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AN INTEGRAL FORMULA FOR CONVEX SETS IN THE EUCLIDEAN PLANE

C. PERI

Abstract. *Let K be a bounded, closed, convex region in the Euclidean plane. The main purpose of this paper is to determine the measure of all sets congruent to K which are contained in a convex polygon. By the application of this result some geometric probability problems involving lattices of figures will be solved.*

1. INTRODUCTION

Let K, K_0 be two bounded, closed, convex sets in the Euclidean plane of area S, S_0 and perimeter L, L_0 respectively.

We denote by $\mu(K; K \cap K_0 \neq \emptyset)$ the measure of all sets congruent to K which intersect K_0. L. A. Santaló has shown that [2]

$$\mu(K; K \cap K_0 \neq \emptyset) = 2\pi(S + S_0) + LL_0.$$

It may be a more difficult problem to find a general formula representing the measure of all sets congruent to K which are contained in K_0. However such a measure is known in the case where the boundaries of K and K_0 have continuous radii of curvature, so that the greatest radius of the boundary of K is equal to or less than the least radius of the boundary of K_0, [2].

Moreover M. J. Stoka has determined the measure of all circles with a given radius which are contained in a parallelogram K_0, [4]. Stoka's formula has been extended by A. Vassallo to the general case where K_0 is a convex polygon, [5].

Here we assume that K is a generic bounded, closed, convex set and K_0 is a convex polygon in the plane. Then we derive an integral formula representing the measure of all sets congruent to K which are contained in K_0 (formula 3). This result enables us to calculate such a measure when K_0 is a parallelogram and K is either a convex set of constant breadth or a regular convex polygon (formulae 6 and 7).

These formulae have a natural application in geometric probability, especially in connection with problems related to lattices of figures, where the convex polygon K_0 is identified as the fundamental cell of a given lattice. This will be discussed in section 3 where we shall consider in detail lattices of parallelograms.

2. «CONVEX SET » SHALL HEREINAFTER MEAN «PLANE BOUNDED, CLOSED, CONVEX SET WITH INTERIOR POINTS»

The position of a convex set K in the Euclidean plane is determined by the position of a point $P(x, y)$ fixed in K and the angle φ formed by a direction r, rigidly associated to K, and a fixed direction of the plane. Then in order to measure sets of convex sets congruent to K we can introduce the *kinematic density* as

$$\mathrm{d}K = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\varphi. \tag{1}$$

This measure is chosen as being the only measure, up to a constant factor, which remains invariant under the group of the motions in the plane, [2].

Let us consider a convex set K having a boundary ∂K of length L. We recall that a *support line* of K is a line which intersects ∂K and does not cut K. Such a line can be coordinatized as follows. Let us draw the perpendicular straight line from P, let p be the distance and θ be its angle with the direction r. Then a support line is uniquely determined by (θ, p). Clearly, if we consider any support line, p is a function of θ. This function $p(\theta)$ is called the *support function* of K.

It follows, from Cauchy's formula [2], that:

$$\int_0^{2\pi} p(\theta)\,\mathrm{d}\theta = L. \tag{2}$$

We shall now determine the measure of all sets congruent to K which are contained in a convex polygon K_0. Naturally it will be assumed throughout that K can be contained in K_0.

Theorem 1. *Let K_0 be a convex n-gon of area S_0 and perimeter L_0. Let K be a convex set of area S and perimeter L which never intersects two nonconsecutive sides of K_0 if $n > 3$ or all the sides of K_0 if $n = 3$. Denote by $\mu(K; K \subset K_0)$ the measure of all sets congruent to K which are contained in K_0. Then*

$$\begin{aligned}\mu(K; K \subset K_0) &= 2\pi S_0 - L_0 L + \sum_{i=1}^{n}(1/\sin\alpha_i)\int_0^{2\pi} p(\theta_i)p(\theta_{i-1})\,\mathrm{d}\varphi + \\ &+ (1/2)\sum_{i=1}^{n}(\mathrm{cotg}\,\alpha_i + \mathrm{cotg}\,\alpha_{i+1}) + \int_0^{2\pi} p^2(\theta_i)\,\mathrm{d}\varphi,\end{aligned} \tag{3}$$

where, for $i = 1, \ldots, n$, α_i are the interior angles of K_0 and $(\theta_i, p(\theta_i))$ are the coordinates of the support lines of K parallel to the sides of K_0.

Proof. We have to integrate with respect to the density (1) over all values x, y, φ corresponding to the positions of K for which K is contained in K_0. By fixing φ, we have

