Note di Matematica Vol. X, Suppl. n. 1, 229-241(1990)

CONTINUOUS FAMILIES OF LINEAR FUNCTIONALS

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *We begin by recalling an elementary (but not widely advertised) theorem concerning finite families of linear functionals. After that, the principle aim is an analogous theorem (again not widely advertised) concerning suitable infinite continuous families of linear functionals. The discussion might well find a place in a first course on linear functional analysis, especially if it is desired to provide some concrete applications of general theorems.*

In all that follows E *is assumed to be a complex linear space,* f *a linear functional on* E, T *a set, and* $(f_i)_{i\in T}$ *a family of liner functigonals on* E. *Trivial changes take care of the case in which* E *is a real linear space.*

1. THE FINITE CASE

Here we take T to be $\{1,2,\ldots,n\}$ for some natural number $n>0$. The theorem in question is

Theorem 0. *Assume* E, f *and* $(f_i)_{1\leq i\leq n}$ *to be as stated above. Define the seminorm* ν *on* E *by*

$$\nu(x) = \sup_{1\leq i\leq n} |f_i(x)| \ for\ all\ x\in E.$$

The following conditions are pairwise equivalent:

$$(I_0) \qquad (\exists m\in \mathbb{R}_+)(\forall x\in E)(|f(x)|\leq m.\nu(x));$$

$$(I_0') \qquad \bigcap_{1\leq i\leq n} ker(f_i)\subseteq ker(f);$$

$$(II_0) \qquad f \text{ is a linear combination of } (f_i)_{1\leq i\leq n};$$

$$(III_0) \qquad (\forall s\in E^{\mathbf{N}})[(\forall i\in\{1,2,\ldots,n\}(\lim_{k\to\infty} f_i(s_k)=0)\Rightarrow \lim_{k\to\infty} f(s_k)=0\}.$$

The proof is omitted.

As an illustration, suppose that $n \in \mathbf{N}$, that E is the linear space (pointwise operations) of all complex valued polynomial functions on $\mathbf{R}$, that $(a_k)_{0 \leq k \leq n}$ is an injective sequence of real numbers, and that, for all $k \in \{0, 1, \ldots, n\}, f_k$ denotes the linear functional on E defined by

$$f_k(x) = x(a_k) \quad for\ all\ x \in E.$$

Then

$$\bigcap_{0 \leq k \leq n} ker(f_k) = \{\mathbf{R} \times \{0\}\},$$

the zero element of E. Accordingly, Theorem 0 ensure that to every linear functional f on E corresponds $(\alpha_k)_{0 \leq k \leq n} \in \mathbb{C}^{n+1}$ such that

$$(0) \qquad f(x) = \sum_{k=0}^{n} a_k x(a_k) \quad for\ all\ x \in E.$$

To make the result more specific, one may introduce the appropriate Lagrange polynomials $L_0, L_1, \ldots, L_n$ defined by

$$L_i(\xi) = \prod_{k=0, k \neq i}^{n} (a_i - a_k)^{-1}(\xi - a_k),$$

for which

$$L_i(a_j) = \delta_{ij} \quad for\ all\ 0 \leq i, j \leq n.$$

Then $L_i \in E$ for all $i \in \{0, 1, \ldots, n\}$, and (0) implies that

$$a_k = f(L_k) \quad for\ all\ k \in \{0, 1, \ldots, n\}$$

and hence that

$$(0_1) \qquad f(x) = \sum_{k=0}^{n} f(L_k) x(a_k) \quad for\ all\ x \in E.$$

In particular, choosing $\xi \in \mathbf{R}$ and $f(x) = x(\xi)$ for all $x \in E, (0_1)$ yields

$$(0_2) \qquad x(\xi) = \sum_{x=0}^{n} L_k(\xi) x(a_k) \quad for\ all\ x \in E\ and\ all\ \xi \in \mathbf{R}.$$

Choosing $f(x) = \int_0^1 x(\xi)\,d\xi$ for all $x \in E$ (or directly from (0_2))

$$(0_3) \qquad \int_0^1 x(\xi)\,d\xi = \sum_{k=0}^{n} \int_0^1 L_k(\xi)\,d\xi\, x(a_k) \quad \textit{for all } x \in E.$$

2. THE INFINITE CASE

We now pass to the study of analogues of Theorem 0 in which the finite family $(f_i)_{1\le i\le n}$ is replaced by possibly infinite families $(f_t)_{t\in T}$.

