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A Quantitative Characterization of Some Finite Simple Groups Through Order and Degree Pattern

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Abstract. Let G be a finite group with $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_h^{\alpha_h}$, where $p_1 < p_2 < \cdots < p_h$ are prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_h, h$ are natural numbers. The prime graph $\Gamma(G)$ of G is a simple graph whose vertex set is $\{p_1, p_2, \ldots, p_h\}$ and two distinct primes p_i and p_j are joined by an edge if and only if G has an element of order $p_i p_j$. The degree $\deg_G(p_i)$ of a vertex p_i is the number of edges incident on p_i, and the h-tuple $(\deg_G(p_1), \deg_G(p_2), \ldots, \deg_G(p_h))$ is called the degree pattern of G. We say that the problem of OD-characterization is solved for a finite group G if we determine the number of pairwise non-isomorphic finite groups with the same order and degree pattern as G. The purpose of this paper is twofold. First, it completely solves the OD-characterization problem for every finite non-Abelian simple groups their orders having prime divisors at most 17. Second, it provides a list of finite (simple) groups for which the problem of OD-characterization have been already solved.

Keywords: Prime graph, degree pattern, OD-characterization.

MSC 2000 classification: 20D05, 20D06, 20D08.

Introduction

Throughout this paper, all groups discussed are *finite* and simple groups are *non-Abelian*. Given a group G, denote by $\omega(G)$ the set of order of all elements in G, and by $\mu(G)$ the set of numbers in $\omega(G)$ that are maximal with respect to divisibility. We also denote by $\pi(n)$ the set of all prime divisors of a positive integer n. For a finite group G, we shall write $\pi(G)$ instead of $\pi(|G|)$. To every finite group G, we associate a simple graph known as *prime graph* (also often called the *Gruenberg–Kegel graph*) and denoted by $\Gamma(G)$. In this graph the vertex set is the set $\pi(G)$, and two distinct vertices p and q are joined by an edge if and only if $pq \in \omega(G)$. Let $s(G)$ be the number of connected components of $\Gamma(G)$. We denote the set of all the connected components of the graph $\Gamma(G)$ by $\{\pi_i(G) : i = 1, 2, \dots, s(G)\}$ and, if the order of G is even, we denote by $\pi_1(G)$ the component containing 2. The *degree* $\deg_G(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident on p. In the case the distinct prime divisors of $|G|$ are $p_1, p_2, \dots, p_h$, where h is a positive integer and $p_1 < p_2 < \cdots < p_h$, we define

$$\mathrm{D}(G) := \big(\deg_G(p_1), \deg_G(p_2), \dots, \deg_G(p_h)\big),$$

and call this h-tuple the *degree pattern of* G.

Let $\mathcal{OD}(G)$ be the collection of pairwise non-isomorphic groups with the same order and degree pattern as G, that is

$$\mathcal{OD}(G) = \big\{H : \ |H| = |G|, \ D(H) = D(G)\big\}.$$

We put $h_{\mathrm{OD}}(G) = |\mathcal{OD}(G)|$. In terms of the function $h_{\mathrm{OD}}(\cdot)$, we have the following definition.

Definition 1. A finite group G is said to be k-fold OD-characterizabale if $h_{\mathrm{OD}}(G) = k$. G is OD-characterizabale if $h_{\mathrm{OD}}(G) = 1$. Moreover, we will say that the OD-characterization problem is solved for a group G, if the value of $h_{\mathrm{OD}}(G)$ is known.

According to Cayley's theorem, for each positive integer n there are only *finitely* many non-isomorphic groups of order n normally denoted by $\nu(n)$. Hence the following corollary is immediate.