$$\mu(K; K \subset K_0) = \int_0^{2\pi} S_0(\varphi)\,\mathrm{d}\varphi,$$

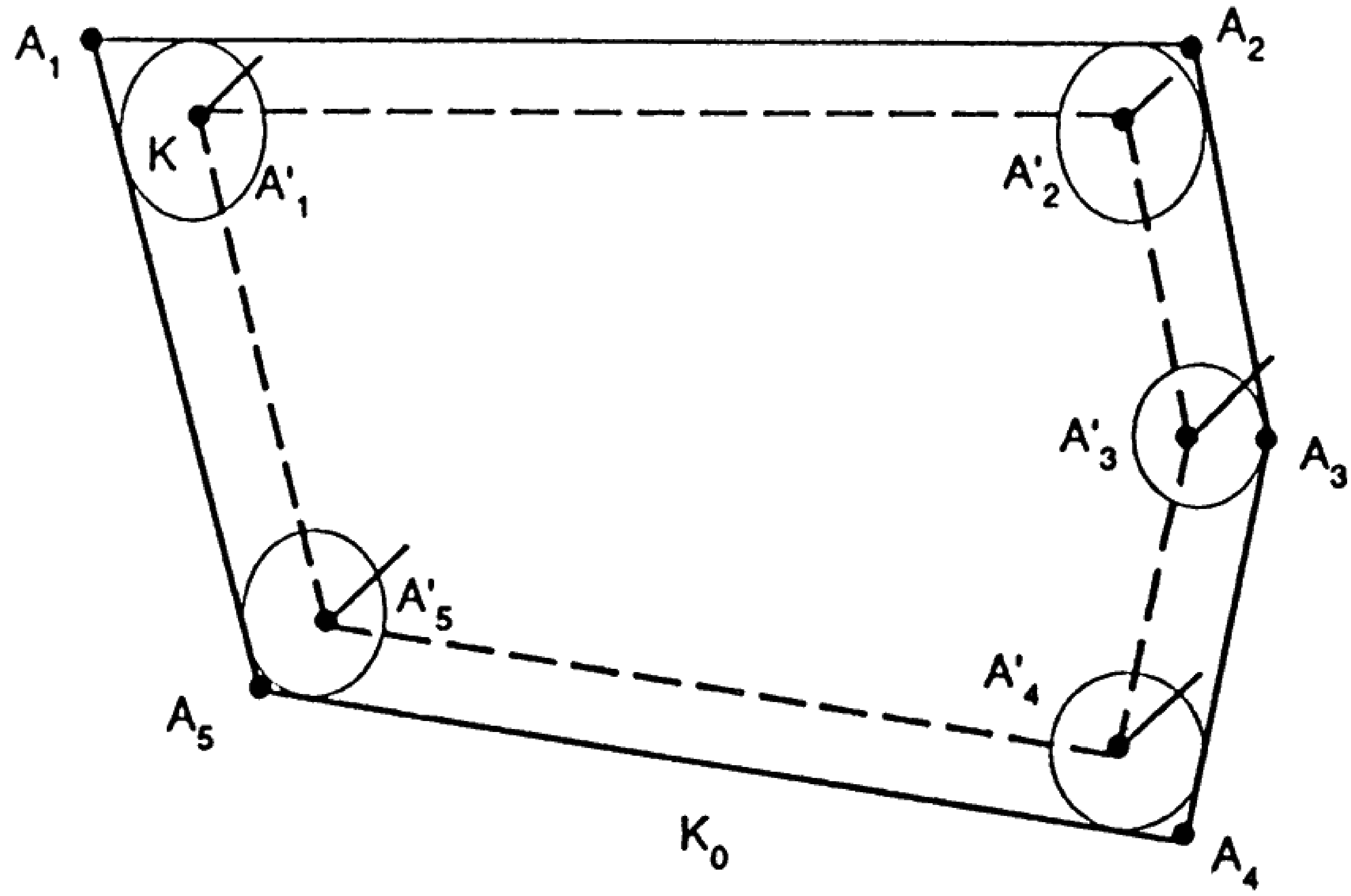

Figure 1

where $S_0(\varphi)$ denotes the area of the domain drawn by the position point P of K when K moves by translation inside K_0.

Let $A_i, i = 1, 2, \ldots, n$, be the vertices of K_0 and let ℓ_i be the length of the side with endpoints A_i, A_{i+1}, where A_{n+1} is interpreted as A_1.

Let us consider the positions of K, corresponding to a fixed orientation φ, for which K is contained in K_0 and touches two consecutive sides of K_0 as shown in figure 1.

We denote by A'_i the position of P for which K touches the sides $\overline{A_{i-1}A_i}$ and $\overline{A_iA_{i+1}}$, where A_0 is interpreted as A_n. The points A'_i are the vertices of a convex polygon K'_0 having sides parallel to the sides of K_0.

Then, for fixed φ, the convex set K is contained in K_0 if and only if P falls in K'_0. Thus $S_0(\varphi)$ coincides with the area of K'_0.

In order to calculate $S_0(\varphi)$ we consider the points B_i and C_i determined on the boundary of K_0 by the sides of K'_0 drawn from A'_i, as shown in figure 2.

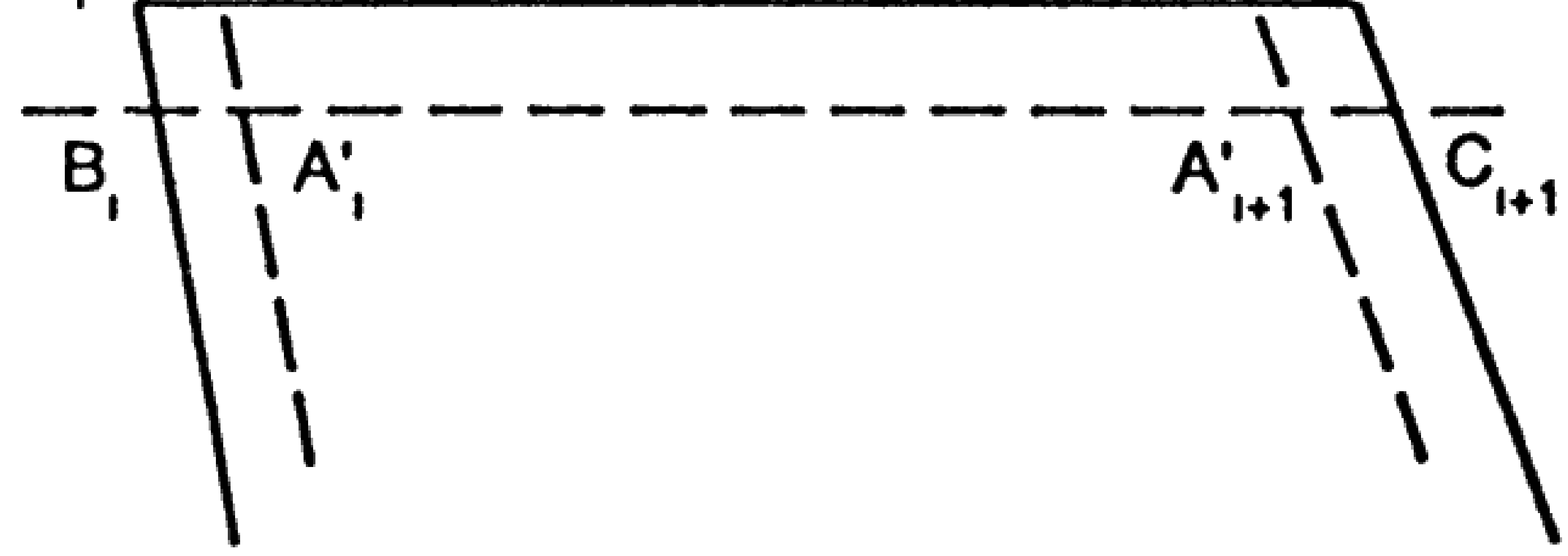

Figure 2

Let $\mathcal{I}_i$ be the trapezium of vertices $A_iA_{i+1}C_{i+1}B_i$ and let $S(\mathcal{I}_i)$ be the area of $\mathcal{I}_i$. Moreover let $\mathcal{P}_i$ be the parallelogram of vertices $A_iA'_iC_iB_i$ and let $S(\mathcal{P}_i)$ be the area of $\mathcal{P}_i$. Then

$$S_0(\varphi) = S_0 - \sum_{i=1}^{n} S(\mathcal{I}_i) + \sum_{i=1}^{n} S(\mathcal{P}_i).$$

Let us denote by $p(\theta)$ the support function of K and by $(\theta_i, p(\theta_i))$ the coordinates of the support lines of K parallel to the sides of K_0. Then the distance between the sides $\overline{A_iA_{i+1}}$ and $\overline{A'_iA'_{i+1}}$ is given by $p(\theta_i)$ and the length of the side $\overline{B_iC_{i+1}}$ is $\ell_i - p(\theta_i)(\operatorname{cotg}\alpha_i + \operatorname{cotg}\alpha_{i+1})$.

Thus we have

$$S(\mathcal{I}_i) = (1/2)[2l_i - p(\theta_i)\operatorname{cotg}\alpha_i - p(\theta_i)\operatorname{cotg}\alpha_{i+1}]p(\theta_i),$$
$$S(\mathcal{P}_i) = p(\theta_i)p(\theta_{i-1})/(\sin\alpha_i),$$

so that

$$S_0(\varphi) = S_0 - \sum_{i=1}^{n} \ell_i p(\theta_i) + (1/2)\sum_{i=1}^{n} p^2(\theta_i)(\operatorname{cotg}\alpha_i + \operatorname{cotg}\alpha_{i+1}) +$$
$$+ \sum_{i=1}^{n}\{p(\theta_i)p(\theta_{i-1})/(\sin\alpha_i)\}.$$

We note that the angle θ_i can be expressed by $\theta_i = (\pi/2) + \gamma_i - \varphi$, where γ_i denotes the angle inclination of the side $\overline{A_iA_{i+1}}$ from the x axis. Hence by (2) we get

$$\int_0^{2\pi}\left[\sum_{i=1}^{n} \ell_i p(\theta_i)\right] d\varphi = \sum_{i=1}^{n} \ell_i \int_0^{2\pi} p(\psi)\, d\psi = L_0 L.$$

Finally, if we integrate $S_0(\varphi)$ with respect to $d\varphi$ over the range $[0, 2\pi]$, by the last formula we obtain the desired formula.