One may expect (II_0) to be replaced by the assertion that f is some sort of limit of finite linear combinations fo the f_t with $t \in T$. Examples will show that at least two simple minded analogues are false.

For every compact Hausdorff space S, denote by $C(S)$ the linear space (pointwise algebra) of all continuous complex-valued functions on S.

Example 1. Take $T = \mathbf{N}$, $E = C([0,1])$. Enumerate the rationals in $]0,1[$ as the range of the injective sequence $(r_n)_{n\in\mathbf{N}}$, and define (f_n) and f by

$$f_n(x) = x(r_n) \text{ for all } n \in \mathbf{N} \text{ ; and all } x \in E$$

$$f(x) = x(0) \text{ for all } x \in E.$$

Define further

$$\nu(x) = \sup_{n\in\mathbf{N}} |f_n(x)| \quad \textit{for all } x \in E.$$

Plainly

$$|f(x)| \le \nu(x) \quad \textit{for all } x \in E.$$

However, if $s \in E^{\mathbf{N}}$ is defined to be $(s_k)_{k\in\mathbf{N}}$, where

$$s_k(\xi) = \max\{1 - k\xi, 0\} \quad \textit{for all } \xi \in [0,1] \textit{ and all } k \in \mathbf{N},$$

then

$$\lim_{k\to\infty} f_n(s_k) = 0 \quad \textit{for all } n \in \mathbf{N}$$

and yet

$$\lim_{k\to\infty} f(s_k) = \lim_{k\to\infty} s_k(0) = \lim_{k\to\infty} 1 = 1.$$

Example 2. Take E and f_k as in Example 1, now regarding E as a Banach space with the uniform norm

$$||x|| = \sup_{\xi \in [0,1]} |x(\xi)| \quad for\ all\ x \in E.$$

Define f by

$$f(x) = \int_0^1 x d\lambda \quad for\ all\ x \in E,$$

where λ denotes Lebesgue measure on $\mathbf{R}$. Plainly

$$\bigcap_{n \in \mathbf{N}} ker(f_n) = \{[0,1] \times \{0\}\},$$

$\{[0,1] \times \{0\}\}$ being the zero element of $C([0,1])$.

Nevertheless one can prove that there is no complex-valued sequence $(\alpha_n)_{n \in \mathbf{N}}$ such that

$$\sum_{n \in \mathbf{N}} \alpha_n f_n(x)$$

is convergent in $\mathbf{R}$ for all $x \in E$ and

$$f(x) = \sum_{n \in \mathbf{N}} \alpha_n f_n(x) \quad for\ all\ x \in E.$$

3. Turning from negative to positive results, we remark that if it be assumed that E is a barrelled topological linear space, that each f_t is a *continuous* linear functional on E, and that

$$\nu(x) = \sup_{t \in T} |f_t(x)| < \infty \quad for\ all\ x \in E,$$

then ν is a lower semicontinuous, hence continuous, seminorm on E. Hence the condition: for some $m \in \mathbf{R}_+$,

$$|f(x)| \leq m.\nu(x) \quad for\ all\ x \in E,$$

implies continuity of f, which in turn implies that

$$\lim_{\alpha} f(s_\alpha) = 0$$

for every net (s_α) of elements of E such that

$$\lim_{\alpha} s_\alpha = 0$$

weakly in E. However, we do not wish to assume *ab initio* that E is a barrelled topological linear space etc.

