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HOLOMORPHIC FUNCTIONS ON C^I, I UNCOUNTABLE

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *In this article we show that $H(C^I)$, the (Fréchet) holomorphic functions on C^I, is complete with respect to the topologies τ_0, τ_ω and τ_δ. The same result for countable I is well known (see [2]) since in this case C^I is a Fréchet space. The extension to uncountable I requires a different approach. For the compact open topology τ_0 we use induction to reduce the problem to the countable case. Next we use the result for τ_0 to reduce the problem for τ_ω and τ_δ to the case of homogeneous polynomials. Using a method developed for holomorphic functions on nuclear Fréchet spaces with a basis and, once more, the result for the compact open topology we complete the proof for τ_ω and τ_δ. We refer to [2] for background information.*

1. HOLOMORPHIC FUNCTIONS ON LOCALLY CONVEX SPACES

Let E denote a locally convex space over C.

A C-valued function on a domain Ω is said to be holomorphic (or Fréchet holomorphic) if

(i) it is continuous;

(ii) its restriction to each finite dimensional section of Ω is holomorphic as a function of several complex variables.

A function which satisfies (ii) is said to be Gâteaux holomorphic. We let $H(\Omega)$ denote the vector space of all holomorphic functions on Ω. The compact open topology on $H(\Omega)$, τ_0, is the topology of uniform convergence on the compact subsets of Ω. A semi-norm p on $H(\Omega)$ is said to be ported by the compact subset K of Ω is for every open set $V, K \subset V \subset \Omega$, there exists $C(V) > 0$ such that

$$p(f) \leq C(V)\|f\|_V$$

for all f in $H(\Omega)$.

The τ_ω topology on $H(\Omega)$ is the topology generated by the τ_ω-continuous semi-norms. A semi-norm p on $H(\Omega)$ is said to be τ_δ-continuous if for every increasing open cover of $\Omega, (V_n)_{n=1}^{\infty}$, there exists a positive integer n_0 and $C > 0$ such that

$$p(f) \leq C\|f\|_{V_{n_0}}$$

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for all $f \in H(\Omega)$.

The τ_δ topology is the topology generated by all τ_δ-continuous semi-norms on $H(\Omega)$. We always have $\tau_0 \leq \tau_\omega \leq \tau_\delta$.

We let $P({}^nE)$ denote the (vector) subspace of $H(E)$ consisting of all (continuous) n-homogeneous polynomials. By [2, proposition 2.41] τ_ω and τ_δ induce the same topology on $P({}^nE)$ for all n.

We shall need the following result which can be easily deduced form [2, definition 3.32, the remarks following this definition and proposition 3.36].

Proposition 1. *Let E denote a locally convex space and suppose $(H(E), \tau_0)$ is complete. The following are equivalent:*

(a) $(H(E), \tau_\delta)$ is complete,

(b) $(H(E), \tau_\omega)$ is complete,

(c) $(P({}^nE), \tau_\omega)$ is complete for all n.

2. HOLOMORPHIC FUNCTIONS ON C^I

A function $f : C^I \to C$ is said to depend on finitely many variables if there exists a finite subset J of I such that

$$f((x_i)_{i\in I}) = f((y_i)_{i\in I})$$

whenever $x_i = y_i$ for all i in J. By Liouville's theorem every element of $H(C^I)$ depends on finitely many variables and a Gâteaux holomorphic function on C^I is holomorphic if and only if it depends on finitely many variables. On $H(C^I)$ (see [1]) we have $\tau_0 < \tau_\omega < \tau_\delta$. Let $I^{(N)} = \{(m_i)_{i\in I}; m_i \in Z^+ \text{ and } m_i = 0 \text{ for all except a finite number of } i\}$. For $a \in C$ we let $a^0 = 1$. For $m = (m_i)_{i\in I} \in I^{(N)}$ we denote by z^m the $|m| = \sum_i |m_i|$-homogeneous polynomial which maps

$$(z_i)_{i\in I} \quad to \quad \prod_{i\in I} z_i^{m_i}.$$

If P is an n-homogeneous polynomial on C^I then, since P depends on finitely many variables, there exists a set of scalars, $(a_m)_{m\in I^{(N)}}$, with $a_m = 0$ for all but a finite number of elements of $I^{(N)}$ such that

$$P((z_i))_{i\in I} = \sum_{m\in I^{(N)}} a_m z^m.$$

Now let p denote τ_ω-continuous semi-norm on $P({}^n(C^I))$. If $b_m = p(z^m)$ for all m in $I^{(N)}$ then

$$p\left(\sum_{m\in I^{(N)}} a_m z^m\right) \leq \sum_{m\in I^{(N)}} |a_m| b_m$$

for all $\sum_{m\in I^{(N)}} a_m z^m$ in $P(^n(C^I))$. Let

$$q\left(\sum_{m\in I^{(N)}} a_m z^m\right) = \sum_{m\in I^{(N)}} |a_m| b_m$$

and, for each finite subset F of $I^{(N)}$, let

$$q_F\left(\sum_{m\in I^{(N)}} a_m z^m\right) = \sum_{\substack{m\in I^{(N)}\\ m\in F}} |a_m| b_m$$

