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ON BANACH ALGEBRAS WITH A JORDAN INVOLUTION

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Dedicated to the memory of Professor Gottfried Köthe

Let A be a Banach algebra. By a Jordan involution $x \to x^{\#}$ on A we mean a conjugate-linear mapping of A onto A where $x^{\#\#} = x$ for all x in A and

$$(xy + yx)^{\#} = x^{\#}y^{\#} + y^{\#}x^{\#}$$

for all x, y in A. Of course any involution is automatically a Jordan involution. An easy example of a Jordan involution which is not an involution is given, for the algebra of all complex two-by-two matrices, by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\#} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

In this note we provide one instance where a Jordan involution is compelled to be an involution.

Say $x \in A$ is #-normal if x permutes with $x^{\#}$ and #-self-adjoint if $x = x^{\#}$. Let y be #-normal. Then

$$2\,(y^{\#}y)^{\#} = (y^{\#}y + yy^{\#})^{\#} = 2y^{\#}y$$

so that $y^{\#}y$ is #-self-adjoint. By [5, pp. 481-2] we know that

$$(1) \qquad (x^n)^{\#} = (x^{\#})^n$$

for all $x \in A$ and all positive integers n. Also $e^{\#} = e$ if A has an identity e.

Berkson [1] and Glickfeld [4] independently showed the following

Theorem. *Let A be a Banach*-algebra with an identity. If $\| x^*x \| = \| x^* \| \| x \|$ for all normal elements x in A then A is a C^*-algebra.*

Our aim is to extend this result in the following way.

Theorem 1. *Let A be a Banach algebra with an identity and a Jordan involution $x \to x^{\#}$. Suppose that*

$$(2) \qquad \| x^{\#}x \| = \| x^{\#} \| \| x \|$$

for every #-normal x in A. Then A is a C^*-algebra with $x \to x^\#$ as its involution.

Lemma. *Suppose that the Jordan involution $x \to x^\#$ has the property that $x = 0$ whenever $x^\# x = 0$ for a #-normal $x \in A$. Then any #-normal element is contained in a maximal commutative #-subalgebra of A which is closed.*

Proof. Suppose that $x \neq 0$ is a central nilpotent. Then $w = x^\# x$ is a non-zero #-self-adjoint nilpotent. Let $m > 1$ be the smallest positive integer such that $w^m = 0$. As w^{m-1} is #-self-adjoint by (1) then $0 = (w^{m-1})^\# w^{m-1}$ and $w^{m-1} \neq 0$. This is contrary to our hypotheses so that A has no non-zero central nilpotent.

It follows from [5, p. 481] that if $xy = yx$ then $x^\# y^\# = y^\# x^\#$ and therefore

$$2(xy)^\# = (xy + yx)^\# = x^\# y^\# + y^\# x^\#.$$

Thus

$$(xy)^\# = x^\# y^\# = y^\# x^\# \tag{3}$$

whenever $xy = yx$.

We see that $Z^\# = Z$ when Z is the center of A. Let V be a commutative #-subalgebra of A. If $y \in A$ permutes with every $z \in V$ then $y^\#$ permutes with every $z \in V$. Suppose that this y is # normal. Let K be the subalgebra generated by V, y and $y^\#$. Each element of K is a finite sum of elements of the form $w = vy^r(y^\#)^s$ where r and s are non-negative integers and $v \in V$. In view of (1) and (3) we see that $w^\# \in K$ so that K is a #-subalgebra.

By standard arguments each #-normal element is contained in a maximal commutative #-subalgebra which must be closed by the reasoning of [6, Theorem 4.1.3].

We now turn to the proof of Theorem 1. Let H be the set of #-self-adjoint element of A. Of course $A = H \oplus iH$. Let $h \in H$. In view of (2) the lemma applies to show that h lies in a maximal commutative #-subalgebra B of A. But we know that B is a commutative C^*-algebra in the involution $x \to x^\#$ on B. Therefore $\| \exp(ith) \| = 1$ for all real t. Hence A is a V-algebra in the notation of [2, p. 205]. Applying the Vidav-Palmer Theorem ([2, Theorem 14, p. 211] or [3, Theorem 45.1]) we see that A is a C^*-algebra with involution $x \to x^\#$.

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