4. In order to formulate Theorem 1 below we list the following definitions and hypotheses.

For every locally compact Hausdorff space T, denote by $C_0(T)$ the linear space (pointwise operations) of complex-valued continuous functions u on T such that

$$\lim_{t\to\infty} u(t) = 0,$$

equipped with the uniform norm

$$||u|| = \sup_{t\in T} |u(t)|.$$

(It T is compact, $C_0(T)$ is usually denoted by $C(T)$; its elements are just the complex-valued continuous functions on T). We denote by $\kappa(T)$ the set of all compact subsets of T; and by $\rho(T)$ the set of all bounded complex Radon measures on T.

As hypotheses we consider:

(H_1) E is a complex linear space and f a linear functional on E

(H_2) T is a locally compact Hausdorff space

(H_3) $(f_t)_{t\in T}$ is a family of linear functionals on E and

$$\nu(x) = \sup_{t\in T} |f_t(x)| < \infty \quad \textit{for all } x \in E$$

(H_4) For all $x \in E$, the function

$$u_x : t \to f_t(x)$$

with domain T, belongs to $C_0(T)$.

5.

Theorem 1. *Assuming* $(H_1) - (H_4)$ *above, the following statements are pairwise equivalent:*

(I) $$(\exists m \in \mathbf{R}_+)(\forall x \in E)(|f(x)| \leq m.\nu(x))$$

(II) $$(\exists m \in \mathbf{R}_+)(\exists \mu \in \rho(T))$$

$$\left(|\mu|(T) \le m \wedge (\forall x \in E)\left(f(x) = \int_T f_t(x)\,d\mu(t)\right)\right)$$

(III) $$(\forall s \in E^{\mathbf{N}})\{[\sup_{k\in\mathbf{N}} \nu(s_k) < \infty \wedge (\forall F \in \kappa(T))(\lim_{k\to\infty}(\sup_{t\in F}|f_t(s_k)|) = 0)] \Rightarrow \lim_{k\to\infty} f(s_k) = 0\}$$

(III') $$(\forall s \in E^{\mathbf{N}})\{[\sup_{k\in\mathbf{N}} \nu(s_k) < \infty \wedge (\forall t \in T)(\lim_{k\to\infty} f_t(s_k) = 0)] \Rightarrow \lim_{k\to\infty} f(s_k) = 0\}$$

(IV) $$(\exists m \in \mathbf{R}_+)(\forall \varepsilon > 0)(\exists F \in \kappa(T))(\forall x \in E)(|f(x)| \le \varepsilon.\nu(x) + m\sup_{t\in F}|f_t(x)|).$$

Sketch Proof. The major component is the proof that (I) $\iff$ (II). Assuming (I), choose $m \in \mathbf{R}^+$ as indicated. Define

$$M = \{u_x : x \in E\} \subseteq C_0(T).$$

By (I), $f(x) = f(y)$ for all $x, y \in E$ such that $u_x = u_y$. Hence there exists a linear functional g on M such that

(i) $$g(u_x) = f(x) \quad for\ all\ x \in E$$

and thus

(ii) $$|g(u)| \le m\sup_{t\in T}|u(t)| \quad for\ all\ u \in M.$$

By the Hahn-Banach theorem ([1], Theorem 1.7.1) g can be extended to $C_0(T)$ in such a way that (ii) holds for all $u \in C_0(T)$. By the Reisz representation theorem ([1], Chapter 4), there exists $\mu \in \rho(T)$ such that

(iii) $$g(u) = \int_T u(t)\,d\mu(t) \quad for\ all\ u \in C_0(T),$$

which implies via (ii) that

(iv) $$|\mu|(T) \le m.$$

Finally, by (i),

$$f(x) = g(u_x) = \int_T f_x(t)\,d\mu(t) \quad for\ all\ x \in E.$$

Thus (I) $\Rightarrow$ (II). The implication (II) $\Rightarrow$ (I) is immediate. In connection with (III') one invokes the Lebesgue dominated convergence theorem.

