterization of quasi interior points in $B_+(C(K,E))$. Then, in Section 4 we apply these results to the space of compact operators $\underline{K}(E, C(K))$.*

1. BASIS FACTS

Let E be a Banach space and let E^* be its dual. By $B(E)$ we denote the unit ball of E. If moreover E is an ordered space then we denote the positive cone by E_+, and the positive part of $B(E)$ we denote by $B_+(E)$. For a closed convex set Q we put

$$Q^* = \{\xi : E \to \mathbb{R} : \xi \in E^*,\ inf\, \xi(Q) < \sup \xi(Q)\}.$$

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If E is a real Banach lattice then $(B_+(E))^* = E^* \setminus \{0\}$, because $\xi(B_+(E)) = \{0\}$ implies $\xi(B(E)) = \{0\}$ (more generally, this is true for every ordered Banach space with a total positive cone).

Definition. *We say that $q_0 \in Q$ is a smooth point of a convex set $Q \subset E$ if there exists a unique (up to multiplicative constant) $\xi_0 \in Q^*$ such that $\xi_0(q_0) = \sup \xi_0(Q)$. The set of all smooth points of Q we denote by smooth Q.*

If Q is the unit ball of E then smoothness of a point $q \in Q$ means that the norm is Gateaux differentiable (weakly differentiable) at q. We need consider smoothness of support points of Q only. Recall that a point $q_0 \in Q$ is a support point of Q if there exists $\xi \in Q^*$ such that $\xi(q_0) = \sup \xi(Q)$ and we say that ξ supports Q at q_0. By supp Q we denote the set of all support points of Q. Obviously, in general not all elements of Q belong to supp Q. An element which is not a support point is called a quasi-interior point of Q. The set of all quasi-interior points of Q we denote by q-int Q (i.e. q-int $Q = Q \setminus$ supp Q). Note that if $Q = \{x_0\}$, then $x_0 \in q$-int Q. We have smooth $Q \subset$ supp $Q = Q \setminus q$-int Q. Obviously $x \in q$-int $B(E)$ if and only if $\| x \| < 1$ (i.e. q-int $B(E) = int B(E)$), and $x \in q$-int $B_+(E)$ implies $\| x \| < 1$.

We shall henceforth suppose that E is a real Banach lattice.

Remark 1. Let $x \in$ supp $B_+(E)$ with $\| x \| < 1$. If η supports $B_+(E)$ at x then $\eta(x) = 0$. Moreover, $\lambda x \in$ supp E_+ for all $\lambda > 0$. If η supports E_+ at x then $\eta(x) = 0$.

Indeed, we need consider only the case $\| x \| \in (0,1)$. Then $\eta(x/\| x \|) \le \sup \eta(B_+(E)) = \eta(x)$, so $\eta(x) \le 0$. Moreover $0 = \eta(0) \le \sup \eta(B_+(E)) = \eta(x)$. Hence $\eta(x) = 0$. For $x \in E_+$ we use the same arguments.

Remark 2. Let $\| x \| = 1$. If $x \in$ smooth $B_+(E)$, then $\alpha x \in q$-int $B_+(E)$ for $\alpha \in (0,1)$.

Indeed, to get a contradiction suppose that $x \in$ smooth $B_+(E)$ and $\alpha x \in$ supp $B_+(E)$ for some $\alpha \in (0,1)$. Then there exists η_1 supporting $B_+(E)$ at αx. By Remark 1 we have $\eta_1(x) = 0$. On the other hand, by the Hahn-Banach Theorem, there exists η_2 such that $\sup \eta_2(B(E)) = \sup \eta_2(B_+(E)) = \eta_2(x) = \| \eta_2 \| = 1$. Obviously $\eta_1 \neq \eta_2$, so $x \notin$ smooth $B_+(E)$.

Remark 3. If η supports $B_+(E)$ at $x \in$ smooth $B_+(E)$ with $\| x \| = 1$ then $\eta(x) > 0$.

Indeed, smoothness of x combined with the Hahn-Banach Theorem guarantees that $\eta(x) = \| \eta \| > 0$.

Remark 4. If $x \in q$-int $B_+(E)$ then $\alpha x \in q$-int $B_+(E)$ for $\alpha \in (0, 1/\| x \|)$.