Theorem 1. *Every finite group is k-fold OD-characterizable for some natural number k.*

However, the situation will be interesting when we restrict ourselves to finite *simple* groups. As a matter of fact, there are many non-Abelian simple groups which are OD-characterizable or 2-fold OD-characterizable (see Table 3 at the end of the paper). The first examples of OD-characterizable simple groups were found in [18]. In [15], Moghaddamfar and Zokayi obtained some infinite series of OD-characterizable simple groups such as: $\mathrm{Sz}(2^{2n+1})$, $L_2(2^n)$, $\mathbb{A}_p$, $\mathbb{A}_{p+1}$ and $\mathbb{A}_{p+2}$, where p is a prime. Recently, Zhang and Shi in [34] obtained another infinite series of OD-characterizable simple groups, that is $L_2(q)$ for q odd. At present, the OD-characerization problem is solved for many finite non-Abelian simple and almost simple groups (a new list of such groups is available in Tables 3-4 at the end of the paper). Nevertheless, we do not know of any *simple* group S for which $h_{\mathrm{OD}}(S) \notin \{1, 2\}$. Therefore, the following problem may be of interest.

Problem 0.2. Is there a finite simple group S for which $h_{\mathrm{OD}}(S) \geqslant 3$?

In connection the finite simple groups which are k-fold OD-characterizable, for $k \geq 2$, it was shown in [3], [17] and [18] that:

$\mathcal{OD}(\mathbb{A}_{10}) = \{\mathbb{A}_{10},\ \mathbb{Z}_3 \times J_2\}$,
$\mathcal{OD}(S_6(5)) = \{S_6(5), O_7(5)\}$,
$\mathcal{OD}(S_{2m}(q)) = \{S_{2m}(q), O_{2m+1}(q)\}$, $m = 2^f \geqslant 2$, $|\pi\big(\frac{q^m+1}{2}\big)| = 1$, q odd prime power,
$\mathcal{OD}(S_{2p}(3)) = \{S_{2p}(3), O_{2p+1}(3)\}$, $|\pi\big((3^p-1)/2\big)| = 1$, p odd prime.

It should be of interest to investigate the question: Let G be a finite group. How many *simple* groups are there in $\mathcal{OD}(G)$? Evidently, two simple groups in $\mathcal{OD}(G)$ must have the same order. The complete list of pairs of non-isomorphic finite simple groups having the same order is well-known (see [8, 19]).

Proposition 1. *Two finite simple groups of the same order are isomorphic, except exactly in the cases:* $\{\mathbb{A}_8, L_3(4)\}$ *and* $\{O_{2n+1}(q), S_{2n}(q)\}$ *for* $n \geqslant 3$ *and* q *odd.*

An immediate consequence of Proposition 1 is the following.

Corollary 1. *For every group* G*, the set* $\mathcal{OD}(G)$ *has at most two simple groups.*

Generally, the orthogonal groups $O_{2n+1}(q)$ and the symplectic groups $S_{2n}(q)$ have the same order and prime graph ([23, Proposition 7.5]), hence $|O_{2n+1}(q)| = |S_{2n}(q)|$ and $\mathrm{D}(O_{2n+1}(q)) = \mathrm{D}(S_{2n}(q))$. Notice that $O_{2n+1}(2^m) \cong S_{2n}(2^m)$ and $O_5(q) \cong S_4(q)$ for each q, and hence, if $n \geq 3$ and q is odd, then the simple groups $O_{2n+1}(q)$ and $S_{2n}(q)$ are non-isomorphic groups). Now, we have the following result.

Proposition 2. *If* $n \geq 3$ *and* q *is odd, then* $h_{\mathrm{OD}}(O_{2n+1}(q)) = h_{\mathrm{OD}}(S_{2n}(q)) \geqslant 2$.

Remark 1.1 Although, the simple groups $\mathbb{A}_8$ and $L_3(4)$ have the same order, but they have different degree patterns, in fact, $D(\mathbb{A}_8) = (1,2,1,0)$ and $D(L_3(4)) = (0,0,0,0)$. It was proved in [15, 18] that $h_{\mathrm{OD}}(\mathbb{A}_8) = h_{\mathrm{OD}}(L_3(4)) = 1$.

