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A SHORT PROOF OF ALEXANDROV-FENCHEL'S INEQUALITY

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In memoriam my teacher Gottfried Köthe

1. INTRODUCTION

More than half a century ago Alexandrov [1] and Fenchel [8] proved a generalization of Minkowski's inequalities on volume and surface area of convex bodies: Let $K, L, K_1, \ldots, K_{n-2}$ be convex bodies in $\mathbf{R}^n$, and let $V(\cdot, \ldots, \cdot)$ denote mixed volume. Then

$$(AF) \qquad V(K, L, K_1, \ldots, K_{n-2})^2 \geq V(K, K, K_1, \ldots, K_{n-2}) V(L, L, K_1, \ldots, K_{n-2})$$

(For proofs see also Busemann [4], and Leichtweiss [9]).

New interest in (AF) has been stimulated recently, partly by the discovery of its equivalence with the Hodge inequality in case of compact projective toric varieties (see Teissier [13], Khovanskij in Burago-Zalgaller [3]).

The problem of characterizing equality in (AF) is still unsolved, though progress has been made during the last five years by R. Schneider ([10], [11], [12]), E. Tondorf, and the author ([5], [6], [7]). The method we have introduced hereby in [5] has meanwhile turned out to be applicable to a short and relatively elementary proof of (AF); we present it in this note. We are hopeful it will also contributed to a better understanding of (AF) and open problems connected with the inequality.

2. EXPLANATION OF METHOD AND FACTS USED

The basic idea of our method can be explained as follows. Let P be an n-dimensional polytope in $\mathbf{R}^n (n \geq 2)$, and let B be the unit ball of $\mathbf{R}^n$. The Minkowski sum $P + B$ («parallel body» of P) can be decomposed as follows (compare Figure 1 for P a cube in $\mathbf{R}^3$): Let $p(\cdot)$ denote the nearest point map which assigns to each $x \in P + B$ its nearest point on P.