6. REMARKS

(1) If we assume merely that T is an arbitrary nonvoid set, and retain (H_1) and (H_2), then (I) implies that there exists a finitely additive complex measure γ of total variation at most m such that

$$f(x) = \int_T f_t(x)\, d\gamma(t) \quad for\ all\ x \in E.$$

But this is insufficient to imply (III).

In a similar way, we might assume (in place of (H_2)) that T is a completely regular topological space, and (in place of (H_4)) that for every $x \in E$ the function u_x belongs to $BC(T)$ (the space of all bounded continuous complex-valued functions on T). Regarding $BC(T)$ as a Banach algebra (pointwise algebra and uniform norm), denote by $\widetilde{T}$ the maximal ideal space of $BC(T)$ with its Gelfand topology. ($\widetilde{T}$ is homeomorphic to βT, the Stone-Čech compactification of T; see [2], Chapter 7). Then T can be regarded as an everywhere dense subspace of $\widetilde{T}$, and every $u \in BC(T)$ has a unique continuous extension $\widetilde{u}$ which is an element of $C(\widetilde{T})$. (I) will then imply that there exists a complex Radon measure μ on the compact space $\widetilde{T}$ such that

$$f(x) = \int_{\widetilde{T}} \widetilde{u}_x(\widetilde{t})\, d\mu(\widetilde{t}) \quad for\ all\ x \in E.$$

Once again, however, this is insufficient to imply (III).

(2) If E is a real linear space and

$$\sup_{t \in T} f_t(x) < \infty \quad for\ all\ x \in E,$$

and if (I) is replaced by

$$m \in \mathbf{R}_+ \wedge (\forall x \in E)(f(x) \le m. \sup_{t \in T} f_t(x)),$$

then in (II) μ be taken to be a non-negative Radon measure on T such that $\mu(T) \le m$.

(3) Example 2 provides an instance in which $E = C([0,1])$ is a Banach space; $T = \mathbf{N}$ is a locally compact, non-compact Hausdorff space with discrete topology; and $(H_1) - (H_4)$ are satisfied. Yet in this case (I) is true and (III) and (III') are false. The breakdown is due, at least in part, to a violation of (H_4).

7. ANALOGUES OF THEOREM 1

Suppose that E and f are as in (H_1) and T is as in (H_2). Suppose also that λ is a chosen non-negative Radon measure on T, that p is chosen from $[1,\infty[$, and that in $(H_3)\nu$ is replaced by

$$(1) \qquad \nu_p(x) = \left(\int_T |f_t(x)|^p d\lambda(t)\right)^{\frac{1}{p}} < \infty \quad for\ all\ x \in E.$$

If it be assumed that $m \in \mathbf{R}_+$ and

$$(2) \qquad |f(x)| \leq m.\nu_p(x) \quad for\ all\ x \in E,$$

then one may infer that there exists $\omega \in L^{p'}(T,\lambda)$ such that $\|\omega\|_{L^{p'}} \leq m$ and

$$(3) \qquad f(x) = \int_T \omega(t) f_t(x) d\lambda(t) \quad for\ all\ x \in E;$$

here p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$. If also $p \in]1,\infty[$, it follows that for every $\varepsilon > 0$ there exists $F \in \kappa(T)$ such that

$$(4) \qquad |f(x)| \leq \varepsilon.\nu_p(x) + m\left(\int_F |f_t(x)|^p d\lambda(t)\right)^{\frac{1}{p}} \quad for\ all\ x \in E.$$

From (3) it also follows that

$$(5) \qquad \lim_{k\to\infty} f(s_k) = 0$$

for every $s \in E^{\mathbf{N}}$ such that

$$(6) \qquad \sup_{k\in\mathbf{N}} \nu_p(s_k) < \infty$$

and

$$(7) \qquad \lim_{k\to\infty} \int_T f_t(s_k) v(t) d\lambda(t) = 0 \quad for\ all\ v \in V,$$

where V denotes any subset of $L^{p'}(T,\lambda)$ whose linear span is dense in $L^{p'}(T,\lambda)$.