Remark 1. It is easily seen that

$$\sum_{i=1}^{n}\{[p(\theta_i)p(\theta_{i-1})/(\sin\alpha_i)] + (1/2)p^2(\theta_i)(\operatorname{cotg}\alpha_i + \operatorname{cotg}\alpha_{i+1})\} =$$
$$= \sum_{i=1}^{n}\{[p(\theta_i)p(\theta_{i+1})/(\sin\alpha_{i+1})] + (1/2)[p^2(\theta_i) + p^2(\theta_{i+1})]\operatorname{cotg}\alpha_{i+1}\}.$$

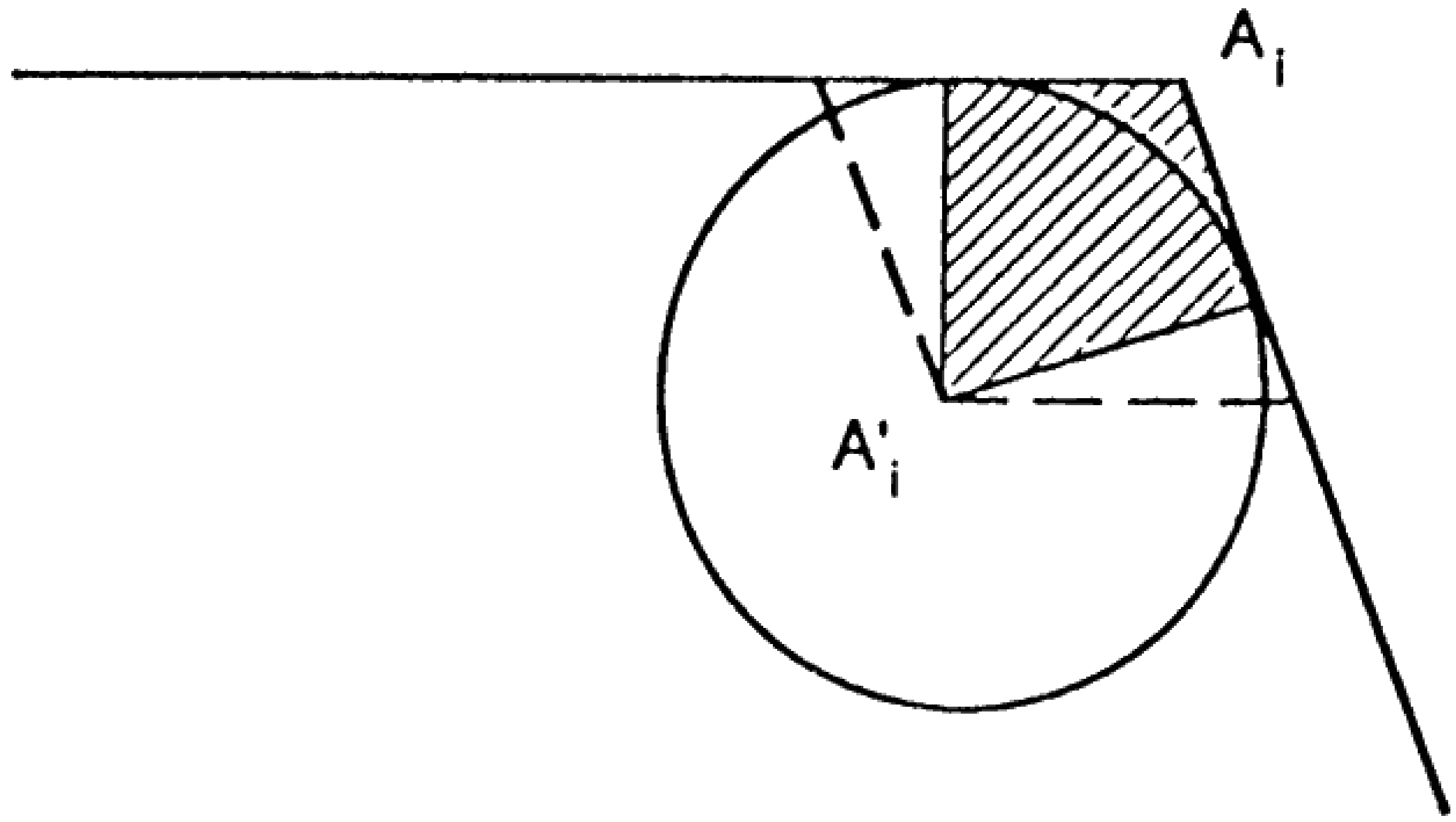

Figure 3

Therefore if we write

$$S(\varphi)=\sum_{i=1}^{n}\{[p(\theta_i)p(\theta_{i+1})/(\sin\alpha_{i+1})]+(1/2)[p^2(\theta_i)+p^2(\theta_{i+1})]\mathrm{cotg}\,\alpha_{i+1}\},$$

formula (3) becomes

$$\mu(K;K\subset K_0)=2\pi S_0-L_0L+\int_0^{2\pi}S(\varphi)\,\mathrm{d}\varphi. \tag{4}$$

Notice that the expression $[p(\theta_i)p(\theta_{i+1})/(\sin\alpha_{i+1})]+(1/2)[p^2(\theta_i)+p^2(\theta_{i+1})]\mathrm{cotg}\,\alpha_{i+1}$ represents the area of the region shaded in figure 3.

In addition we assume that K is either a convex set whose boundary does not include rectilinear components or a polygon of m sides, with $m\geq n-1$.

Then $S(\varphi)$ can be interpreted as the area of the polygon circumscribing K with sides parallel to the sides of K_0.

Next we apply formula (3) to particular classes of convex sets.

Proposition 1. *Let K_0 be a convex n-gon of area S_0 and perimeter L_0. Let K be a circle of radius R which can never intersect two nonconsecutive sides of K_0 if $n>3$ or all the sides of K_0 if $n=3$. Then*

$$\mu(K;K\subset K_0)=2\pi S_0-2\pi RL_0+2\pi R^2\sum_{i=1}^{n}\mathrm{cotg}(\alpha_i/2). \tag{5}$$

The proof follows directly from formula (3) by substituting $R=p(\theta_i)$. This result has been already obtained by S. Vassallo in [5].

Proposition 2. *Let K_0 be a parallelogram of sides a, b and angle α, with $\alpha \leq \pi/2$. Let K be a convex set of area S and perimeter L having a constant breadth D, with $D < \min(a \sin\alpha, b \sin\alpha)$. Then*

$$\mu(K; K \subset K_0) = 2\pi ab \sin\alpha - 2\pi D(a+b) + (2\pi D^2/\sin\alpha). \tag{6}$$

Proof. We recall that K is a convex set of constant breadth D if $p(\theta) + p(\theta+\pi) = D =$ constant, where $p(\theta)$ is the support function of K. Then we have $L = \pi D$ by formula (2).