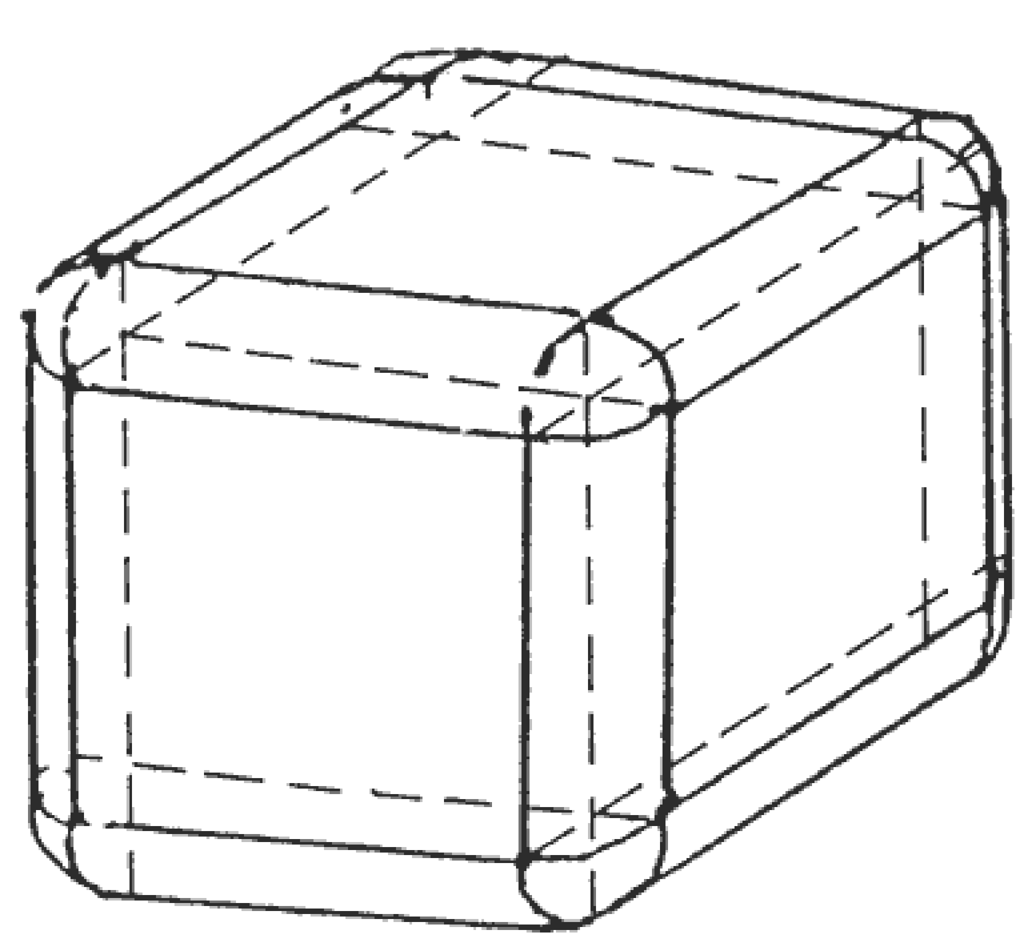

Figure 1.

The polytope P itself is the «inner» part of $P+B$; its volume is $V(P)=V(P,\dots,P)$. If a is a vertex of P, $-a+p^{-1}(a)$ is a sector of B, the outer angle of P in a. The union of such sets is B.

If F is a facet of P, that is, an $(n-1)$-dimensional face of $P, p^{-1}(F)$ is a prism above F whose volume equals the $(n-1)$-dimensional volume of F. The union of these sets has volume $O(P)$, the surface area measure of P which is easily seen to satisfy

$$nV(B,P,\dots,P,)=O(P).$$

In a similar way we may characterize $\binom{n}{k}V(B,\dots,B,P,\dots,P), P$ occuring k times, as the total volume of all «wedges» $p^{-1}(F^{(k)})$ over the k-dimensional faces $F^{(k)}$ of P.

Let $a\in$ relint $F^{(k)}$ (relative interior point). Then $\Theta_{F^{(k)}}:=p^{-1}(a)$ is the outer angle of P in $F^{(k)}$. By $v_i(\cdot)$ we denote i-dimensional volume; so we have

$$V(p^{-1}(F^{(k)}))=v_{n-k}(\Theta_{F^{(k)}})\cdot v_k(F^{(k)}),$$

and

$$(1)\qquad \binom{n}{k}V(B,\dots,B,P,\dots,P)=\sum_{k\text{ fixed}} v_{n-k}(\Theta_{F^{(k)}})\cdot v_k(F^{(k)}),$$

where B occurs $n-k$ times.

Formula (1) can be generalized in such a way that B is replaced by an arbitrary convex body C. We choose $0\in C$. All above arguments can be applied (which is technically carried out in [5]). In particular, the outer angle $\Theta_{F^{(k)}}$ is replaced by an outer angle $\Theta^C_{F^{(k)}}$ with respect to C and the choice of 0 in C. So (1) remains valid if we set C for B and $\Theta^C_{F^{(k)}}$ for $\Theta_{F^{(k)}}$.

Now we set $P=\lambda_1K_1+\dots+\lambda_kK_k$ for convex polytopes $K_1,\dots,K_k$ and any nonnegative real numbers $\lambda_1,\dots,\lambda_k$. Then by Minkowski's formula on polynomial expansion of mixed volumes and by comparing coefficients we find ($v_i(\cdot,\dots,\cdot)$ denoting i-dimensional mixed volume):

$$(2)\qquad \binom{n}{k}V(C,\dots,C,K_1,\dots,K_k)=\sum_{k\text{ fixed}} v_{n-k}(\Theta^C_{F^{(k)}})v_k(F_1^{(k)},\dots,F_k^{(k)})$$

where

$$F^{(k)}=\lambda_1F_1^{(k)}+\dots+\lambda_kF_k^{(k)},\quad F_i^{(k)}\quad a\ face\ of\quad K_i,\quad i=1,\dots,k.$$

If in $(AF)\, V(K, K, K_1, \dots, K_{n-2}) = 0$ or $V(L, L, K_1, \dots, K_{n-2}) = 0$, there is nothing to prove. So let both terms be $\neq 0$.

By a homothety of L we can arrange $V(K, K, K_1, \dots, K_{n-2}) = V(L, L, K_1, \dots, K_{n-2})$.