In the special case in which T is an arbitrary non-void set (with its discrete topology) and λ is conting measure on T, the integrals are to be replaced by the corresponding sums. Then (2) reads

$$\nu_p(x) = \left(\sum_{t\in T} |f_t(x)|^p\right)^{\frac{1}{p}} < \infty \quad \textit{for all } x \in E;$$

in (3), $\omega \in \ell^{p'}(T)$ and

$$f(x) = \sum_{t\in T} \omega(t) f_t(x) \quad \textit{for all } x \in E;$$

(4) reads

$$|f(x)| \leq \varepsilon.\nu_p(x) + m\left(\sum_{t\in F} |f_t(x)|^p\right)^{\frac{1}{p}} \quad \textit{for all } x \in E;$$

and (7) may be replaced by

$$\lim_{k\to\infty} f_t(s_k) = 0 \quad \textit{for all } t \in T.$$

Interested readers will be able to formulate still more analogues of Theorem 1.

8. TWO ILLUSTRATIONS OF THEOREM 1

(a) Consider the following situation: $E = C_c^\infty(\mathbf{R})$, the space of indefinitely differentiable complex-valued functions on $\mathbf{R}$ which have compact supports; $T = K$, a compact subset of $\mathbf{R}$;

$$f_t(x) = \int_{\mathbf{R}} exp(it\xi)x(\xi)\,d\xi = \widehat{x}(t) \quad \textit{for all } t \in \mathbf{R} \textit{ and all } x \in E;$$

$$f(x) = \int_{\mathbf{R}} \phi(\xi)x(\xi)\,d\xi \quad \textit{for all } x \in E,$$

ϕ denoting a given locally integrable complex-valued function on $\mathbf{R}$.

Hypotheses $(H_1) - (H_4)$ are all satisfied and Theorem 1 and its proof affirm the equivalence of the following two statements:

(i) $m \in \mathbf{R}_+$ and $K \in \kappa(\mathbf{R})$ and

$$(8) \qquad \left|\int_{\mathbf{R}} \phi(\xi)x(\xi)\,d\xi\right| \leq m.\sup_{t\in K} |\widehat{x}(t)| \quad \textit{for all } x \in E;$$

(ii) $m \in \mathbf{R}_+$ and $K \in \kappa(\mathbf{R})$ and there exists $\mu \in \rho(\mathbf{R})$ supported by K and having total mass at most m and such that

$$(9) \qquad \phi(\xi) = \int_K exp(it\xi)\, d\mu(t) \quad \textit{for almost all } \xi \in \mathbf{R}.$$

A consequence of (ii) is that, by suitable modification of ϕ on a negligible subset of $\mathbf{R}$, one may assume that the equality in (9) holds for all $\xi \in \mathbf{R}$.

A similar appeal to the substance of §7 shows that if in addition $1 \leq p < \infty$, then the statement

(iip) $m \in \mathbf{R}$ and $K \in \kappa(\mathbf{R})$ and

$$(10) \qquad \left|\int_{\mathbf{R}} \phi(\xi)x(\xi)\,d\xi\right| \leq m\left(\int_K |\widehat{x}(t)|^p dt\right)^{\frac{1}{p}} \quad \textit{for all } x \in E$$

is equivalent to the statement

(iip) $m \in \mathbf{R}$ and $K \in \kappa(\mathbf{R})$ and there exists a complex-valued function $\omega \in L^{p'}(\mathbf{R})$ which vanishes on $\mathbf{R} \setminus K$ and satisfies $||\omega||_L^{p'}(\mathbf{R}) \leq m$ such that

$$(11) \qquad \phi(\xi) = \int_{\mathbf{R}} \omega(t)\, exp(it\xi)\, dt \quad \textit{for almost all } \xi \in \mathbf{R}.$$

Here again a suitable modification of ϕ on a negligible subset of $\mathbf{R}$ will arrange that the equality in (11) holds for all $\xi \in R$.