In order to apply formula (3) we note that since K_0 is a parallelogram we have $\operatorname{cotg}\alpha_i + \operatorname{cotg}\alpha_{i+1} = 0$, $\sin\alpha_i = \sin\alpha$, $p(\theta_{i+2}) = p(\theta_i + \pi)$ for all i.

Thus formula (3) becomes in this case

$$\mu(K; K \subset K_0) = 2\pi ab \sin\alpha - 2\pi D(a+b) + (1/\sin\alpha) \int_0^{2\pi} \sum_{i=1}^{4} p(\theta_i)p(\theta_{i-1})\, d\varphi.$$

Moreover

$$\int_0^{2\pi} \sum_{i=1}^{4} p(\theta_i)p(\theta_{i-1})\, d\varphi = \int_0^{2\pi} Dp(\theta_1)\, d\varphi + \int_0^{2\pi} Dp(\theta_1 + \pi)\, d\varphi = 2\pi D^2.$$

This implies (6).

In the case where K is a circle of diameter D, formula (6) has been obtained by M. I. Stoka in [4].

Proposition 3. *Let K_0 be a parallelogram of side a, b and angle α, with $\alpha \leq \pi/2$. Let K be a regular polygon with n sides of length ℓ which can never intersect two nonconsecutive sides of K_0. Then*

$$\mu(K; K \subset K_0) = 2\pi ab \sin\alpha - 2nl(a+b) + \{nl^2/[\sin\alpha \sin^2(\pi/n)]\}F(n, \beta), \tag{7}$$

where the function $F(n, \beta)$ is defined as follows.

(i) For n even, if $j\pi/n < \alpha \leq \pi(j+1)/n$, with j even, and $\beta = \alpha - j\pi/n$, then:

$$F(n,\beta) = [(\pi/n) - (\beta/2)]\cos\beta + (\beta/2)\cos[(2\pi/n) - \beta] + (1/2)\sin[(2\pi/n) - \beta] + (1/2)\sin\beta.$$

(ii) For n even, if $j\pi/n < \alpha \leq \pi(j+1)/n$, with j odd, and $\beta = \alpha - j\pi/n$, then:

$$F(n,\beta) = (1/2)[(\pi/n) + \beta]\cos[(\pi/n) - \beta] + (1/2)[(\pi/n) - \beta]\cos[(\pi/n) + \beta] + \sin(\pi/n)\cos\beta.$$

(iii) For n odd, if $j\pi/2n < \alpha \le \pi(j+1)/2n$, with j even, and $\beta = \alpha - j\pi/2n$, then:

$$\begin{aligned} F(n,\beta) &= (1/4)\{[(2\pi/n) - \beta]\cos\beta + [(\pi/n) + \beta]\cos[(\pi/n) - \beta] + \\ &+ [(\pi/n) - \beta]\cos[(\pi/n) + \beta] + \beta\cos[(2\pi/n) - \beta] + \sin\beta + \\ &+ \sin[(2\pi/n) - \beta] + \sin[(\pi/n) - \beta] + \sin[(\pi/n) + \beta]\}. \end{aligned}$$

(iv) For n odd, if $j\pi/2n < \alpha \le \pi(j+1)/2n$, with j odd, and $\beta = \alpha - j\pi/2n$, then:

$$\begin{aligned} F(n,\beta) &= (1/4)\{[(\pi/2n) + \beta]\cos[(3\pi/2n) - \beta] + [(3\pi/2n) + \beta]\cos[(\pi/2n) - \\ &- \beta] + [(\pi/2n) - \beta]\cos[(3\pi/2n) + \beta] + [(3\pi/2n) - \beta]\cos[(\pi/2n) + \beta] + \\ &+ \sin[(3\pi/2n) - \beta] + \sin[(\pi/2n) - \beta] + \sin[(3\pi/2n) + \beta] + \sin[(\pi/2n) + \\ &+ \beta]\}. \end{aligned}$$

Proof. Since K_0 is a parallelogram formula (3) becomes

$$\mu(K; K \subset K_0) = 2\pi ab\sin\alpha - 2nl(a+b) + (1/\sin\alpha)\int_0^{2\pi}\sum_{i=1}^{4} p(\theta_i)p(\theta_{i+2})\,d\varphi$$

where $p(\theta_{i+1}) = p(\theta_i + \pi)$ for all i. We recall that θ_i denotes the angle formed by the direction perpendicular to the i-th side of K_0 and the direction r.

Then we label the sides of K_0 so that $\theta_2 = \theta_1 + \alpha$. For the sake of brevity we write θ instead of θ_1. Since $\theta = \gamma - \varphi$, for a suitable constant γ, we can express the above integral by means of the new variable θ as follows:

$$\mu(K; K \subset K_0) = 2\pi ab\sin\alpha - 2nl(a+b) + (1/\sin\alpha)\int_0^{2\pi}\sum_{i=1}^{4} p(\theta_i)p(\theta_{i+1})\,d\theta.$$

From a geometric point of view, by remark 1, this is equivalent to fixing K in the plane and to considering all the parallelograms of angle α which circumscribe K. This approach seems to be more convenient in order to calculate $p(\theta_i)$.

To fix K in the plane is just the same as to fix a frame in the plane. Therefore we take the centre of the circumcircle of K as the origin O and a line drawn from O to a vertex of K as the x-axis, as shown in figure 4 where $n = 6$.

Moreover we identify the position point P of K with O and the direction r with the x-axis. Hence $p(\theta_i)$ represents the length of the projection of the radius of the circumcircle