The validity of (AF) remains invariant under the homothety. It is not difficult to show (see [5]), that (AF) is then equivalent to

$$(3) \qquad \sum [v_2(\Theta^{K+L}_{F^{(n-2)}}) - 2v_2(\Theta^{k}_{F^{(n-2)}}) - 2v_2(\Theta^{L}_{F^{(n-2)}})]\, v(F_1^{(n-2)}, \dots, F_{n-2}^{(n-2)}) \geq 0$$

The side condition $V(K, K, K_1, \dots, K_{n-2}) = V(L, L, K_1, \dots, K_{n-2})$ can be expressed as

$$(4) \qquad \sum [v_2(\Theta^{k}_{F^{(n-2)}}) - v_2(\Theta^{L}_{F^{(n-2)}})]\, v(F_1^{(n-2)}, \dots, F_{n-2}^{(n-2)}) = 0$$

The summations in (3), (4) may be restricted to all $F^{(n-2)}$ which are *edge sum faces* of P, that is, are such that each $F_i^{(n-2)}$ contains a line segment s_i where

$$\dim(s_1 + \dots + s_{n-2}) = n - 2.$$

In all other case $v(F_1^{(n-2)}, \dots, F_{n-2}^{(n-2)}) = 0$.

We make also use of three classical facts about convex bodies:

(a) Finitely many convex bodies $K_1, \dots, K_r$ can always simultaneously be approximated by n-dimensional convex polytopes $K_1^{(i)}, \dots, K_r^{(i)}$, respectively, such that $K_1^{(i)}, \dots, K_r^{(i)}$ are, for the same i, strictly combinatorially isomorphic, that is, have isomorphic boundary complexes and the same outer normals in facets.

(b) Let A, B be convex bodies in $\mathbf{R}^2$ which have a common width (that is, possess pairs of parallel supporting lines with the same distance and all parallel to each other). Then (see Bonnesen-Fenchel [2], p. 99):

$$2v_2(A, B) - v_2(A) - v_2(B) \geq 0.$$

The following is easily obtained by direct calculation:

(c) (AF) is true for K, L if and only if it is true for $K_\lambda := (1-\lambda)K + \lambda L$ and $K_\mu := (1-\mu)K + \mu L, 0 < \lambda < \mu < 1$, instead of K, L, respectively.

3. PROOF OF THE INEQUALITY

Let $K^{(i)}, L^{(i)}, K_1^{(i)}, \ldots, K_{n-2}^{(i)}$ according to (a) be strictly combinatorially isomorphic and have limits $K, L, K_1, \ldots, K_{n-2}$, respectively, for $i \to \infty$. In the following we hold i fixed. Accordingt to (c) we replace $K^{(i)}, L^{(i)}$ by $K_\lambda^{(i)}, K_\mu^{(i)}$, respectively. We can write again $K := K_\lambda^{(i)}, L := K_\mu^{(i)}, K_j := K_j^{(i)}, j = 1, \ldots, n-2$, and assume $V(K, K, K_1, \ldots, K_{n-2}) = V(L, L, K_1, \ldots, K_{n-2}) > 0$. Let, furthermore, $0 \in (\text{relint} L)$. Now we choose $\mu - \lambda > 0$ so small that the following becomes true:

(d) Given any two facets F_K, F_L of K, L, respectively, with the same outer normal u there exists a ray ρ_u emanating from 0 which cuts F_K and F_L in relative interior points.

Let now u, v be outer facet normals of $P := K_1 + \ldots + K_{n-2}$ such that the two facets intersect in an $(n-2)$-face F. Since $K_1, \ldots, K_{n-2}$ are strictly combinatorially isomorphic, F is always an edge sum face. Let π_F be the projection parallel to aff F (affine hull) onto the 2-dimensional subspace E of $\mathbf{R}^n$ perpendicular to aff F. In E we obtain the sets Θ_F^K, Θ_F^L as bounded by $\pi_F(\rho_u) = \rho_u, \pi_F(\rho_v) = \rho_v$, and pairs of line segments which are projections of facets of K, L, respectively (Figure 2). Let the first pair of those line segments intersect in q_K, the second in q_L. Up to a translation of L we can assume that either

(I) q_K lies outside Θ_F^K, and q_L lies outside Θ_F^L
or one of the relations

(II) $q_K \in \text{relint } \Theta_F^K, q_L \in \text{relint } \Theta_F^L$
is true (Figure 2a, b).

