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A REMARK ON LOCALLY DETERMINING SEQUENCES IN INFINITE DIMENSIONAL SPACES

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Abstract. *We obtain a real version of a result of J. M. Ansemil and S. Dineen [2], i.e., every locally determining at* 0 *set for analytic functions in a metrizable separable real locally convex space contains a locally determining at* 0 *null sequence.*

Chmielowski and Lubczonok in [6] proved that every locally determining at 0 set in $\mathbb{C}^n$ contains a locally determining at 0 null sequence. Chmielowski [5] also asked if this was true for metrizable separable locally convex spaces. In [2] Ansemil and Dineen solve this question for holomorphic functions. In this note we give an affirmative answer for analytic functions.

Thanks are given to J. M. Ansemil who asked the authors about this case.

We refer to Köthe [8] for the theory of locally convex spaces and to Dineen [7] for the theory of infinite dimensional holomorphy.

Definition 1. *Let* E *be a metrizable complex locally convex space. Let* $(V_n)_{n\in\mathbf{N}}$ *be a decreasing basis of balanced open neighbourhoods of* 0. *For every* $n \in \mathbf{N}$, *let* $\mathscr{H}_n$ *be a* τ_o*-closed subspace of* $\mathscr{H}^\infty(V_n)$ *such that* $\mathscr{H}_n|_{V_m} \subset \mathscr{H}_m$, *for* $m \geq n$. *We let* $H := (\mathscr{H}_n)_{n\in\mathbf{N}}$.

A subset L *of* E *is said to be* H*-locally determining at the origin if for any connected open subset* U *of* E *with* $0 \in U$ *and for any* $f \in \mathscr{H}(U)$ *such that* $f|_{V_n} \in \mathscr{H}_n$ *for some* $n \in \mathbf{N}$, *we have that* $f|_{U\cap L} \equiv 0$ *implies* $f \equiv 0$.

Theorem 2. *Let* E *be a separable metrizable complex locally convex space. Let* $H := (\mathscr{H}_n)_{n\in\mathbf{N}}$ *be as in Definition 1. If* L *is an* H*-locally determining at* 0 *subset of* E, *then there is a null sequence in* L *which is also* H*-locally determining at* 0.

Proof. We follow the pattern of proof in Ansemil and Dineen [2, Th. 3].

By Montel's theorem the set $B_n := \{f \in \mathscr{H}^\infty(V_n) : \|f\|_{V_n} \leq 1\}$ is τ_o-compact, hence $C_n := \{f \in \mathscr{H}_n : \|f\|_{V_n} \leq 1\} = B_n \cap \mathscr{H}_n$ is τ_o-compact.

Now, by a result of Ng [9], the space $F_n := \{\Phi \in (\mathscr{H}_n, \|\cdot\|_{V_n})' : \Phi|_{C_n}$ is τ_o − continuous$\}$ endowed with the norm $\|\Phi\|_n' := \sup\{|\Phi(f)| : f \in \mathscr{H}_n, \|f\|_{V_n} \leq 1\}$ is a Banach space and

$$(F_n')_\beta = (F_n, \|\cdot\|_n')' = \mathscr{H}_n,\ n = 1, 2, \dots .$$

For every $x \in V_n$, the evaluation mapping at x

$$\delta_x : \mathscr{H}^\infty(V_n) \longrightarrow \mathbb{C}$$
$$f \longrightarrow \delta_x(f) := f(x)$$

is τ_o-continuous, so $\widetilde{\delta}_x : \mathscr{H}_n \longrightarrow \mathbb{C}$, $\widetilde{\delta}_x := \delta_x|_{\mathscr{H}_n}$, is τ_o-continuous, therefore $\widetilde{\delta}_x \in F_n$.

Since E is metrizable and separable there exists in V_n a dense sequence $(x_{n,m})_{m\in\mathbf{N}}$.

Let us show that $(F_n, \|.\|'_n)$ is a separable space. If $f \in \mathscr{H}_n$ and $< f, \widetilde{\delta}_{x_{n,m}} >= 0$ for every $m \in \mathbf{N}$ then $f(x_{n,m}) = 0$ for every $m \in \mathbf{N}$, hence $f \equiv 0$. This mean that $(\widetilde{\delta}_{x_{n,m}})_{m\in\mathbf{N}}$ separates points of $\mathscr{H}_n$, so it is $\sigma(F_n, \mathscr{H}_n)$-total and therefore is $\mu(F_n, \mathscr{H}_n) = \beta(F_n, \mathscr{H}_n) = \|.\|'_n$-total.