Clearly, by the Cauchy inequalities q_F is a τ_0-continuous semi-norm, q is always finite since each polynomial has only a finite number of non-zero terms and

$$q = \sup_F q_F.$$

Since τ_ω is a barrelled topology on $P(^n(C^I))$ ([2, p. 24]) q is a τ_ω-continuous semi-norm on $P(^n(C^I))$.

We summarize the above in the following proposition:

Proposition 2. *If p is a τ_ω-continuous semi-norm on $P(^n(C^I))$ then there exists a τ_ω-continuous semi-norm q on $P(^n(C^I))$ and a collection of τ_0-continuous semi-norms $(q_\alpha)_{\alpha\in A}$ such that:*

(i) $p \leq q$,

(ii) $q = \sup_{\alpha\in A} q_\alpha$.

3. COMPLETENESS OF $(H(C^I), \tau_0)$

Proposition 3. *$(H(C^I); \tau_0)$ is complete.*

Proof. Let $(f_\alpha)_{\alpha\in\Gamma}$ denote a Cauchy net in $(H(C^I), \tau_0)$. Since the Banach space $C(K)$, K compact, with the supremum norm is complete there exists a function f on C^I, continuous on compact subset of C^I, such that $f_\alpha \to f$ as $\alpha \to \infty$, uniformly on compact sets. Since $f_\alpha \to f$ uniformly on the finite dimensional compact subsets of C^I and each f_α is holomorphic it follows that f is Gâteaux holomorphic. Hence, to complete the proof we must show that f is continuous. By our remarks in §2 this is equivalent to showing that f depends on a finite number of variables. Suppose otherwise. Let J_1 denote a non-empty finite subset of I. Then there exist $x' = (x'_i)_{i\in I}$ and $y' = (y'_i)_{i\in I}$ in C^I such that $f(x'+y') \neq f(x')$ and $y'_i = 0$ for $i \in J_1$. Let $\delta = |f(x'+y') - f(x')|$ and let $K_1 = \{(\omega_i)_{i\in I}; |\omega_i| \leq |x'_i| + |y'_i| \text{ for all } i \text{ in } I\}$. Then K_1 is a compact subset of C^I, x' and $x'+y'$ belong to K_1. Now choose $\alpha \in \Gamma$ such that

$$\|f - f_\alpha\|_{K_1} \leq \delta/8.$$

Since f_α is holomorphic it depends on a finite number of variables I_1. Let

$$\widetilde{x}'_i = \begin{cases} x'_i & \text{if } i \in I_1 \cup J_1, \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\widetilde{y}'_i = \begin{cases} y'_i & \text{if } i \in I_1 \cup J_1, \\ 0 & \text{otherwise} \end{cases}$$

Then $\widetilde{x}'$ and $\widetilde{x}' + \widetilde{y}'$ belong to K_1 and since x' and $\widetilde{x}'$ agree on I_1 and y' and $\widetilde{y}'$ agree on I_1 we have

$$f_\alpha(x'+y') = f_\alpha(\widetilde{x}'+\widetilde{y}') \quad \text{and} \quad f_\alpha(x') = f_\alpha(\widetilde{x}').$$

Hence

$$\begin{aligned} |f(\widetilde{x}'+\widetilde{y}') - f(\widetilde{x}')| &\geq |f(x'+y') - f(x')| - |f(\widetilde{x}'+\widetilde{y}') - f_\alpha(\widetilde{x}'+\widetilde{y}')| \\ &- |f_\alpha(\widetilde{x}'+\widetilde{y}') - f_\alpha(x'+y')| - |f_\alpha(x'+y') - f(x'+y')| \\ &- |f(x') - f_\alpha(x')| - |f_\alpha(x') - f_\alpha(\widetilde{x}')| - |f_\alpha(\widetilde{x}') - f(\widetilde{x}')| \geq \delta/2 \end{aligned}$$

both $\widetilde{x}'$ and $\widetilde{y}'$ have their support in $I_1 \cup J_1$ and $\widetilde{y}'_i = 0$ if $i \in J_1$. Let $J_2 = I_1 \cup J_1$. Using the same method we can find a finite subset I_2 of I and vectors $\widetilde{x}^2$ and $\widetilde{y}^2$ with support in $I_2 \cup J_2$ such that

$$f(\widetilde{x}^2 + \widetilde{y}^2) \neq f(\widetilde{x}^2) \quad \textit{and} \quad \widetilde{y}^2_i = 0 \quad \textit{if} \quad i \in J_2.$$

By induction we can generate an increasing sequence of finite subset of I, $(J_n)_{n=1}^\infty$, and sequences of vectors $(\widetilde{x}^n)$ and $(\widetilde{y}^n)$ in C^I such that

(i) $f(\widetilde{x}^n + \widetilde{y}^n) \neq f(\widetilde{x}^n)$ for all n,
(ii) $\widetilde{x}^n$ and $\widetilde{y}^n$ have their support in J_{n+1},
(iii) $\widetilde{y}^n_i = 0$ if $i \in J_n$.