In what follows we will consider the finite non-Abelian simple groups S with the property $\pi(S) \subseteq \{2,3,5,7,11,13,17\}$. We denote the set of all these simple groups by $\mathcal{S}_{\leqslant 17}$. Using the classification of finite simple groups it is not hard to obtain a full list of all groups in $\mathcal{S}_{\leqslant 17}$. Actually, there are 73 such groups (see [14, Table 4] or [28, Table 1]). For convenience, the values of $|S|$, $\mu(S)$, $D(S)$, $s(S)$ and $h_{\mathrm{OD}}(S)$ are listed in Table 2 (see [2, 5, 10, 11, 20, 21, 23, 24]). The comparison between simple groups listed in Table 3 and the simple groups in $\mathcal{S}_{\leqslant 17}$ shows that there are only 13 groups which we must solve the OD-characterization problem for them; namely, the groups $L_3(16)$, $L_5(3)$, $U_3(17)$, $U_4(4)$, $S_4(8)$, $S_4(13)$, $S_6(4)$, $G_2(4)$, $F_4(2)$, $O_8^-(2)$, $O_{10}^-(2)$, $O_8^+(3)$ and $O_8^+(4)$. The goal of the present paper is to prove that these groups are OD-characterizable.

Table 2. *The simple groups in $\mathcal{S}_{\leqslant 17}$ except alternating ones.*

S	$\|S\|$	$\mu(S)$	$D(S)$	$s(S)$	$h_{\rm OD}(S)$
$U_4(2) \cong S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	$\{12, 9, 5\}$	$(1, 1, 0)$	2	2
$L_2(7) \cong L_3(2)$	$2^3 \cdot 3 \cdot 7$	$\{7, 4, 3\}$	$(0, 0, 0)$	3	1
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	$\{9, 7, 2\}$	$(0, 0, 0)$	3	1
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	$\{12, 8, 7\}$	$(1, 1, 0)$	2	1
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	$\{25, 24, 7\}$	$(1, 1, 0, 0)$	3	1
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$\{10, 8, 7, 6\}$	$(2, 1, 1, 0)$	2	1
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\{7, 5, 4, 3\}$	$(0, 0, 0, 0)$	4	1
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$\{15, 12, 10, 8, 7\}$	$(2, 2, 2, 0)$	2	1
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$\{12, 9, 8, 7, 5\}$	$(1, 1, 0, 0)$	3	1
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	$\{56, 42, 25, 24\}$	$(2, 2, 0, 2)$	2	1
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	$\{15, 12, 10, 9, 8, 7\}$	$(2, 2, 2, 0)$	2	1
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$\{15, 12, 10, 9, 8, 7\}$	$(2, 2, 2, 0)$	2	1
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	$\{11, 6, 5\}$	$(1, 1, 0, 0)$	3	1
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$\{11, 8, 6, 5\}$	$(1, 1, 0, 0)$	3	1
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$\{11, 10, 8, 6\}$	$(2, 1, 1, 0)$	2	1
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	$\{18, 15, 11, 8\}$	$(1, 2, 1, 0)$	2	1
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$\{11, 10, 8, 6\}$	$(2, 1, 1, 0)$	2	1
M^cL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	$\{30, 14, 12, 11, 9, 8\}$	$(3, 2, 2, 1, 0)$	2	1