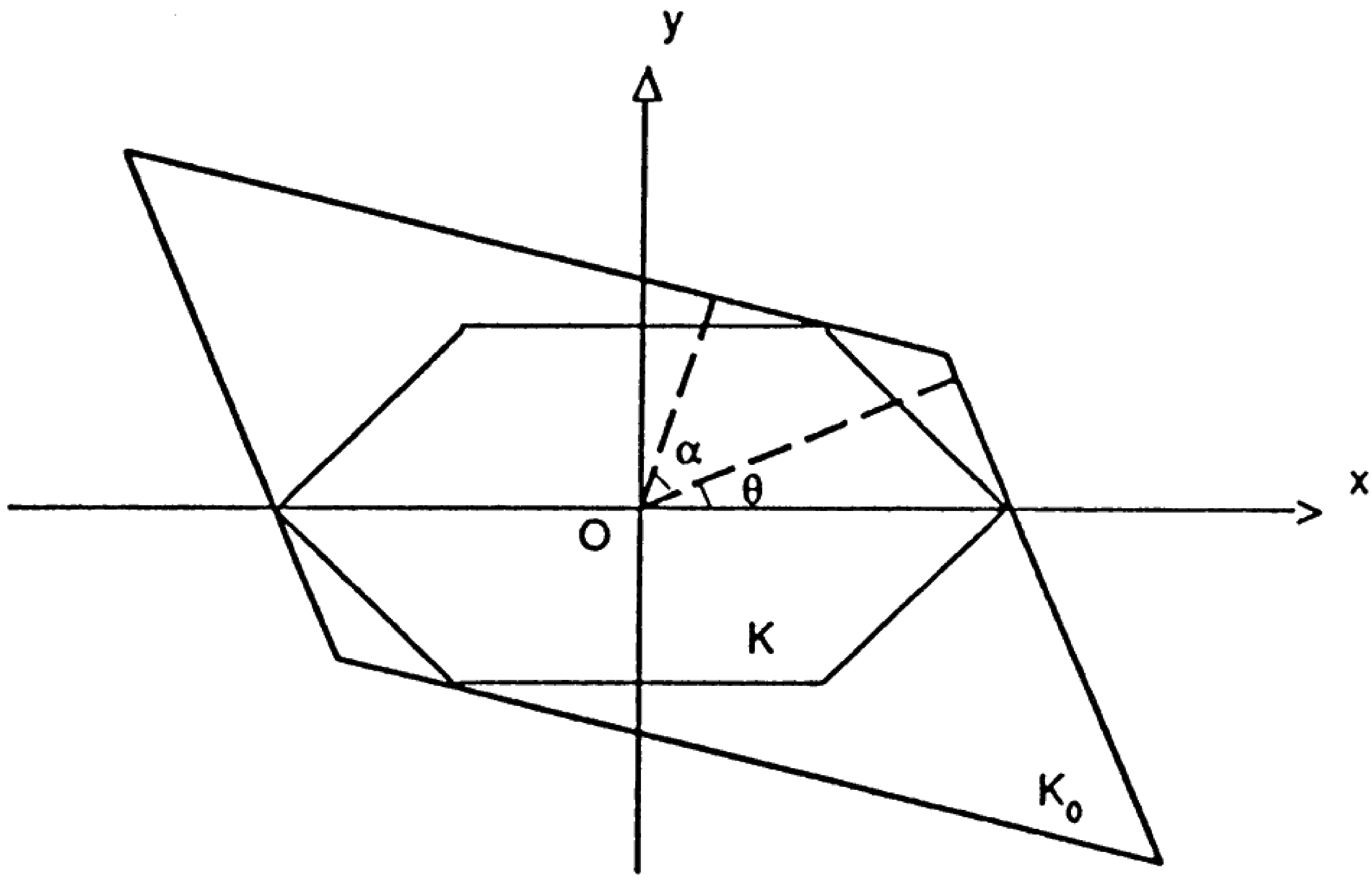

Figura 4

of K in the direction determined by θ_i. Further, any configuration corresponding to a value θ, with $\theta > 2\pi/n$, can be obtained from a configuration corresponding to a suitable value of θ, with $0 \leq \theta \leq 2\pi/n$, by a rotation of a multiple of $2\pi/n$ around the origin O.

Therefore the function $p(\theta_i)$ is a periodic function of period $2\pi/n$. Thus it is sufficient to determine $p(\theta_i)$ for $\theta \in [0, 2\pi/n]$.

Let us assume that **n** is even.

Then, if $j\pi/n < \alpha \leq \pi(j+1)/n$ and $\beta = \alpha - (j\pi/n)$, we distinguish the following cases.

I) *Let j be even.* Then:

a) for $\theta \in [0, (\pi/n) - \beta], p(\theta_1) = p(\theta_3) = R\cos\theta, p(\theta_2) = p(\theta_4) = R\cos(\theta + \beta)$;

b) for $\theta \in [(\pi/n) - \beta, \pi/n], p(\theta_1) = p(\theta_3) = R\cos\theta, p(\theta_2) = p(\theta_4) =$

$$= R\cos[(2\pi/n) - \theta - \beta];$$

c) for $\theta \in [\pi/n, (2\pi/n) - \beta], p(\theta_1) = p(\theta_3) = R\cos[(2\pi/n) - \theta],$

$$p(\theta_2) = p(\theta_4) = R\cos[(2\pi/n) - \theta - \beta];$$

d) for $\theta \in [(2\pi/n) - \beta, 2\pi/n], p(\theta_1) = p(\theta_3) = R\cos[(2\pi/n) - \theta],$

$$p(\theta_2) = p(\theta_4) = R\cos[\theta + \beta - (2\pi/n)]$$

where R denotes the radius of the circumcircle of K.

II) *Let j be odd.* Then:

a) for $\theta \in [0, (\pi/n) - \beta], p(\theta_1) = p(\theta_3) = R\cos\theta, p(\theta_2) = p(\theta_4) =$

$= R\cos[(\pi/n) - \theta - \beta];$

b) for $\theta \in [(\pi/n) - \beta, \pi/n], p(\theta_1) = p(\theta_3) = R\cos\theta, p(\theta_2) = p(\theta_4) =$

$= R\cos[\theta + \beta - (\pi/n)];$

c) for $\theta \in [\pi/n, (2\pi/n) - \beta], p(\theta_1) = p(\theta_3) = R\cos[(2\pi/n) - \theta],$

$p(\theta_2) = p(\theta_4) = R\cos[\theta + \beta - (\pi/n)];$

d) for $\theta \in [(2\pi/n) - \beta, 2\pi/n], p(\theta_1) = p(\theta_3) = R\cos[(2\pi/n) - \theta],$

$p(\theta_2) = p(\theta_4) = R\cos[(3\pi/n) - \theta - \beta].$

Let us assume that **n is odd.**

Then, if $j\pi/2n < \alpha \leq \pi(j+1)/2n$ and $\beta = \alpha - (j\pi/2n)$, we distinguish the following cases.

I) *Let* $j = 4h$. Then:

a) for $\theta \in [0, (\pi/n) - \beta], p(\theta_1) = R\cos\theta, p(\theta_2) = R\cos(\theta + \beta), p(\theta_3) =$

$= R\cos[(\pi/n) - \theta], p(\theta_4) = R\cos[(\pi/n) - \beta - \theta];$

b) for $\theta \in [(\pi/n) - \beta, \pi/n], p(\theta_1) = R\cos\theta, p(\theta_2) = R\cos[(2\pi/n) - \beta - \theta], p(\theta_3) =$

$= R\cos[(\pi/n) - \theta], p(\theta_4) = R\cos[\theta + \beta - (\pi/n)];$

c) for $\theta \in [\pi/n, (2\pi/n) - \beta], p(\theta_1) = R\cos[(2\pi/n) - \theta], p(\theta_2) = R\cos[(2\pi/n) -$

$- \beta - \theta], p(\theta_3) = R\cos[\theta - (\pi/n)],$

$p(\theta_4) = R\cos[\theta + \beta - (\pi/n)];$

d) for $\theta \in [(2\pi/n) - \beta, 2\pi/n], p(\theta_1) = R\cos[(2\pi/n) - \theta], p(\theta_2) = R\cos[\theta + \beta -$

$- (2\pi/n)], p(\theta_3) = R\cos[\theta - (\pi/n)],$

$p(\theta_4) = R\cos[(3\pi/n) - \theta - \beta].$

II) *Let* $j = 4h + 2$. Then:

a) for $\theta \in [0, (\pi/n) - \beta], p(\theta_1) = R\cos\theta, p(\theta_2) = R\cos[(\pi/n) - \beta - \theta], p(\theta_3) =$