Indeed, to get a contradiction suppose that $B_+(E)$ is supported by η at αx. Because of $\| \alpha x \| < 1$, by Remark 1 we have $\eta(\alpha x) = 0 \ge \sup \eta(B_+(E))$, so then $\eta(x) = 0 \ge \sup \eta(B_+(E))$, i.e. $x \in$ supp $B_+(E)$, contradicting $x \in q$-int $B_+(E)$.

Remark 5. If ξ supports $B_+(E)$ at x and $\xi_+ \neq 0$, then ξ_+ supports $B_+(E)$ at x, too. Hence if ξ supports $B_+(E)$ at $x \in$ smooth $B_+(E)$, then $\xi = \xi_+ \geq 0$.

Indeed, we have $\xi_+(x) = \sup\{\xi(y) : 0 \leq y \leq x\} \leq \xi(x)$; hence $\xi(x) = \xi_+(x)$, which means that ξ_+ supports $B_+(E)$ at x. Smoothness of x implies uniqueness of the supporting functional, i.e. $\xi = \xi_+$.

Remark 6. Let $x \in$ supp $B_+(E)$ be such that $\alpha x \in q$-int $B_+(E)$ for some $\alpha \in (0, 1)$. If η supports $B_+(E)$ at x then $\eta = \eta_+$.

Indeed, suppose that $\eta_- \neq 0$. Then $\alpha x \in q$-int $B_+(E)$ implies that $\eta_-(\alpha x) > 0$ so $\eta_-(x) > 0$. On the other hand $\eta_+(x) = \sup\{\eta(y) : 0 \leq y \leq x\} = \eta(x)$ ($\eta_+(x) \leq \eta(x)$ by supporting property and $\eta_+(x) \geq \eta(x)$ by definition of η_+), so $\eta_-(x) = 0$, and we get a contradiction.

Remark 7. Let $x \in$ smooth $B(E)$ and let $\alpha x \in q$-int $B_+(E)$ for some $\alpha \in (,0, 1)$. Then $x \in$ smooth $B_+(E)$.

Indeed, let $x \in$ smooth $B(E)$ and let $\alpha x \in q$-int $B_+(E)$ for some $\alpha \in (0, 1)$. To get a contradiction suppose that $x \notin$ smooth $B_+(E)$. Then there exist two different functionals η and ξ with equal norms which support $B_+(E)$ at x. Because $x \in$ smooth $B(E)$ we have $\| x \| = 1$ and $x \in$ supp $B_+(E)$.

By Remark 6 $\eta = \eta_+$ and $\xi = \xi_+$. We have $\eta_+(x) = \| \eta_+ \| = \sup \eta_+(B(E))$ and $\xi_+(x) = \| \xi_+ \| = \sup \xi_+(B(E))$, hence η_+ and ξ_+ support $B(E)$ and $x \in$ smooth $B(E)$, a contradiction.

Remark 8. $x \in$ smooth $B_+(E)$ with $\| x \| = 1$ implies $x \in$ smooth $B(E)$.

Indeed, if x is not a smooth point of $B(E)$ then there exist two different functionals supporting $B(E)$ at x which also support $B_+(E)$ at x.

Remark 9. If Q is convex then q-int Q is convex, too.

Indeed, to get a contradiction, suppose that there exists $z = \lambda x + (1 - \lambda) y \in$ supp Q for some $x, y \in q$-int Q and $\lambda \in (0, 1)$.

Then there exists a non trivial functional $\eta \in Q^*$ supporting Q at z, i.e. $\eta(z) = \sup \eta(Q)$. We have $\eta(z) = \lambda\eta(x) + (1-\lambda)\eta(y) \leq \lambda \sup \eta(Q) + (1-\lambda) \sup \eta(Q) = \eta(z)$, so $\eta(x) = \eta(y) = \eta(z) = \sup \eta(Q)$. Hence $x, y \in$ supp Q and we get a contradiction.

Remark 10. If $x, y \in q$-int $B_+(E)$ then $\lambda x \in q$-int E_+ for all $\lambda > 0$, and conversely if $x \in q$-int E_+ then $\lambda x \in q$-int $B_+(E)$ for all $\lambda \in (0, 1/ \| x \|)$. Therefore we have

$$int\, B(E) \cap q\text{-int}\, E_+ = q\text{-int}\, B_+(E) \text{ and}$$

$$int\, B(E) \cap \text{ supp } B_+(E) = int\, B(E) \cap \text{ supp } E_+$$

Remark 11. Let $\| x \| < 1$. Then $x \in$ smooth $B_+(E) \cap$ $int B(E)$ if and only if $\lambda x \in$ smooth E_+ for all $\lambda > 0$ if and only if $\lambda x \in$ smooth E_+ for some $\lambda > 0$.