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	$\{20, 15, 12, 11, 8, 7\}$	$(2, 2, 2, 0, 0)$	3	1
$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	$\{18, 15, 12, 11, 10, 8, 7\}$	$(2, 2, 2, 0, 0)$	3	1
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	$\{13, 8, 6\}$	$(1, 1, 0)$	2	1
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$\{13, 12, 5\}$	$(1, 1, 0, 0)$	3	1
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$\{15, 13, 10, 4\}$	$(1, 1, 2, 0)$	2	1
$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	$\{30, 20, 13, 12\}$	$(2, 2, 2, 0)$	2	1
$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	$\{20, 13, 12, 9, 8\}$	$(2, 1, 1, 0)$	2	1
${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	$\{16, 13, 12, 10\}$	$(2, 1, 1, 0)$	2	1
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$\{13, 7, 6\}$	$(1, 1, 0, 0)$	3	1
$L_2(27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	$\{14, 13, 3\}$	$(1, 0, 1, 0)$	3	1
$G_2(3)$	$2^6 \cdot 3^6 \cdot 7 \cdot 13$	$\{13, 12, 9, 8, 7\}$	$(1, 1, 0, 0)$	3	1
${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	$\{28, 21, 18, 13, 12, 8\}$	$(2, 2, 2, 0)$	2	1
$\mathrm{Sz}(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	$\{13, 7, 5, 4\}$	$(0, 0, 0, 0)$	4	1
$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$\{65, 63, 2\}$	$(0, 1, 1, 1, 1)$	3	1
$U_4(5)$	$2^7 \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13$	$\{63, 60, 52, 24\}$	$(3, 3, 2, 1, 1)$	1	1
$L_3(9)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	$\{91, 80, 24\}$	$(2, 1, 1, 1, 1)$	2	1
$S_6(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	$\{36, 30, 24, 20, 14, 13\}$	$(3, 2, 2, 1, 0)$	2	2
$O_7(3)$	$2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$	$\{20, 18, 15, 14, 13, 12, 8\}$	$(3, 2, 2, 1, 0)$	2	2
$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	$\{21, 15, 13, 12, 10, 8\}$	$(2, 3, 2, 1, 0)$	2	1
$S_4(8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	$\{65, 63, 18, 14, 4\}$	$(2, 2, 1, 2, 1)$	2	1
$O_8^+(3)$	$2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13$	$\{20, 18, 15, 14, 13, 12, 8\}$	$(3, 2, 2, 1, 0)$	2	1
$L_5(3)$	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^2 \cdot 13$	$\{121, 104, 80, 78, 24, 18\}$	$(3, 2, 1, 0, 2)$	2	1
$L_6(3)$	$2^{11} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	$\{182, 121, 120, 104, 80, 78, 36\}$	$(4, 3, 2, 2, 0, 3)$	2	1
Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$\{24, 21, 20, 18, 15, 14, 13, 11\}$	$(3, 3, 2, 2, 0, 0)$	3	1
Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	$\{30, 24, 22, 21, 20, 18, 16, 14, 13\}$	$(4, 3, 2, 2, 1, 0)$	2	1
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	$\{17, 9, 8\}$	$(0, 0, 0)$	3	1
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	$\{17, 15, 2\}$	$(0, 1, 1, 0)$	3	1
$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	$\{17, 15, 10, 6, 4\}$	$(2, 2, 2, 0)$	2	1