$= R\cos[(\pi/n) - \theta], p(\theta_4) = R\cos(\theta + \beta);$

b) for $\theta \in [(\pi/n) - \beta, \pi/n], p(\theta_1) = R\cos\theta, p(\theta_2) = R\cos[\theta + \beta - (\pi/n)], p(\theta_3) =$
$= R\cos[(\pi/n) - \theta], p(\theta_4) = R\cos[(2\pi/n) - \beta - \theta];$

c) for $\theta \in [\pi/n, (2\pi/n) - \beta], p(\theta_1) = R\cos[(2\pi/n) - \theta], p(\theta_2) = R\cos[\theta + \beta - (\pi/n)], p(\theta_3) = R\cos[\theta - (\pi/n)], p(\theta_4) =$
$= R\cos[(2\pi/n) - \beta - \theta];$

d) for $\theta \in [(2\pi/n) - \beta, 2\pi/n], p(\theta_1) = R\cos[(2\pi/n) - \theta], p(\theta_2) = R\cos[(3\pi/n) - \theta - \beta], p(\theta_3) = R\cos[\theta - (\pi/n)],$
$p(\theta_4) = R\cos[\theta + \beta - (2\pi/n)].$

III) *Let* $j = 4h + 1$. Then:

a) for $\theta \in [0, (\pi/2n) - \beta], p(\theta_1) = R\cos\theta, p(\theta_2) = R\cos[(\pi/2n) + \beta + \theta], p(\theta_3) =$
$= R\cos[(\pi/n) - \theta], p(\theta_4) = R\cos[(\pi/2n) - \beta - \theta];$

b) for $\theta \in [(\pi/2n) - \beta, \pi/n], p(\theta_1) = R\cos\theta, p(\theta_2) = R\cos[(3\pi/2n) - \theta - \beta],$
$p(\theta_3) = R\cos[(\pi/n) - \theta], p(\theta_4) = R\cos[\theta + \beta - (\pi/2n)];$

c) for $\theta \in [\pi/n, (3\pi/2n) - \beta], p(\theta_1) = R\cos[(2\pi/n) - \theta],$
$p(\theta_2) = R\cos[(3\pi/2n) - \beta - \theta],$
$p(\theta_3) = R\cos[\theta - (\pi/n)],$
$p(\theta_4) = R\cos[\theta + \beta - (\pi/2n)];$

d) for $\theta \in [(3\pi/2n) - \beta, 2\pi/n], p(\theta_1) = R\cos[(2\pi/n) - \theta], p(\theta_2) = R\cos[\theta + \beta - (3\pi/2n)], p(\theta_3) = R\cos[\theta - (\pi/n)], p(\theta_4) =$
$= R\cos[(5\pi/2n) - \theta - \beta].$

IV) *Let* $j = 4h + 3$. Then:

a) for $\theta \in [0, (\pi/2n) - \beta], p(\theta_1) = R\cos\theta, p(\theta_2) = R\cos[(\pi/2n) - \beta - \theta], p(\theta_3) =$
$= R\cos[(\pi/n) - \theta], p(\theta_4) = R\cos[(\pi/2n) + \theta + \beta];$

b) for $\theta \in [(\pi/2n) - \beta, \pi/n], p(\theta_1) = R\cos\theta, p(\theta_2) = R\cos[\theta + \beta - (\pi/2n)],$

$p(\theta_3) = R\cos[(\pi/n) - \theta], p(\theta_4) = R\cos[(3\pi/2n) -$

$- \beta - \theta];$

c) for $\theta \in [\pi/n, (3\pi/2n) - \beta], p(\theta_1) = R\cos[(2\pi/n) - \theta], p(\theta_2) = R\cos[\theta + \beta -$

$- (2\pi/n)], p(\theta_3) = R\cos[\theta - (\pi/n)],$

$p(\theta_4) = R\cos[(3\pi/2n) - \beta - \theta];$

d) for $\theta \in [(3\pi/2n) - \beta, 2\pi/n], p(\theta_1) = R\cos[(2\pi/n) - \theta],$

$p(\theta_2) = R\cos[(5\pi/2n) - \theta - \beta],$

$p(\theta_3) = R\cos[\theta - (\pi/n)],$

$p(\theta_4) = R\cos[\theta + \beta - (3\pi/2n)].$

Notice that by integrating with respect to $d\theta$ the cases I and II, as well as the cases III and IV, give the same integral.

Hence we obtain the formula (7) satisfying the requirements (i) - (iv) by integrating the function $\sum_{i=1}^{4} p(\theta_i)p(\theta_{i+1})$ with respect to $d\theta$ and by substituting $\ell/[2\sin(\pi/n)]$ for R.

Now, as an example of this result, we consider the case where K_0 is a rectangle of sides a and b.

• If $n = 2(2k + 1)$ then $\alpha = \pi(j + 1)/n$ with $j = 2k$ and there follows from (7) $\mu(K; K \subset K_0) = 2\pi ab - 2nl(a + b) + [nl^2/\sin^2(\pi/n)][(\pi/n)\cos(\pi/n) + \sin(\pi/n)]$.

If we denote by S the area of the circumcircle of K and by S the area of K, then the above formula becomes

$$\mu(K; K \subset K_0) = 2\pi ab - 2nl(a + b) + 4\sec(\pi/n)(S + S).$$

• If $n = 4k$ then $\alpha = \pi(j + 1)/n$, with $j = 2k - 1$, and there follows from (7) $\mu(K; K \subset K_0) = 2\pi ab - 2nl(a + b) + [nl^2/\sin^2(\pi/n)][(\pi/n) + \sin(\pi/n)\cos(\pi/n)]$. Thus, by using the same notation as above, we get

$$\mu(K; K \subset K_0) = 2\pi ab - 2nl(a + b) + 4(S + S).$$

• Finally, if n is odd and consequently $\alpha = \pi(j + 1)/2n$, with $j = n - 1$, there follows from (7)

$$\mu(K; K \subset K_0) = 2\pi ab - 2nl(a + b) + [nl^2/\sin^2(\pi/2n)]\{[2\sin(\pi/2n)][1 - \sin^3(\pi/2n)]\}.$$

3. The results of section 2 will apply to problems involving lattices of figures with a convex fundamental cell, as described below. We recall that a «lattice of fundamental regions» is a sequence $\{A_n\}_{n\in\mathbf{N}}$ of regions A_n such that:

i) every point of the plane belongs to exactly one region A_n;

ii) every region A_n can be transformed into the region A_0 by a motion of the Euclidean plane which transforms any A_i into another region A_j, i.e. a motion which leaves the lattice invariant as a whole.

The region A_0 will be referred to as the «fundamental cell» of the lattice.

We now prove:

Theorem 2. *Let K_0 be a convex polygon of area S_0 and perimeter L_0 which is the fundamental cell of a lattice $\mathcal{L}$. Let K be a convex set of area S and perimeter L which can never intersect two nonconsecutive sides of K_0 if $n > 3$ of all the sides of K_0 if $n = 3$. Suppose that K is randomly located on $\mathcal{L}$. Then the probability that K intersects the boundary of a fundamental region of $\mathcal{L}$ is given by*

$$p = \frac{L_0 L - \int_0^{2\pi} S(\varphi)\, d\varphi}{2\pi S_0}, \tag{8}$$

where $S(\varphi)$ has the same meaning as in remark 1.