For every $n \in \mathbf{N}$, select a dense sequence $(w_{n,m})_{m\in\mathbf{N}}$ in F_n. Using the identity theorem for open sets we see that, for every $n \in \mathbf{N}$, the set $\{\widetilde{\delta}_x : x \in L \cap V_k\}$ spans a dense subspace of F_n for each integer $k \geq n$.

For each pair of positive integers (n,k), $k \geq n$, choose a finite sequence of points in $L \cap V_k$, $x_{n,kj}$, $1 \leq j \leq \alpha(n,k)$ such that

$$\sup_{1\leq m\leq k} \mathrm{d}_n(w_{n,m}, F_{n,k}) \leq 1/k,$$

where $F_{n,k}$ is the subspace of F_n spanned by $\{\widetilde{\delta}_{x_{n,kj}}, 1 \leq j \leq \alpha(n,k)\}$ and d_n is the distance defined by the norm on F_n.

Let

$$L^* := \bigcup_{\substack{(n,k)\\ k\geq n\\ 1\leq j\leq \alpha(n,k)}} \{x_{n,kj}\}$$

By construction L^* is a subset of L and $|\{\xi \in L^* : \xi \notin V_n\}| < \infty$ for each $n \in \mathbf{N}$. Hence L^* can be rearranged to form a null sequence in E. If f is a holomorphic function defined on a connected open subset U of E with $0 \in U$ such that $g := f|_{V_n} \in \mathscr{H}_n$ and $f|_{L^*\cap V_n} \equiv 0$ for some $n \in \mathbf{N}$, then $< F_{n,k}, g >= 0$ for each positive integer $k \geq n$. Since $\cup_{k\geq n} F_{n,k}$ spans a dense subspace of F_n it follows that $< F_n, g >= 0$ and hence $g \equiv 0$. Since U is connected the identity theorem for open sets implies that $f \equiv 0$. This completes the proof. ∎

For every real locally convex space E we consider its complexification $E + iE$.

The following Proposition is due to A. Alexiewicz and W. Orlicz [1] for real Banach spaces. (See also [3], [4]). Since analytic functions on locally convex spaces factor locally through normed linear spaces and we can extend locally to the completion of normed linear spaces, Proposition 3 can be deduced from Theorem 5.7 of Alexiewicz and Orlicz. For the sake of completness we include a proof.

Proposition 3. *Let E be a real locally convex space. Let $U \subset E$ denote an open set with $0 \in U$. For each $f \in \mathcal{A}(U)$ (analytic function) there are an absolutely convex open neighbourhood of 0 $V \subset U$ and $\hat{f} \in \mathcal{H}^\infty(V + iV)$ such that $\hat{f}(x) = f(x)\ \forall x \in V$.*

Proof. Since f is analytic there are polynomials $P_n \in \mathcal{P}({}^nE; \mathbf{R})$, $n = 0,1,2,\ldots$ and an absolutely convex open neighbourhood of 0, W, contained in U such that

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

uniformly on W. We may also suppose that f is bounded on W. Select $p \in \mathbf{N}$ such that

$$|f(x) - \sum_{n=0}^{q} P_n(x)| < 1,\ \forall x \in W,\ \forall q \geq p.$$

For each $j = 1,2,\ldots,p$ choose an absolutely convex open neighbourhood of $0, W_j$, such that $\|P_j\|_{W_j} := \sup\{|P_j(x)| : x \in W_j\} < \infty$. Putting $V := \frac{1}{4e}(W \cap W_1 \cap \ldots \cap W_p)$ we have that

$$\|P_n\|_{4eV} < \max\{|P_o(0)|, \|P_1\|_{W_1}, \ldots, \|P_p\|_{W_p}, 2\} := M < \infty,\ n = 0,1,2,\ldots$$

Let $A_n (n \geq 1)$ denote the unique real n-linear symmetric form associated to P_n, and $\hat{A}_n$ denote the complexification of A_n defined on $E + iE$. Define $\hat{P}_n(z) := \hat{A}_n(z,\ldots,z)$, $\forall z \in E + iE$. Since

$$\|\hat{P}_n\|_{V+iV} \leq \|\hat{A}_n\|_{V+iV} \leq 2^n\|A_n\|_V \leq 2^n\frac{n^n}{n!}\|P_n\|_V = \frac{2^n n^n}{n! 4^n e^n}\|P_n\|_{4eV} < \frac{CM}{2^n},$$