Because of strict combinatorial isomorphy we have:

$$\Theta_F^K + \Theta_F^L = \Theta_F^{K+L}$$

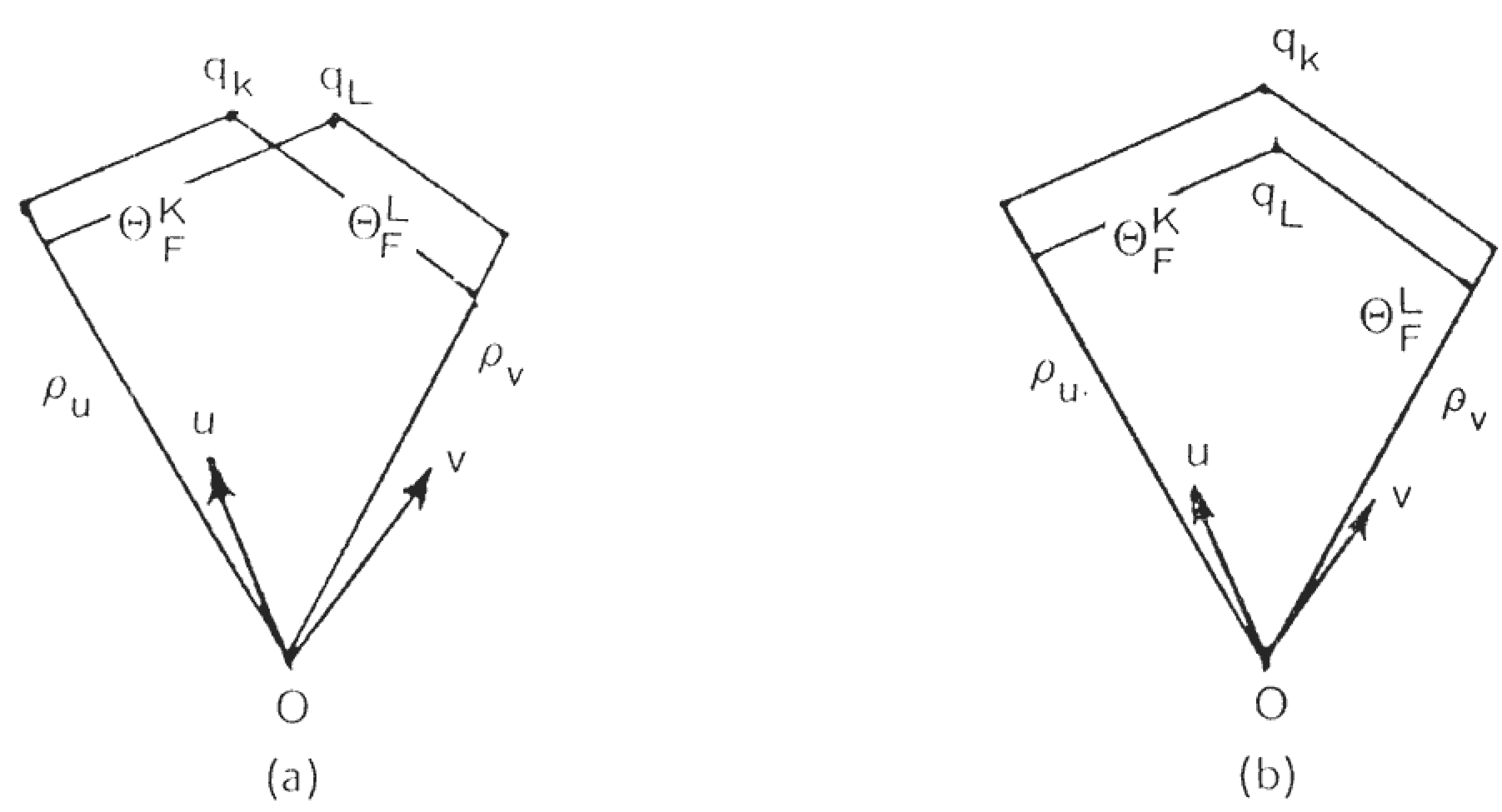

Figure 2.