Let $J = \cup_n J_n$. We now restrict all f_α and f to the Fréchet space $C^J \times 0^{I\setminus J}$. Since $f_\alpha|_{C^J\times 0^{I\setminus J}} \to f|_{C^J\times 0^{I\setminus J}}$ uniformly on compact sets it follows that $f|_{C^J\times 0^{I\setminus J}}$ is holomorphic and hence depends on a finite number of variables in J. This is impossible, however, by (i), (ii) and (iii), since any finite subset of J is contained in some J_n. This completes the proof.

4. COMPLETENESS FOR THE τ_ω and τ_δ TOPOLOGIES

Proposition 4. *$(H(C^I);\tau_\omega)$ and $(H(C^I);\tau_\delta)$ are complete locally convex spaces.*

Proof. By propositions 1 and 3 it suffices to show that $(P({}^n(C^I)),\tau_\omega)$ is complete for all n. Let $(P_\alpha)_{\alpha\in\Gamma}$ denote a τ_ω Cauchy net in $(P({}^n(C^I)),\tau_\omega)$. Since $\tau_\omega \geq \tau_0$, proposition 3 implies that there exists a polynomial P in $P({}^n(C^I))$ such that $P_\alpha \to P$ in $(P({}^n(C^I)),\tau_0)$ as $\alpha \to \infty$. Let p denote a τ_ω-continuous semi-norm on $P({}^n(C^I))$. By proposition 2 we may suppose in the following argument that

$$p = \sup_{\beta\in B} p_\beta$$

where each p_β is a τ_0-continuous semi-norm and B is some indexing set. Given $\varepsilon > 0$ there exists $\alpha_0 \in \Gamma$ such that $p(P_{\alpha_1} - P_{\alpha_2}) \leq \varepsilon$ for all $\alpha_1, \alpha_2 \geq \alpha_0$. Hence $p_\beta(P_{\alpha_1} - P_{\alpha_2}) \leq \varepsilon$ for all $\beta \in B$ and all $\alpha_1, \alpha_2 \geq \alpha_0$. Since p_β is τ_0-continuous and $P_\alpha \to P$ as $\alpha \to \infty$ in the compact open topology we have

$$p_\beta(P_\alpha - P) \leq \varepsilon \text{ for all } \beta \in B \text{ and all } \alpha \geq \alpha_0.$$

Hence

$$p(P_\alpha - P) = \sup_{\beta\in B} p_\beta(P_\alpha - P) \leq \varepsilon$$

and $P_\alpha \to P$ in $(P({}^n(C^I)),\tau_\omega)$ as $\alpha \to \infty$. This completes the proof.

5. BALANCED DOMAINS IN $\mathbb{C}^I$

If U is a balanced open subset of a locally convex space E and τ is a locally convex topology on $H(U)$ then $(H(U),\tau)$ is said to be T.S. (Taylor series) complete if for any sequence $(P_n)_{n=0}^\infty$, $P_n \in \mathrm{P}({}^nE)$ all n, $\sum_{n=0}^\infty p(P_n) < \infty$ for every τ-continuous seminorm p implies $\sum_{n=0}^\infty P_n \in H(U)$ [2, p. 128]. The hypothesis in proposition 1 are used to show that $(H(E),\tau_0)$ is T.S. complete and from this it follows that $(H(E),\tau_\omega)$ and $(H(E),\tau_\delta)$ are also T.S. complete. Now, if U is a balanced open subset of C^I and $(P_n)_{n=0}^\infty$ is a sequence of continuous polynomials, $P_n \in \mathrm{P}({}^nE)$ all n, then since each polynomials only depends on finitely many variables the sequence $(P_n)_{n=0}^\infty$ only depends on countably many variables and hence, using the fact that C^N, N countable, is a Fréchet space we see that $(H(U),\tau_0)$ is T.S. complete for any balanced open subset U of E. Propositions 3 and 4 thus imply the following.

Proposition 5. *If* U *is a balanced domain in* $\mathbb{C}^I$ *then* $(H(U), \tau)$ *is complete for* $\tau = \tau_0, \tau_\omega$ *and* τ_δ.

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