Proof. The probability p corresponds to the probability that K, with its position point P inside K_0, intersects the boundary of K_0.

Let q be the probability that K is contained in K_0. Then $p = 1 - q$.

Let us denote by $\mu(K; P \in K_0)$ the measure of all positions of K for which $P \in K_0$. Then

$$q = \frac{\mu(K; K \subset K_0)}{\mu(K; P \in K_0)}.$$

By (1) we get

$$\mu(K; P \in K_0) = \int_{\{P\in K_0\}} dx\, dy\, d\varphi = 2\pi S_0.$$

Thus, there follows from formula (3) and remark 1

$$p = 1 - q = \frac{L_0 L - \int_0^{2\pi} S(\varphi)\, d\varphi}{2\pi S_0}.$$

We now consider the particular case where the fundamental cell of the lattice is a parallelogram.

More precisely we prove

Proposition 4. *Let $\mathcal{L}$ be a lattice of parallelograms of sides a, b and angle α, with $\alpha \leq \pi/2$. Let K be a convex set of constant breadth D, with $D < \min\ (a \sin\alpha, b \sin\alpha)$, randomly located on $\mathcal{L}$. Denote by $p_i, i = 0, 1, 2$, the probability that K intersects exactly i lines of $\mathcal{L}$. Then*

(9)
$$p_0 = \frac{ab\sin^2\alpha - D(a+b)\sin\alpha + D^2}{ab\sin^2\alpha},$$

(10)
$$p_1 = \frac{D(a+b)\sin\alpha - 2D^2}{ab\sin^2\alpha},$$

(11)
$$p_2 = \frac{D^2}{ab\sin^2\alpha}.$$

Proof. Since $p_0 = q$, where q is the probability determined in the previous proposition, then by (6) and (8) we have

$$p_0 = \frac{\mu(K; K \subset K_0)}{\mu(K; P \in K_0)} = \frac{ab\sin^2\alpha - D(a+b)\sin\alpha + D^2}{ab\sin^2\alpha}.$$

Let us consider the polygonal line Γ_0 consisting of two consecutive sides of a parallelogram of $\mathcal{L}$. All sides of the parallelograms can be obtained by moving Γ_0 so that $\mathcal{L}$ remains invariant as a whole.

Them from a well-known formula involving lattices of curves [2], we get

$$E(n) = \frac{2(a+b)D}{ab\sin\alpha},$$

where n is the number of intersections of K with the straight lines of $\mathcal{L}$ and $E(n)$ is the mean value of n.

Notice that the positions of K with n odd constitute a set of zero measure, because they are positions of contact, determined by two parameters (set of motions of dimension 2) while the kinematic measure refers to sets of motions of dimension 3.

On the other hand since $D < \min(a \sin\alpha, b \sin\alpha)$, it follows that $n \leq 4$. Therefore the above formula becomes

$$2p_1 + 4p_2 = \frac{2(a+b)D}{ab\sin\alpha}.$$

Hence we can derive the desired formulas (10) and (11) noting that $p_0 + p_1 + p_2 = 1$. Similarly we can prove the following

Proposition 5. *Let $\mathcal{L}$ be a lattice of parallelograms of sides a, b and angle α, with $\alpha \leq \pi/2$. Let K be a regular polygon with n sides of length l. Assume that K is randomly located on $\mathcal{L}$ and can never intersect two nonconsecutive sides of the parallelograms of $\mathcal{L}$. If p_i is defined as in the previous proposition, then*

$$p_0 = 1 - \frac{nl(a+b)}{\pi ab \sin\alpha} + \frac{nl^2}{2\pi ab \sin^2\alpha \sin^2(\pi/n)} F(n,\beta), \tag{12}$$

$$p_1 = \frac{nl(a+b)}{\pi ab \sin\alpha} - \frac{nl^2}{\pi ab \sin^2\alpha \sin^2(\pi/n)} F(n,\beta), \tag{13}$$

$$p_2 = \frac{nl^2}{2\pi ab \sin^2\alpha \sin^2(\pi/n)} F(n,\beta), \tag{14}$$

where $F(n,\beta)$ is defined as in Theorem 1.

The proof follows from the same argument as in the previous proposition, noting that in this case

$$E(n) = \frac{2nl(a+b)}{\pi ab \sin\alpha}, \text{ and } \mu(K; K \subset K_0) \text{ is represented by } (7).$$

We now regard a lattice of parallelograms as a union of two lattices consisting of strips of constant breadth.

Then we have the following results.

Proposition 6. *Let $\mathcal{L}$ be a lattice of parallelograms of sides a, b and angle α, with $\alpha \leq \pi/2$. Let $\mathcal{L}_1$ be the lattice of strips determined by the lines of $\mathcal{L}$ parallel to the side of length a. Similarly, let $\mathcal{L}_2$ be the lattice of strips determined by the lines of $\mathcal{L}$ parallel to the side of length b.*

Let K be a convex set of constant breadth D, with $D < \min(a \sin\alpha, b \sin\alpha)$, randomly located on $\mathcal{L}$.

Denote by I_i, with $i = 1, 2$, the event which happens when K intersects the straight lines of $\mathcal{L}_i$. Then I_1 and I_2 are independent events.

Proof. Let $p(I_i)$ be the probability associated to I_i. Since the density (1) is invariant under inversion of the motion, we can assume that K is fixed in the plane and $\mathcal{L}$ moves at random in the plane. Then $p(I_i)$ is equal to the probability that a line of $\mathcal{L}_i$ intersects K.

Let us assume that K is surrounded by two circles C_1 and C_2 of diameter $a \sin\alpha$ and $b \sin\alpha$ respectively. The circle C_i is always intersected by one straight line of $\mathcal{L}_i$. Then $p(I_i)$ represents the probability that a straight line of $\mathcal{L}_i$, intersecting C_i, also intersects K.

By a well-known formula, [2], the probabilities $p(I_1)$ and $p(I_2)$ are given by

$$p(I_1) = D/(b \sin\alpha) \qquad \text{and} \qquad p(I_2) = D/(a \sin\alpha).$$

Moreover let $I = I_1 \cup I_2$ and let $p(I)$ be the probability associated to I. From (8) and (6) we get

$$p(I) = \frac{D(a+b)\sin\alpha - D^2}{ab\sin^2\alpha}.$$

Thus

$$p(I_1 \cap I_2) = p(I_1) + p(I_2) - p(I_1 \cup I_2) = \frac{D^2}{ab\sin^2\alpha} = p(I_1)p(I_2),$$

giving the desired result.

In the particular case where K is a disk of diameter D, this result has been already obtained by M. I. Stoka in [4].

Proposition 7. *Let $\mathcal{L}$ be a lattice of parallelograms of sides a, b and angle α, with $\alpha \leq \pi/2$. Let K be a regular convex polygon with n sides of length ℓ, which can never intersect two nonconsecutive sides of the parallelograms of $\mathcal{L}$.*

Suppose that $\mathcal{L}_1, \mathcal{L}_2, I_1, I_2$ have the same meaning as in the previous proposition. Then there exists a value of α such that I_1 and I_2 are independent events.