Therefore, by (b),

$$(5)\quad v_2(\Theta_F^{K+L}) - 2v_2(\Theta_F^K) - 2v_2(\Theta_F^L) = 2v_2(\Theta_F^K, \Theta_F^L) - v_2(\Theta_F^K) - v_2(\Theta_F^L) \geq 0,$$

provided Θ_F^K, Θ_F^L have a common width. In situation (I) a common width exists if we choose the approximating sequences $\{K^{(i)}\}$ etc. such that the angle between ρ_u, ρ_v becomes small enough so that the line through 0 parallel to the line $q_K\ q_L$ is a supporting line of Θ_F^K and Θ_F^L.

In case (II) we proceed as follows, where $q_L \in$ relint Θ_F^K is assumed. If $\mu - \lambda$ is sufficiently small there exists a line g in E which separates properly q_K from all other vertices of $\pi_F(K)$ and q_L from all other vertices of $\pi_F(L)$. Let Δ_K, Δ_L be the triangles which g cuts off from Θ_F^K, Θ_F^L, respectively. Referring back to $K^{(i)}, L^{(i)}$ we find triangles $\Delta_{K^{(i)}}, \Delta_{L^{(i)}}$ such that

$$(6)\qquad \Delta_K = (1-\lambda)\Delta_{K^{(i)}} + \lambda\Delta_{L^{(i)}}, \quad \Delta_L = (1-\mu)\Delta_{K^{(i)}} + \mu\Delta_{L^{(i)}}$$

(see Figure 3). Then $\pi_F^{-1}(g)$ is a hyperplane H which cuts off the $(n-2)$-faces $\pi_F^{-1}(q_K), \pi_F^{-1}(q_L)$ from K, L, respectively.

We note that no other set Θ_F^K, or Θ_F^L, is affected by there cutting offs (since P, K, L are strictly combinatorially isomorphic).

Let $\check{\Theta}_F^K, \check{\Theta}_F^L$ be the pentagons obtained from Θ_F^K, Θ_F^L by cutting off q_K, q_L, respectively. As in (I) we may assume the parallel line of g through 0 to be a supporting line of $\check{\Theta}_F^K$ and $\check{\Theta}_F^L$.