Table 2. (Continued)

S	$\|S\|$	$\mu(S)$	$D(S)$	$s(S)$	$h_{\rm OD}(S)$
He	$2^{10}\cdot 3^3\cdot 5^2\cdot 7^3\cdot 17$	$\{28,21,17,15,12,10,8\}$	$(3,3,2,2,0)$	2	1
$O_8^-(2)$	$2^{12}\cdot 3^4\cdot 5\cdot 7\cdot 17$	$\{30,21,17,12,9,8\}$	$(2,3,2,1,0)$	2	1
$L_4(4)$	$2^{12}\cdot 3^4\cdot 5^2\cdot 7\cdot 17$	$\{85,63,30,12\}$	$(2,3,3,1,1)$	1	1
$S_8(2)$	$2^{16}\cdot 3^5\cdot 5^2\cdot 7\cdot 17$	$\{30,24,21,20,18,17,14\}$	$(3,3,2,2,0)$	2	1
$U_4(4)$	$2^{12}\cdot 3^2\cdot 5^3\cdot 13\cdot 17$	$\{65,51,30,20\}$	$(2,3,3,1,1)$	1	1
$U_3(17)$	$2^6\cdot 3^4\cdot 7\cdot 13\cdot 17^3$	$\{102,96,91,18\}$	$(2,2,1,1,2)$	2	1
$O_{10}^-(2)$	$2^{20}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\cdot 17$	$\{35,33,30,24,21,20,18,17,14\}$	$(3,4,3,3,1,0)$	2	1
$L_2(169)$	$2^3\cdot 3\cdot 5\cdot 7\cdot 13^2\cdot 17$	$\{85,84,13\}$	$(2,2,1,2,0,1)$	3	1
$S_4(13)$	$2^6\cdot 3^2\cdot 5\cdot 7^2\cdot 13^4\cdot 17$	$\{182,156,85,84\}$	$(3,3,1,3,3,1)$	2	1
$L_3(16)$	$2^{12}\cdot 3^2\cdot 5^2\cdot 7\cdot 13\cdot 17$	$\{91,85,15,10,4\}$	$(1,1,3,1,1,1)$	2	1
$S_6(4)$	$2^{18}\cdot 3^4\cdot 5^3\cdot 7\cdot 13\cdot 17$	$\{85,65,63,51,34,30,20,12,8\}$	$(3,4,4,1,1,3)$	1	1
$O_8^+(4)$	$2^{24}\cdot 3^5\cdot 5^4\cdot 7\cdot 13\cdot 17^2$	$\{255,65,63,34,30,20,12,8\}$	$(3,4,4,1,1,3)$	1	1
$F_4(2)$	$2^{24}\cdot 3^6\cdot 5^2\cdot 7^2\cdot 13\cdot 17$	$\{30,28,24,21,20,18,17,16,13\}$	$(3,3,2,2,0,0)$	3	1

Proposition 3. *The simple groups $L_3(16)$, $L_5(3)$, $U_3(17)$, $U_4(4)$, $S_4(8)$, $S_4(13)$, $S_6(4)$, $G_2(4)$, $F_4(2)$, $O_8^-(2)$, $O_{10}^-(2)$, $O_8^+(3)$ and $O_8^+(4)$, are OD-characterizable.*

This completes the problem of OD-characterization for all simple groups in $\mathcal{S}_{\leqslant 17}$. More precisely, we have the following corollary.

Corollary 2. *All simple groups in $\mathcal{S}_{\leqslant 17}$, except $U_4(2)$, $\mathbb{A}_{10}$, $S_6(3)$ and $O_7(3)$, are OD-characterizable.*

We conclude the introduction with some further notation. Let $\Gamma=(V,E)$ be a simple graph. A set of vertices $I\subseteq V$ is said to be an independent set of Γ if no two vertices in I are adjacent in Γ. The independence number of Γ, denoted by $\alpha(\Gamma)$, is the maximum cardinality of an independent set among all independent sets of Γ. Given a group G, for convenience, we will denote $\alpha(\Gamma(G))$ as $t(G)$. Moreover, for a vertex $r\in\pi(G)$, let $t(r,G)$ denote the maximal number of vertices in independent sets of $\Gamma(G)$ containing r. Our notation for simple groups is borrowed from [5]. Especially, we denote by $\mathbb{A}_m$ and $\mathbb{S}_m$, the alternating and symmetric group of degree m, respectively.

1 Preliminaries

We start with a well-known theorem due to Gruenberg and Kegel.

Theorem 2 (Theorem A, [25]). *Let G be a finite group such that $s(G)\geqslant 2$. Then one of the following statements holds:*

(1) *G is a Frobenius group or a 2-Frobenius group,*

(2) *G is an extension of a nilpotent normal $\pi_1(G)$-group N by a group G_1, where $P\leqslant G_1\leqslant \mathrm{Aut}(P)$, P is a non-Abelian simple group and G_1/P is a $\pi_1(G)$-group. Moreover $s(P)\geqslant s(G)$, and for every i, $2\leqslant i\leqslant s(G)$, there exists j, $2\leqslant j\leqslant s(P)$, such that $\pi_i(G)=\pi_j(P)$.*

Remark 2.1 (a) A group $G = ABC$ is a 2-Frobenius group if AB is a Frobenius group with complement B and $G/A = (BC)/A$ is a Frobenius group with complement C/A. Note that a 2-Frobenius group is always solvable.