Indeed, let $\| x \| < 1$. If $x \notin$ smooth $B_+(E)$ then there exist two different functionals η_i $(i = 1,2)$ supporting $B_+(E)$ at x. By Remark 1 $\eta_i(x) = 0$, so η_i supports E_+ at λx for all $\lambda > 0$, too. Hence $\lambda x \notin$ smooth E_+.

Now if $\lambda x \notin$ smooth E_+ for some $\lambda > 0$ and $\| x \| < 1$ then there exist two different functionals η_i $(i = 1,2)$ supporting E_+ at λx. By Remark 1 $\eta_i(x) = 0$ i.e. η_i supports $B_+(E)$ at x, too. Hence $x \notin$ smooth $B_+(E)$.

Remark 12. Let $x \neq 0$. If $x \in$ smooth $B_+(E) \cap int B(E)$ (or if $x \in$ smooth E_+) then $x / \| x \| \notin$ smooth $B_+(E)$.

Indeed, let η support $B_+(E)$ (or E_+) at x. Then by Remark 1 $\eta(x) = 0$ and η supports $B_+(E)$ at $x / \| x \|$. The existence of a second functional is guaranteed by the Hahn-Banach Theorem applied to $B(E)$ at $x / \| x \|$. Hence $x / \| x \| \notin$ smooth $B_+(E)$.

Remark 13. Let $x \in$ smooth $B_+(E)$ with $\| x \| = 1$. If η with $\| \eta \| = 1$ supports $B_+(E)$ at x then $\eta \in ext\ B(E^*)$. Moreover $\eta \geq 0$ and $\eta \in ext\ B_+(E^*)$.

Indeed, by the Hahn-Banach Theorem and by smoothness of x, we must to have $\eta(x) = 1$. If η were not extreme then there would be two distinct η_1, η_2 such that $\eta = (\eta_1 + \eta_2)/2$. Because $\eta_i(x) \leq 1$ $(i = 1,2)$ we would get $\eta_i = \| \eta \| = 1$ i.e. η_i would be two distinct functionals supporting $B_+(E)$ at x which is impossible. By Remark 5, we have $\eta \geq 0$. Hence $\eta \in ext\ B_+(E^*)$.

Remark 14. Let $x \in$ smooth $B_+(E)$ with $\| x \| < 1$ (or $x \in$ smooth E_+). If η supports $B_+(E)$ (E_+) at x then $-\eta$ belongs to an extreme ray of E^*.

Indeed, let η support $B_+(E)$ at x with $\| x \| < 1$. By Remark 1 $\eta(x) = 0$, so $-\eta \geq 0$. Suppose that $-\eta = \alpha_1 \eta_1 + \alpha_2 \eta_2$ where $\alpha_i > 0$ and $\eta_i \in E^*_+$. Then $\eta_i(x) = 0$, so $-\eta_i$ supports $B_+(E)$ at x. Because $x \in$ smooth $B_+(E)$, $\eta_i = \lambda_i \eta$ for some constants λ_i. Hence $-\eta$ is an element of an extreme ray.

Note that $x \in q$-int E_+ if and only if the ideal generated by x (the principal ideal E_x) is dense in E, or equivalently, if the order interval $[0, x] = \{y : 0 \leq y \leq x\}$ is total in E (see [5, II.6. Corollary 1].

2. THE DUAL SPACE OF $C(K, E)$

Let K be a compact Hausdorff space, and let E be a Banach lattice. By $C(K, E)$ we denote the Banach space of all continuous functions from K into E equipped with the supremum norm. For $f \in C(K, E)$ we have $f \geq 0$ if $f(k) \geq 0$ for all $k \in K$. The dual space $C(K, E)^*$ can be represented as the space of all set functions γ defined on the Borel sets $B \subset K$ with values in E^* countably additive, of bounded variation and regular, endowed with the usual vector operations and with the norm

$$\| \gamma \| = \operatorname*{Var}_K \gamma = \sup \sum_i \| \gamma(A_i) \|$$

where the supremum is taken over all finite Borel partitions (A_i) of K (see e.g. [2] or [7]). The equivalence between these spaces is given by the relation

$$\xi(f) = \int_K \langle f(k), \mathrm{d}\gamma(k)\rangle, \qquad f \in C(K,E).$$

Note that Variation of γ is finitely additive with respect to disjoint sets (i.e. $\underset{M\cup L}{Var}\,\gamma = \underset{M}{Var}\,\gamma + \underset{L}{Var}\,\gamma$ for M, L disjoint).