$n = 1,2,\ldots$, where $C > 0$ satisfies $\frac{n^n}{n!\, e^n} \leq C$ for every $n \in \mathbf{N}$, it follows that $\hat{P}_n \in \mathcal{P}({}^n(E + iE); \mathbb{C})$ and that the series

$$\hat{f}(z) := \sum_{n=0}^{\infty} \hat{P}_n(z)$$

converges absolutely and uniformly on $V + iV$. This means that $\hat{f} \in \mathcal{H}^\infty(V + iV)$ and clearly $\hat{f}(x) = f(x)$, $\forall x \in V$. This completes the proof. ∎

Observation 4. If $f, g \in \mathscr{H}(V + iV)$, where V is an absolutely convex neighbourhood of 0 in a metrizable real locally convex space E, and f and g coincide on V then f is equal to g. Indeed, if P is a continuous n-homogeneous polynomial on $E + iE$ which is null on V we have that for each x in E the mapping $\lambda \in \mathbb{C} :\longrightarrow P(\lambda x)$ is an n-homogeneous polynomial of one complex variable whose restriction to $\mathbb{R}$ is zero. By the fundamental theorem of algebra $P \equiv 0$.

Theorem 5. *Let E be a metrizable separable real locally convex space. If $L \subset E$ is a locally determining at 0 set for analytic functions, then L contains a null sequence which is also locally determining at 0 for analytic functions.*

Proof. Let $(V_n)_{n\in\mathbb{N}}$ be a basis of absolutely convex open neighbourhoods of 0 with $V_{n+1} \subset \frac{1}{2}V_n$, for every $n \in \mathbb{N}$. The complexification of E, $E + iE$, is a metrizable separable complex locally convex space for which the sequence $(W_n)_{n\in\mathbb{N}}$ is a basis of neighbourhoods of 0, where W_n denotes the balanced hull of $V_n + iV_n$.

For a given balanced open neighbourhood of 0, W, in $E + iE$, we put $\mathscr{R}(W) := \{f \in \mathscr{H}^\infty(W) :$ the m-homogeneous Taylor polynomials of f at 0, $(P_m)_{m=0}^\infty$, satisfy $P_m(x) \in \mathbb{R}, \forall x \in E\}$.

Let $\mathscr{H}_n := \{h \in \mathscr{H}^\infty(W_n) : \exists W \subset W_n$, balanced open neighbourhood of 0, and $\exists f, g \in \mathscr{R}(W)$ such that $h(z) = f(z) + ig(z), \forall z \in W\}$.

Clearly $\mathscr{H}_n$ is a subspace of $\mathscr{H}^\infty(W_n)$ and $\mathscr{H}_{n|W_{n+1}} \subset \mathscr{H}_{n+1}$, for all $n \in \mathbb{N}$.

We proceed to prove that $\mathscr{H}_n$ is τ_o-closed in $\mathscr{H}^\infty(W_n)$. Given a net $(h_\lambda)_{\lambda\in\Lambda} \subset \mathscr{H}_n$ τ_o-convergent to $h \in \mathscr{H}^\infty(W_n)$ we have, for a fixed $x \in E$, that $(h_\lambda)_{\lambda\in\Lambda}$ converges uniformly to h on the compact set

$$\Delta := \{t\alpha x : |t| \leq 1, t \in \mathbb{C}\}$$

where $\alpha > 0$ is such that $\alpha x \in V_n$. If R_m^λ (resp. R_m) denotes the m-homogeneous Taylor polynomial at 0 of h_λ (resp. h) we have

$$(1) \qquad ||R_m^\lambda - R_m||_\Delta \leq ||h_\lambda - h||_\Delta, \; m = 0, 1, 2, \ldots,$$

by the Cauchy inequalities. Since $h_\lambda \in \mathscr{H}_n$, by definition there are W^λ and $f_\lambda, g_\lambda \in \mathscr{R}(W^\lambda)$ with $h_\lambda(z) = f_\lambda(z) + ig_\lambda(z)$, $\forall z \in W^\lambda$. Denoting by P_m^λ (resp. Q_m^λ) the m-homogeneous Taylor polynomial at 0 of f_λ (resp. g_λ) we have $R_m^\lambda = P_m^\lambda + iQ_m^\lambda$ in $E + iE$. Now, it follows from (1) that

$$\alpha^m R_m(x) = R_m(\alpha x) = \lim_\lambda R_m^\lambda(\alpha x) = \lim_\lambda (P_m^\lambda(\alpha x) + iQ_m^\lambda(\alpha x)) =$$

$$= \alpha^m (\lim_\lambda P_m^\lambda(x) + i \lim_\lambda Q_m^\lambda(x)),$$

hence $R_m(x) = \lim_\lambda P_m^\lambda(x) + i \lim_\lambda Q_m^\lambda(x)$..