(b) For a finite group G, the connected component $\pi_i(G)$ for each $i \geqslant 2$, is a clique.

The following theorem due to Vasilev can be applied to a wide class of finite groups including the groups with connected prime graph.

Theorem 3 (Theorem 1, [22]). *Let G be a finite group with $t(G) \geqslant 3$ and $t(2, G) \geqslant 2$, and let K be the maximal normal solvable subgroup of G. Then the quotient group G/K is an almost simple group, i.e., there exists a finite non-Abelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$.*

We will also need the following lemma which is taken from [14, Table 4].

Lemma 1. *Let S be a simple group and $S \in \mathcal{S}_{17}$. Then either* $\mathrm{Out}(S) = 1$ *or* $\pi(\mathrm{Out}(S)) \subseteq \{2, 3\}$.

2 Main Results

In this section, we will deal with the simple groups $G_2(4)$, $S_4(8)$, $O_8^+(3)$, $L_5(3)$, $O_8^-(2)$, $U_4(4)$, $U_3(17)$, $O_{10}^-(2)$, $S_4(13)$, $L_3(16)$, $S_6(4)$, $O_8^+(4)$ and $F_4(2)$. For convenience, the prime graphs associated with these simple groups are depicted in Fig. 1.

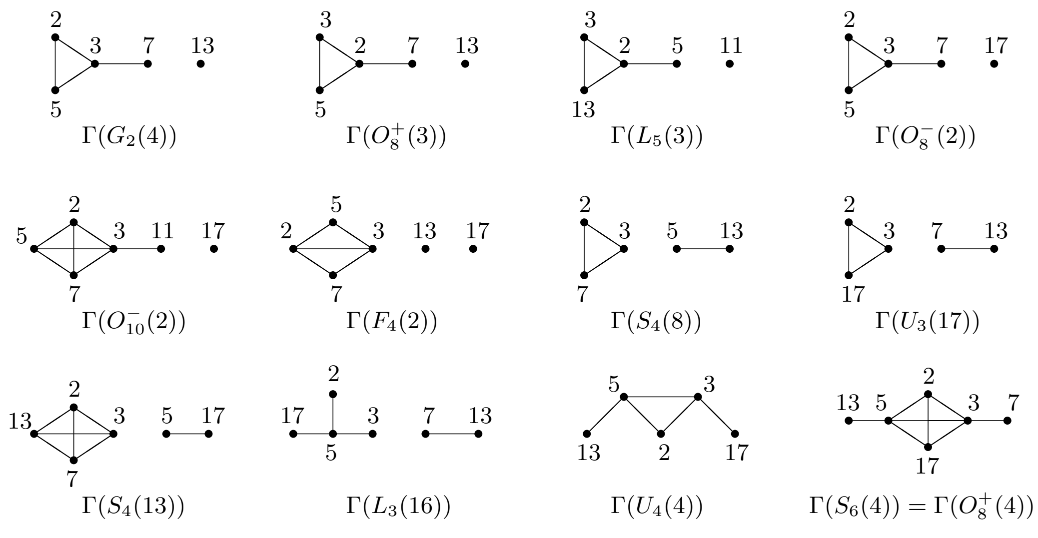

Fig. 1. Prime graphs associated with some simple groups.

Proof of Proposition 3. Let H be one of the following simple groups $G_2(4)$, $S_4(8)$, $O_8^+(3)$, $L_5(3)$, $O_8^-(2)$, $U_4(4)$, $U_3(17)$, $O_{10}^-(2)$, $S_4(13)$, $L_3(16)$, $S_6(4)$, $O_8^+(4)$ or $F_4(2)$. Suppose that G is a finite group such that $|G| = |H|$ and $D(G) = D(H)$. We have to prove that $G \cong H$. We now consider two cases separately.