Lemma. *Let to $\xi \in C(K,E)^*$ correspond the set function γ. Then $\xi \geq 0$ if and only if $\gamma(B) \in E^*_+$ for all Borel sets in K.*

Proof. Obviously if $\gamma(B) \geq 0$ for all Borel subsets of K then for every positive simple function f we have $\int_K \langle f(k), \mathrm{d}\gamma(k)\rangle \geq 0$, so also for every positive $f \in C(K,E)$ as a limit of positive simple functions. Hence $\xi \geq 0$.

Now suppose that $\xi \geq 0$. Fix a Borel set B and $0 \neq x \in B_+(E)$. Because the subspace $C_x = \{h \otimes x : h \in C(K)\}$ of $C(K,E)$ is equivalent to $C(K)$, there exists a Radon measure μ_x on K such that $\xi(h \otimes x) = \int_K h(k)\,\mathrm{d}\mu_x(k)$. The measure μ_x is positive since $\xi \geq 0$. We have $[\gamma(B)](x) = \mu_x(B)$. Hence $\gamma(B) \geq 0$.

Singer [6, see also 7, II.1.4. Lemma 1.7, p. 197] proved that $\xi \in ext\, B(C(K,E)^*)$ if and only if ξ is of the form $\eta \otimes \delta_{k_0}$ (i.e. $\xi(f) = \eta(f(k_0))$), where $k_0 \in K$ and $\eta \in ext\, B(E^*)$. We adapt his proof to get an analogous result for the positive part of the unit ball of $C(K,E)^*$.

Proposition 1. *Let $\xi \neq 0$. Then $\xi \in ext\, B_+(C(K,E)^*)$ if and only if there exist $\eta \in ext\, B_+(E) \setminus \{0\}$ and $k_0 \in K$ such that $\xi = \eta \otimes \delta_{k_0}$.*

Proof. Assume that $\xi \neq 0$ with $\| \xi \| = 1$ is not of the form $\eta \otimes \delta_{k_0}$, where $k_0 \in K$ and $\eta \in E^*$. For ξ there exists a measure γ defined on the Borel sets of K such that

$$\xi(f) = \int_K \langle f(k), \mathrm{d}\gamma(k)\rangle, \qquad f \in C(K,E) \text{ with } \underset{K}{Var}\,\gamma = 1.$$

Then from the proof of Lemma 1.7 in [7, II.1.4 p. 197] it follows that there exist two disjoint open neighborhoods U_i of $k_i (i = 1,2)$ such that $V_i \in (0,1)$, where $V_i = \underset{U_i}{Var}\,\xi_i$. We define $x \in C(K,E)^*$ by

$$x(f) = V_1 \int_{U_2} \langle f(k), \mathrm{d}\gamma(k)\rangle - V_2 \int_{U_2} \langle f(k), \mathrm{d}\gamma(k)\rangle.$$

We have $\| x \| = 2V_1V_2 > 0$ and $\| \xi \mp x \| = 1$. Because $\xi \geq 0$, by the lemma above, we have $\xi_W \geq 0$ (W open or closed subset of K), where ξ_W is defined by $\xi_W(f) = \int_W \langle f(k), \mathrm{d}\gamma(k) \rangle$, $f \in C(K, E)$ and ξ corresponds to γ. Therefore

$$\xi + x = (1 - V_2)\xi_{U_2} + (1 + V_1)\xi_{U_2} + \xi_{K \setminus (U_1 \cup U_2)} \geq 0$$

Analogously we get $\xi - x \geq 0$. Hence $\xi \notin ext\ B_+(C(K,E)^*)$. Therefore we have proved that every $\xi \in ext\ B_+(C(K,E)^*)$ is of the form $\eta \otimes \delta_{k_0}$ where $\eta \in B_+(E^*)$. Because $(\eta_1 \otimes \delta_{k_0} + \eta_2 \otimes \delta_{k_0})/2 = ((\eta_1 + \eta_2)/2) \otimes \delta_{k_0}$, extremality of ξ implies that $\eta \in ext\ B_+(E^*)$.