Proof. Using the same argument as above, by formulas (7) and (8) we obtain

$$p(I_1) = \frac{nl}{\pi b \sin\alpha}, \qquad p(I_2) = \frac{nl}{\pi a \sin\alpha},$$

$$p(I) = \frac{nl(a+b)}{\pi ab \sin\alpha} - \frac{nl^2 F(n,\beta)}{\pi ab \sin^2\alpha \sin^2(\pi/n)}.$$

where $F(n,\beta)$ is defined as in Theorem 1.

Then $p(I_1)p(I_2) = p(I_1 \cap I_2)$ if and only if

$$F(n,\beta) = (2n/\pi)\sin^2(\pi/n). \tag{15}$$

Consequently, we have to prove that equation (15) has a solution $\overline{\beta}$, with $0 < \overline{\beta} < \pi/n$ for any $n > 3$ even and with $0 < \overline{\beta} < \pi/(2n)$ for any n odd.

Let us consider the following functions:

$$\Phi_1(x) = x + (1/2)\sin 2x - (2/x)\sin^2 x;$$

$$\Phi_2(x) = x\cos x + \sin x - (2/x)\sin^2 x;$$

$$\Psi_1(x) = (1/4)(2x + 2x\cos x + \sin 2x + 2\sin x) - (2/x)\sin^2 x;$$

$$\Psi_2(x) = (1/2)[(3/2)x\cos(x/2) + (x/2)\cos(3x/2) + \sin(x/2) + \sin(3x/2)] - (2/x)\sin^2 x.$$

By using the Taylor expansion at 0 of these functions we verify that for any $x \in (0,1]$ there follows

$$\Psi_1(x) \geq \Phi_1(x) \geq 0.01x^5 > 0,$$

$$\Phi_2(x) \leq -0.02x^5 < 0,$$

$$\Psi_2(x) \leq 2\Phi_2(x/2) < 0.$$

Let

$$G(n,\beta) = F(n,\beta) - (2n/\pi)\sin^2(\pi/n).$$

For fixed n, $G(n,\beta)$ is a continuous function of β. Moreover by the definition of $F(n,\beta)$ we have

- in case (i): $G(n,0) = \Phi_1(\pi/n)$, $G(n,\pi/n) = \Phi_2(\pi/n)$;
- in case (ii): $G(n,0) = \Phi_2(\pi/n)$, $G(n,\pi/n) = \Phi_1(\pi/n)$;
- in case (iii): $G(n,0) = \Psi_1(\pi/n)$, $G(n,\pi/2n) = \Psi_2(\pi/n)$;
- in case (iv): $G(n,0) = \Psi_2(\pi/n)$, $G(n,\pi/2n) = \Psi_1(\pi/n)$.

Thus in cases (i), (ii) the function $G(n,\beta)$ changes sign in $[0,\pi/n]$ and in cases (iii), (iv) it changes sign in $[0,\pi/2n]$.

Therefore there exists a value $\overline{\beta}$ such that $G(n,\overline{\beta}) = 0$ where $\overline{\beta} \in (0,\pi/n)$ for n even and $\overline{\beta} \in (0,\pi/2n)$ for n odd.

These conclusions complete the proof.

Following M. I. Stoka [4], we now assume that the set K is random both in its position and in its geometric character.

Then we prove

Proposition 8. *Let $\mathcal{L}$ be a lattice of parallelograms of sides a, b and angle α, with $\alpha \leq \pi/2$. Let K be a convex set of constant breadth Λ, with $0 < \Lambda < D < \min(a\sin\alpha, b\sin\alpha)$, where Λ is a random variable having a probability distribution with density $f(\Lambda)$ and second moment $E(\Lambda^2)$.*

Assume that K is randomly located on $\mathcal{L}$ and denote by p_i, with $i = 0,1,2$, the probability that K intersects $\mathcal{L}$ in exactly i lines.

Then

(16) $$p_0 = \frac{ab\sin^2\alpha - E(\Lambda)(a+b)\sin\alpha + E(\Lambda^2)}{ab\sin^2\alpha},$$

(17) $$p_1 = \frac{E(\Lambda)(a+b)\sin\alpha - 2E(\Lambda^2)}{ab\sin^2\alpha},$$

(18) $$p_2 = \frac{E(\Lambda^2)}{ab\sin^2\alpha}.$$

Proof. Let us denote by $p_0(\lambda)$ the probability that K does not intersect the lines of $\mathcal{L}$ when $\Lambda = \lambda$. Then

$$p_0 = \int_0^D p_0(\lambda) f(\lambda)\,d\lambda.$$

Thus, by virtue of formula (9) we get

$$p_0 = \int_0^D \left[\frac{ab\sin^2\alpha - \lambda(a+b)\sin\alpha + \lambda^2}{ab\sin^2\alpha}\right] f(\lambda)\,d\lambda =$$
$$= \frac{ab\sin^2\alpha - E(\Lambda)(a+b)\sin\alpha + E(\Lambda^2)}{ab\sin^2\alpha}.$$

Similarly, we obtain from (10) and (11) the desired formulas (17) and (18).

In the same way we can prove

Proposition 9. *Let $\mathcal{L}$ be a lattice of parallelograms of sides a, b and angle α, with $\alpha \leq \pi/2$. Let K be a regular convex polygon with n sides of length Λ, where Λ is a bounded random variable having a probability distribution with density $f(\Lambda)$ and with the second moment $E(\Lambda^2)$.*

Assume that K is randomly located on $\mathcal{L}$ and it can never intersect two non-consecutive sides of the parallelograms of $\mathcal{L}$. Then if p_i has the same meaning as above, it follows that

(19) $$p_0 = 1 - \frac{n(a+b)E(\Lambda)}{\pi ab\sin\alpha} + \frac{nF(n,\beta)E(\Lambda^2)}{2\pi ab\sin^2\alpha\sin^2(\pi/n)},$$

(20) $$p_1 = \frac{n(a+b)E(\Lambda)}{\pi ab\sin\alpha} - \frac{nF(n,\beta)E(\Lambda^2)}{\pi ab\sin^2\alpha\sin^2(\pi/n)},$$

(21) $$p_2 = \frac{nF(n,\beta)E(\Lambda^2)}{2\pi ab\sin^2\alpha\sin^2(\pi/n)},$$

where $F(n,\beta)$ is defined as in Theorem 1.

Remark 2. In this section we have assumed that K moves at random in the plane while the lattice $\mathcal{L}$ is fixed. Since the density (1) is invariant under the inversion of the motion, the previous results hold even if we fix K and move $\mathcal{L}$. Then the probability p_i can be regarded as the probability that $\mathcal{L}$ cuts K by exactly i lines.

The latter approach is generally adopted in the stereological applications, where K is a convex set whose geometric properties are studied by exploring K with random lattices of lines (or planes). Then Propositions 8 or 9 enable us to estimate the moments $E(\Lambda)$ and $E(\Lambda^2)$.

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C. Peri
Istituto di Matematica Generale, Finanziaria ed Economica
Università Cattolica del Sacro Cuore
Largo Gemelli, 1
20123 Milano, Italy