Case 1. *H is isomorphic to one of the groups: $G_2(4)$, $O_8^+(3)$, $L_5(3)$, $O_8^-(2)$, $O_{10}^-(2)$ or $F_4(2)$.*

In all cases, we conclude that $\Gamma(G) = \Gamma(H)$ which is disconnected, and so $t(G) \geqslant 3$ and $t(2, G) \geqslant 3$. Now, it follows from Theorem 3 that there exists a finite non-Abelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G. If $S \cong H$, then $K = 1$ and G is isomorphic to H, because $|G| = |H|$. Therefore, in what follows, we will prove that $S \cong H$.

(1) $H \cong G_2(4)$, $O_8^+(3)$, $L_5(3)$ or $O_8^-(2)$. Analysis of different possibilities for H proceeds along similar lines, so, we only handle one case. Assume that $H \cong G_2(4)$. In this case, we have $|G| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$, $D(G) = (2, 3, 2, 1, 0)$ and $\Gamma(G) = \Gamma(G_2(4))$. Since the prime graphs of G and $G_2(4)$ coincide, $\{5, 7, 13\}$ is an independent set in $\Gamma(G)$. Now we claim that K is a $\{5, 7, 13\}'$-group. Let $\{p_1, p_2, p_3\} = \{5, 7, 13\}$. If $\pi(K) \cap \{p_1, p_2, p_3\}$ contains at least 2 primes, say p_i and p_j, then a Hall $\{p_i, p_j\}$-subgroup of K is an Abelian group. Hence $p_i \sim p_j$ in $\Gamma(K)$, and so in $\Gamma(G)$, a contradiction. Assume now that $p_i \in \pi(K)$ and $p_j \notin \pi(K)$. Let $P_i \in \mathrm{Syl}_{p_i}(K)$. By Frattini argument $G = KN_G(P_i)$. Therefore, the normalizer $N_G(P_i)$ contains an element of order p_j, say x. Now, $P\langle x \rangle$ is a subgroup of G, which is again an Abelian group, and so it leads to a contradiction as before. Finally, since K and $\mathrm{Out}(S)$ are $\{5, 7, 13\}'$-groups, $|S|$ is divisible by $5^2 \cdot 7 \cdot 13$. Comparing the orders of simple groups in $\mathcal{S}_{\leqslant 17}$ yields S is isomorphic to $G_2(4)$.

(2) $H \cong O_{10}^-(2)$. In this case, we have $|G| = 2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$, $D(G) = (3, 4, 3, 3, 1, 0)$ and $\Gamma(G) = \Gamma(O_{10}^-(2))$. Again, using similar arguments as those in part (1), we can show that K is a $\{7, 11, 17\}'$-group. Moreover, since K and $\mathrm{Out}(S)$ is a $\{7, 11, 17\}'$-group, $|S|$ is divisible by $7 \cdot 11 \cdot 17$. Comparing the orders of simple groups in $\mathcal{S}_{\leqslant 17}$ yields S is isomorphic to $O_{10}^-(2)$.

(3) $H \cong F_4(2)$. In this case, we have $|G| = 2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$, $D(G) = (3, 3, 2, 2, 0, 0)$ and $\Gamma(G) = \Gamma(F_4(2))$. As before, one can show that K is a $\{2, 3\}$-group. Moreover, since K and $\mathrm{Out}(S)$ is a $\{2, 3\}$-group, $|S|$ is divisible by $5^2 \cdot 7^2 \cdot 13 \cdot 17$. Considering the orders of simple groups in $\mathcal{S}_{\leqslant 17}$, we conclude that S is isomorphic to $F_4(2)$.

Case 2. *H is isomorphic to one of the groups:* $L_3(16)$, $U_3(17)$, $U_4(4)$, $S_4(8)$, $S_4(13)$, $S_6(4)$ *or* $O_8^+(4)$.