Now suppose that $\xi = \eta \otimes \delta_{k_0}$, where $\eta \in ext\ B_+(E) \setminus \{0\}$ and $k_0 \in K$. Suppose that $\xi = (\xi_1 + \xi_2)/2$ where $\xi_i \in B_+(C(K,E)^*)$. Then $\| \xi \| = 1$. Let γ_i be the set function corresponding to ξ_i. Obviously for every neighborhood U of k_0 we have

$$1 = \underset{U}{Var}\ \gamma \leq (\underset{U}{Var}\ \gamma_1 + \underset{U}{Var}\ \gamma_2)/2 \leq 1, \text{ so } \underset{U}{Var}\ \gamma_i = 1.$$

And, by additivity of the variation, $\underset{U^c}{Var}\, \gamma_i = 0$. Therefore ξ_i is of the form $\eta_i \otimes \delta_{k_0}$ where $\eta_i \in B_+(E^*)$. Because $\eta \in ext\ B_+(E)$, we get $\eta = \eta_1 = \eta_2$ and $\xi = \xi_1 = \xi_2$ i.e. $\xi \in ext\ B_+(C(K,E)^*)$.

3. SMOOTH AND QUASI INTERIOR POINTS IN $B_+(C(K,E))$

Theorem 1. *$f \in q\text{-}int\ B_+(C(K,E))$ if and only if $f(K) \subseteq q\text{-}int\ B_+(E)$.*

Proof. Suppose that $f(K) \subseteq q\text{-int}\ B_+(E)$. Then $0 < \| f \| < 1$. If ξ supports $B_+(C(K, E))$ at f then, by Remark 1, $\xi(f) = 0$. Therefore $\xi(B_+(C(K,E))) \leq \xi(f) = 0$, i.e. $-\xi \geq 0$. Put

$$M = \{\xi \in B_+(C(K,E)^*) : \xi(f) = 0\}$$

Note that if ξ with $\| \xi \| \leq 1$ supports $B_+(C(K,E))$ at f, then $-\xi \in M$, and for every element $\xi \in M$ we have $\sup(-\xi)(B_+(C(K,E))) = \xi(f) = 0$. Obviously M is a closed bset of the w^*-compact set $B_+(C(K,E)^*)$. The set M is a face of $B_+(C(K,E)^*)$. Indeed, let $\xi = \alpha\xi_1 + (1-\alpha)\xi_2$ where $\xi_i \in B_+(C(K,E)^*)$, $i = 1,2$, $\alpha \in (0,1)$. Then $\xi_i(f) = 0$, so $\xi_i \in M$. Therefore $ext\ M \subseteq ext\ B_+(C(K,E)^*)$. Let $\xi \in ext\ M$. By Proposition 1 ξ is of the form $\xi(h) = \eta(h(k))$, where $k \in K$, $h \in C(K,E)$, and $\eta \in ext\ B_+(E^*)$. We have $\xi(f) = \eta(f(k)) = 0$. If $0 \neq \eta \in B_+(E^*)$, then $\eta \in (B_+(E))^*$. For $\eta \in (B_+(E))^*$ we have $\eta(f(k)) \neq 0$, because $f(k) \in q\text{-int}\ B_+(E)$. This implies that $\eta = 0$. Hence $\xi = 0$, and $ext\ M = \{0\}$ with $0 \notin (B_+(C(K,E)))^*$. By the Krein-Milman Theorem $M = \{0\}$. Therefore $f \in q\text{-int}\ B_+(C(K,E))$.

Now suppose that $f(k_0) \in q\text{-int}\ B_+(E)$ for some k_0. Then $f(k_0) \in \text{supp}\ B_+(E)$ and there exists η_0 supporting $B_+(E)$ at $f(k_0)$. It is easy to check that $\xi = \eta_0 \otimes \delta_{k_0}$ supports $B_+(C(K,E))$ at f.