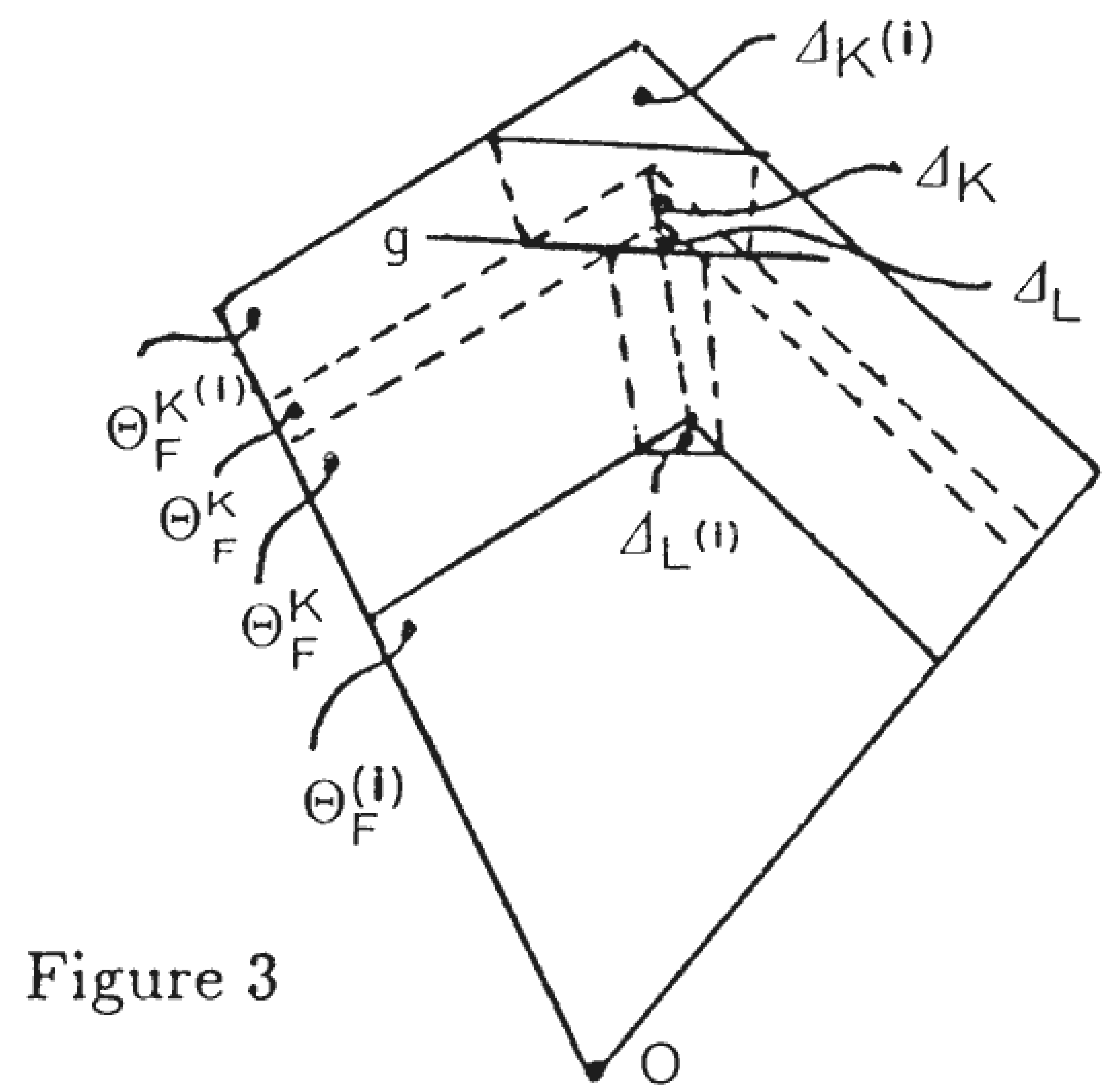

Figure 3.

So we have shown that (5) is valid for the pairs Θ_F^K, Θ_F^L in case (I) and for $\check{\Theta}_F^K, \check{\Theta}_F^L$ in case (II). Let $\check{K}, \check{L}$ be the polytopes obtained from K, L, respectively, after all cuttings offs in cases (II) have been carried out. Then (3) is valid for $\check{K}, \check{L}$.

We wish to establish also (4) for $\check{K}, \check{L}$. For this purpose we make use of a relative freedom in choosing the cutting hyperplane H introduced above. Whenever situation (II) occurs for some pair $\Theta_F^K, \Theta_F^L, q_L \in$ relint Θ_F^K, the left side of (4) attains after the cut a negative value

$$\alpha := -[v(\Delta_K) - v(\Delta_L)]v(F_1^{(n-2)}, \ldots, F_{n-2}^{(n-2)})$$

In order to compensate α we look for an appropriate pair $\Theta_{\tilde{F}}^K, \Theta^L K_{\tilde{F}}$. We have found one if a situation (I) occurs: If $\lambda - \mu$ is a small enough, a line $\tilde{g}$ exist which strictly separates q_K and q_L from the other vertices of $\Theta_{\tilde{F}}^K, \Theta_{\tilde{F}}^L$, respectively. We can choose $\tilde{g}$ such that for the triangles $\tilde{\Delta}_K, \tilde{\Delta}_L$ cut off from Θ_F^K, Θ_F^L, respectively, satisfy $[v(\tilde{\Delta}_K) - v(\tilde{\Delta}_L)]v(F_1^{(n-2)}, \ldots, F_{n-2}^{(n-2)}) = \alpha$.

Then the left side of (4) increases by $-\alpha$ so that equality in (4) is established.

If no pair $\Theta_{\tilde{F}}^K, \Theta_{\tilde{F}}^L$ according to (I) exists, there is a pair satisfying $\tilde{q}_K \in$ relint $\Theta_{\tilde{K}}^F$, where $\tilde{q}_K$ is defined analogously to q_K; otherwise (4) were violated for K, L. Then the compensation is achieved in an obvious fashion. So (AF) is shown for $\check{K}, \check{L}$.

Let ε_0 be the maximal height of all triangles $\Delta_K, \Delta_F, \tilde{\Delta}, \tilde{\Delta}_L$ occuring above in establishing (3). ε_0 can, by appropriate choice of $\lambda - \mu$, be made arbitrarily small. Since $0 < \lambda < 1$ and $0 < \mu < 1$, there exists a constant c such that the maximal hights of the triangles $\Delta_{K^{(i)}}, \Delta_{L^{(i)}}, \tilde{\Delta}_{K^{(i)}}, \tilde{\Delta}_{L^{(i)}}$, remain below $c \cdot \varepsilon_0$. Therefore, the validity of (AF) for $\check{K}, \check{L}$ implies (AF) for $K = K_\lambda^{(i)}, L = K_\mu^{(i)}$, and hence for arbitrary convex bodies K, L.