Put $P_m(x) := \lim_\lambda P_m^\lambda(x)$, and $Q_m(x) := \lim_\lambda Q_m^\lambda(x)$. P_m and Q_m are m-homogeneous polynomials on E. Since $||P_m||_{V_n} \leq ||R_m||_{V_n} \leq ||R_m||_{W_n} \leq ||f||_{W_n} < \infty$, the series $f := \sum_{m=0}^{\infty} P_m$ converges uniformly on $\frac{1}{2}V_n$ and $f \in \mathcal{A}(\frac{1}{2}V_n)$. Analogously $g := \sum_{m=0}^{\infty} Q_m \in \mathcal{A}(\frac{1}{2}V_n)$.

Applying Proposition 3, we can find an absolutely convex open neighbourhood of 0 , Y, in E with $Y \subset \frac{1}{2}V_n$ and $\widehat{f}, \widehat{g} \in \mathscr{H}^\infty(Y + iY)$ such that $\widehat{f}(x) = f(x)$ and $\widehat{g}(x) = g(x)$, for all $x \in Y$. Thus $\widehat{f}|_W, \widehat{g}|_W \in \mathscr{R}(W)$ for a balanced open neighbourhood of 0, W, contained in $Y + iY$. Since $h(x) = \widehat{f}(x) + i\widehat{g}(x)$, $\forall x \in Y$, Observation 4 yields $h \in \mathscr{H}_n$.

We claim that L is $H := (\mathscr{H}_n)_{n\in\mathbf{N}}$-locally determining at 0.

Let $U \subset E + iE$ be a connected open set with $0 \in U$. Consider $h \in \mathscr{H}(U)$ such that $h|_{W_n} \in \mathscr{H}_n$ for some $n \in \mathbf{N}$ and $h|_{U\cap L} \equiv 0$ (we may suppose that $W_n \subset U$).

There exist á balanced open neighbourhood of 0 , W, in $E+iE$, and $f, g \in \mathscr{R}(W)$ such that $h(z) = f(z) + ig(z)$, $\forall z \in W$. Select T an absolutely convex open neighbourhood of 0 in E such that $T + iT \subset W$. For all $x \in T \cap L$, $f(x) + ig(x) = h(x) = 0$, hence $f(x) = g(x) = 0$. Since L is H-locally determining at 0 for analytic functions and $f|_T, g|_T \in \mathcal{A}(T)$, it follows that $f|_T = g|_T \equiv 0$, hence $h|_T \equiv 0$. Again, by Observation 4, $h|_{T+iT} \equiv 0$. So $h \equiv 0$.

Theorem 2 ensures that there is a null sequence $L^* := (x_k)_{k\in\mathbf{N}} \subset L$ which is H-locally determining at 0. Finally we will see that L^* is locally determining at 0 for analytic functions.

Suppose $V \subset E$ a connected open set with $0 \in W$ and $f \in \mathcal{A}(W)$ such that $f|_{L^*\cap W} \equiv 0$. Applying Proposition 3 we can find $V_n \subset V$ and a function $\widehat{f} \in \mathscr{H}^\infty(V_n + iV_n)$ such that $\widehat{f}(x) = f(x)$, for all $x \in V_n$. Moreover, since $W_{n+1} \subset V_n + iV_n$ it follows easily that $\widehat{f}|_{W_{n+1}} \in \mathscr{H}_{n+1}$.

As L^* is H-locally determining at 0 and $\widehat{f}|_{L^*\cap(V_n+iV_n)} = \widehat{f}|_{L^*\cap V_n} = f|_{L^*\cap V_n} \equiv 0$, then $\widehat{f}|_{V_n+iV_n} \equiv 0$, hence $f|_{V_n} \equiv 0$. Thus $f \equiv 0$ on W by the identity theorem. This completes the proof. ∎

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