Corollary 1. *$f \in q$-int $C_+(K, E)$ if and only if $f(K) \subseteq q$-int E_+.*

Theorem 2. *Let $f \in B_+(C(K, E))$. Then $f \in smooth B_+(C(K, E))$ if and only if there exists a point $k_0 \in K$ such that $f(k_0) \in smooth B_+(E)$ and $f(k) \in q$-int $B_+(E)$ for all $k \neq k_0$.*

Proof. Suppose that for different points $k_1 k_2$ we have $f(k_i) \in$ supp $B_+(E)$, $i = 1, 2$. Then there exist (not necessarily distinct) functionals $\eta_i \in B_+(E)^*$ supporting $B_+(E)$ at $f(k_i)$. We obtain two different functionals $\xi_i = \eta_i \otimes \delta_{k_i}$ which support $B_+(C(K, E))$ at f, i.e. f is not a smooth point.

Now suppose that $f(K) \subseteq q$-int $B_+(E)$. Then by Theorem 1 f is not smooth. Thus if f is smooth, then there is exactly one point k_0 such that $f(k_0) \in$ supp $B_+(E)$.

Now suppose that there exists $k_0 \in K$ such that $f(k_0) \in$ supp $B_+(E)$ and $f(K \setminus \{k_0\}) \subseteq q$-int $B_+(E)$. Let ξ with $\| \xi \| = 1$ support $B_+(C(K, E))$ at f. Obviously if $f(k_0) \notin$ smooth $B_+(E)$ then we can find two different functionals $\xi_i = \eta_i \otimes \delta_{k_0}$ $i = 1, 2$, where η_i supports $B_+(E)$ at $f(k_0)$, $\eta_1 \neq \eta_2$, $\| \eta_i \| = 1$. Hence $f \notin$ smooth $B_+(C(K, E))$.

Therefore we can now assume that $f(k_0) \in$ smooth $B_+(E)$. We consider two cases: 1) $\| f(k_0) \| < 1$, and 2) $\| f(k_0) \| = 1$.

1) In this case $\| f \| < 1$ and by Remark 1 $\xi(f) = 0$. Hence for every supporting functinal ξ with normal equal to one, $-\xi \in M$. Every $x \in ext\ M$ is of the form $x(\cdot) = \eta(\cdot(k))$, $k \in K$, $\eta \in B_+(E^*)$. Because $f(k) \in q$-int $B_+(E)$ for all $k \neq k_0$, we get that $x = \eta \otimes \delta_{k_0}$. By the Krein-Milman Theorem every element of M (sp also every supporting functional, too) is of this form. Because $f(k_0) \in$ smooth $B_+(E)$, the functional η is determined uniquely, hence so is ξ, i.e. $f \in$ smooth $B_+(C(K, E))$.

2) In this case for $\alpha \in (0, 1)$ by Remarks 2 and 4 $\alpha f(K) \subseteq q$-int $B_+(E)$ and, by Theorem 1, $\alpha f \in q$-int $B_+(C(K, E))$. Hence $\alpha\xi(f) = \xi(\alpha f) < \sup \xi(B_+(C(K, E))) = \xi(f)$. This implies that $\xi(f) > 0$. We have $\xi_+(h) = \sup\{\xi(g) : 0 \leq g \leq h\} \leq \xi(f) = \xi_+(f)$ for all $h \in B_+(C(K, E))$ i.e. ξ_+ is non zero and supports $B_+(C(K, E))$ at f. Put

$$N = \{\kappa_+ / \| \kappa_+ \| : \kappa \text{ supports } B_+(C(K, E)) \text{ at } f\}.$$

One can easily check that

$$N = \{\kappa \in B_+(C(K, E)^*) \ : \ \kappa(f) = \| \kappa \| = 1\}.$$

The set N is a weak* closed face of $B_+(C(K, E)^*)$ and $ext\ N \subseteq ext\ B_+(C(K, E)^*)$. Every $\kappa \in ext\ N$ is of the form $\kappa = \eta \otimes \delta_{k_1}$ $k_1 \in K$, $\eta \in B_+(E)$. Since $f(K \setminus \{k_0\}) \subseteq q$-int $B_+(E)$ we have $k_1 = k_0$ and $\eta = \eta_0$ is a unique functional with norm one supporting

$B_+(E)$ at $f(k_0)$. Therefore N has only one element $\xi_0 = \eta_0 \otimes \delta_{k_0}$. Consider $\xi_- = \xi_+ - \xi$. We have $\xi_-(B_+(C(K,E))) \geq 0 = \xi_-(f)$. Hence $-\xi_-$ supports $B_+(C(K,E))$ at f. Suppose that $\xi_- \neq 0$. Then $-\xi_- \in (B_+(C(K,E)))^*$. Since $\alpha f \in q\text{-int}\, B_+(C(K,E))$ for $\alpha \in (0,1)$ we get $0 = -\xi_-(\alpha f) < \sup(-\xi_-)(B_+(C(K,E))) = (-\xi_-)(f) = 0$.