Assuming this modification to have been made, either of (ii) or (iip) implies that ϕ is (the restriction to $\mathbf{R}$ of) an entire function of order one and exponential type on $\mathbb{C}$. More specifically, either of (ii) or (iip) implies

$$|\phi^{(k)}(\xi + i\eta)| \leq m. \max_{t\in K}(|t|^k exp(-t\eta))$$

for all $k \in \mathbb{N}$ and all $\xi, \eta \in \mathbf{R}$.

In addition, if $2 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, (iip) implies

$$\left(\int_{\mathbf{R}} |\phi_{(k)}(\xi + i\eta)|^p d\xi\right)^{\frac{1}{p}} \leq m\left(\int_k |t|^{p'k} exp(-p'\eta t)\, |\omega(t)|^{p'} dt\right)^{\frac{1}{p'}}$$

for all $k \in \mathbb{N}$ and all $\xi, \eta \in \mathbf{R}$.

It is perhaps a little surprising that inequalities (i) or (ip) should imply that ϕ is (after modification) so extremely regular.

Compare these results with the Paley-Weiner theorem.

(b) Now consider the situation in which $E = C_c^\infty(]-1,1[); T = \mathbf{N}$ with the discrete topology;

$$f_n(x) = \int_{-1}^{1} \xi^n x(\xi) d\xi \quad \text{for all} \quad n \in \mathbf{N} \quad \text{and all} \quad x \in E;$$

$$f(x) = \int_{-1}^{1} \phi(\xi) x(\xi) d\xi \quad \text{for all} \quad x \in E;$$

ϕ denoting a complex-valued function which is locally integrable on $]-1,1[$.

Hypotheses $(H_1) - (H_4)$ are again satisfied and Theorem 1 and its proof affirm the equivalence of the following two statements:

(iii) $m \in \mathbf{R}_+$ and

$$(12) \qquad \left| \int_{-1}^{1} \phi(\xi) x(\xi) d\xi \right| \leq m . \sup_{n \in \mathbf{N}} \left| \int_{-1}^{1} \xi^n x(\xi) d\xi \right| \quad for\ all\ x \in E;$$

(iv) $m \in \mathbf{R}_+$ and there exists $\omega \in \ell^1(\mathbf{N})$ such that

$$\sum_{n \in \mathbf{N}} |\phi(n)| \leq m$$

and

$$(13) \qquad \phi(\xi) = \sum_{n \in \mathbf{N}} \omega(n) \xi^n \quad \text{for almost all} \quad \xi \in]-1,1[.$$

If also ϕ is continuous on $]-1,1[$, the equality in (13) holds for all $\xi \in]-1,1[$. In any case, ω is uniquely determined by ϕ, and one may denote by ϕ^0 the function

$$\xi \to \sum_{n \in \mathbf{N}} \omega(n) \xi^n$$

with domain the disk

$$\Delta = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}.$$

The function ϕ^0 is continuous on Δ and holomorphic interior to Δ.

Since $\ell^1(\mathbf{N})$ is a commutative Banach algebra with identity relative to pointwise addition and scalar multiplication and a product defined by truncated convolution,

$$\omega * \sigma(n) = \sum_{p+q=n} \omega(p)\sigma(q),$$

it follows that

$$\mathscr{A} = \{\phi^0 : \omega \in \ell^1(\mathbf{N})\}$$

is likewise a commutative Banach algebra with identity relative to pointwise operations and norm

$$||\phi^0|| = \sum_{n \in \mathbf{N}} |\omega(n)|.$$

Further examination shows that the maximal ideals in $\mathscr{A}$ are of the form

$$M_\xi = \{\phi^0 \in \mathscr{A} : \phi^0(\xi) = 0\}$$

as ξ ranges over Δ. From this it follows that an element ϕ^0 of $\mathscr{A}$ has an inverse in $\mathscr{A}$ if and only if

$$\phi^0(\xi) \neq 0 \quad \textit{for all}\ \xi \in \Delta.$$

Again it is a little surprising that (iii), together with the continuity of ϕ, entails such extreme regularity of ϕ.

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Received December 11, 1990

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