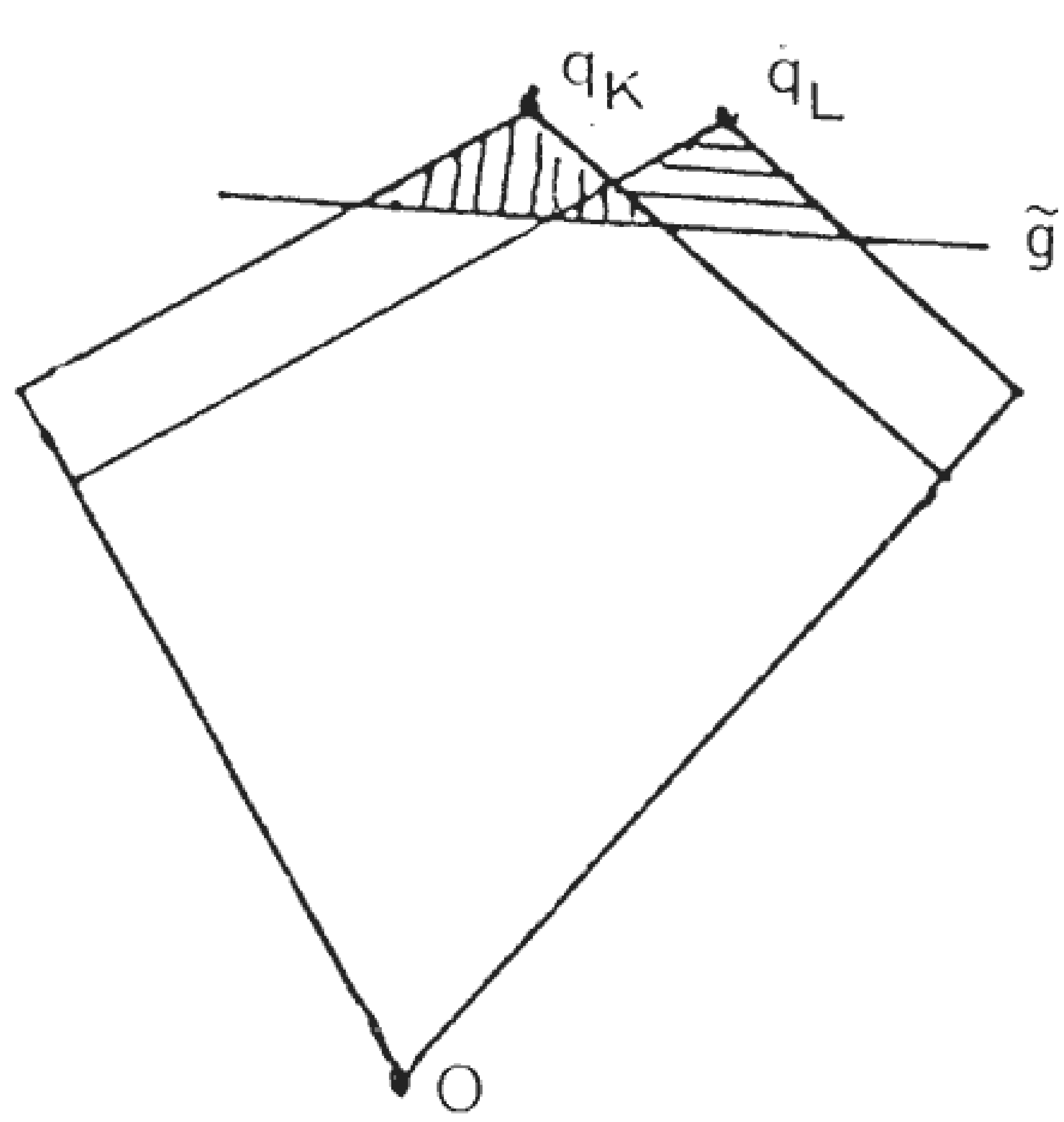

Figure 4.

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