This contradiction shows that $\xi_- = 0$. Therefore $\xi = \lambda\xi_0$, $\lambda \in \mathbf{R}_+$, i.e. $f \in$ smooth $B_+(C(K,E))$. This ends the proof.

Corollary 2. *$f \in$ smooth $B_+(C(K))$ if and only if there exists a unique $k_0 \in K$ such that $f(k_0) = 0$ or 1 and $f(k) \in (0,1)$ for all $k \neq k_0$.*

Here we should add a result from a well-known Banach's monography [1, p. 168]: «Let $f \in C(K)$ (K - compact metric space). For given $k_0 \in K$ the inequality $|f(k_0)| > |f(k)|$ for every $k \neq k_0$ holds if and only if $\lim_{h \to 0} \frac{\|f+hg\|-\|f\|}{h}$ exists for every $g \in C(K)$».

4. THE SPACE OF COMPACT OPERATORS $\underline{K}(E, C(K))$

We denote by $\underline{K}(E, C(K))$ the space of compact linear operators from the Banach lattice E into $C(K)$ with the usual operator norm. Heinrich [4] announced a result which we can briefly write:

$T \in$ smooth $B(\underline{K}(E, C(K)))$ *if and only if there exists a unique* $k_0 \in K$ *such that* $T^*\delta_{k_0} \in$ smooth $B(E^*)$ *and* $T^*\delta_k \in \text{int}B(E^*)$ *for all* $k \neq k_0$.

We can prove an analogous result for the positive part of the unit ball using Theorems 1 and 2 and the well known fact that we can identify $C(K, E^*)$ with $\underline{K}(E, C(K))$ (see [3, VI.7.1, p. 490]).

Theorem 3. *$T \in$ smooth $B_+(\underline{K}(E, C(K)))$ if and only if there exists a unique $k_0 \in K$ such that $T^*\delta_{k_0} \in$ smooth $B_+(E^*)$ and $T^*\delta_k \in q$-int $B_+(E^*)$ for all $k \neq k_0$. And $T \in q$-int $B_+(\underline{K}(E, C(K)))$ if and only if $T^*\delta_k \in q$-int $B_+(E^*)$ for all $k \in K$. Moreover, $T \in q$-int $\underline{K}_+(E, C(K))$ if and only if $T^*\delta_k \in q$-int E^*_+ for all $k \in K$.*

Proposition 2. *Let K be a compact uncountable Hausdorff space. Then*

$$q\text{-int}\, B_+(C(K)^*) = \emptyset \text{ and smooth } B_+(C(K)^*) = \emptyset.$$

Proof. Let K satisfy the assumption of the proposition. It is well known that $C(K)^*$ coincides with the space of all regular Borel measures on K. Fix $\mu \in B_+(C(K)^*)$. Let $\{B_\alpha\}$ be an uncountable family of disjoint Borel subsets of K (for instance singletons). Then there exist at least two disjoint sets B_1, B_2 in $\{B_\alpha\}$ with $\mu(B_\iota) = 0$, $\iota = 1, 2$. It is easy to check that ξ_ι supports $B_+(C(K)^*)$ at μ, where ξ_ι is defined by $\xi_\iota(\varphi) = -\varphi(B_\iota)$, $\varphi \in C(K)^*$. Hence $\mu \in$ supp $B_+(C(K)^*)$, but $\mu \notin$ smooth $B_+(C(K)^*)$ and $\mu \notin q$-int $B_+(C(K)^*)$. Because μ was arbitrary we get that the sets q-int $B_+(C(K)^*)$ and smooth $B_+(C(K)^*)$ are empty.

Proposition 3. *Let $K_1 K_2$ be compact Hausdorff spaces and let K_1 be uncountable. Then*

$$q\text{-int}\, B_+(\underline{K}(C(K_1), C(K_2)) = \emptyset \text{ and } \operatorname{smooth} B_+(\underline{K}(C(K_1), C(K_2)) = \emptyset .$$

The proof is clear from Theorem 3 and Proposition 2.

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