(4) $H \cong U_3(17)$ or $S_4(8)$. Here, we will illustrate only the proof for $U_3(17)$, other case is similar. Assume that $H \cong U_3(17)$. In fact, G is a finite group such that $|G| = 2^6 \cdot 3^4 \cdot 7 \cdot 13 \cdot 17^3$ and $D(G) = (2, 2, 1, 1, 2)$. Notice that, according to our hypothesis there are several possibilities for the prime graph of G, as shown in Fig. 2:

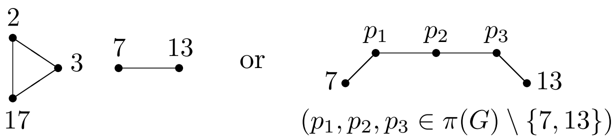

Fig. 2. All possibilities for the prime graph of G.

We now consider two cases separately.

(4.1) Assume first that $\Gamma(G)$ is connected. In this case $7 \nsim 13$ in $\Gamma(G)$. Since $\{7, 13, p_2\}$ is an independent set, $t(G) \geqslant 3$ and so G is a non-solvable group. Moreover, since $\deg_G(2) = 2$ and $|\pi(G)| = 5$, $t(2, G) \geqslant 2$. Thus by Theorem 3 there exists a finite non-Abelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G. We claim now that K is a $\{2, 3\}$-group. If $\{7, 13\} \subseteq \pi(K)$, then a Hall subgroup of K has order $7 \cdot 13$, which is an Abelian subgroup. Hence $7 \sim 13$ in $\Gamma(G)$, a contradiction. Suppose now that $\{p, q\} = \{7, 13\}$, $p \in \pi(K)$ and $q \notin \pi(K)$. Let $P \in \mathrm{Syl}_p(K)$. By Frattini argument $G = KN_G(P)$. Therefore, the normalizer $N_G(P)$ contains an element of order q, say x. Now, $P\langle x\rangle$ is a subgroup of G of order $7 \cdot 13$, which leads again to a contradiction. Therefore, if $17 \in \pi(K)$, then similar arguments as above show that $7 \sim 17 \sim 13$, against our hypothesis on the degree pattern of G. Finally, since K and $\mathrm{Out}(S)$ are $\{2, 3\}$-groups, $|S|$ is divisible by $7 \cdot 13 \cdot 17^3$. Comparing the orders of simple groups in $\mathcal{S}_{17}$ yields S is isomorphic to $U_3(17)$, and so $K = 1$ and G is isomorphic to $U_3(17)$, because $|G| = |U_3(17)|$. But then $\Gamma(G) = \Gamma(U_3(17))$ is disconnected, which is impossible.

(4.2) Assume next that $\Gamma(G)$ is disconnected, which immediately implies that $\Gamma(G) = \Gamma(U_3(17))$. We now apply Theorem 2 to obtain a simple section of G. If G is a Frobenius group with kernel K and complement C, then $|K| = 2^6 \cdot 3^4 \cdot 17^3$ and $|C| = 7 \cdot 13$, which is a contradiction because $|C| \nmid |K| - 1$. If G is a 2-Frobenius group, then G is a solvable group (Remark 2.1 (a)) and we may consider a Hall $\{7, 17\}$-subgroup L of G of order $7 \cdot 17^3$. By Sylow's Theorem it follows that every Sylow subgroup of L is normal in L and so L is a nilpotent group. This forces $7 \sim 17$ in

$\Gamma(G)$, which is a contradiction. Therefore G is an extension of a nilpotent normal $\{2, 3, 17\}$-group N by a group G_1, where $P \leqslant G_1 \leqslant \mathrm{Aut}(P)$, P is a non-Abelian simple group and G_1/P is a $\{2, 3, 17\}$-group. Hence $|P| = 2^\alpha \cdot 3^\beta \cdot 7 \cdot 13 \cdot 17^\gamma$, where $2 \leqslant \alpha \leqslant 6$, $1 \leqslant \beta \leqslant 4$ and $1 \leqslant \gamma \leqslant 3$, and comparing the order and degree pattern of simple groups in $\mathcal{S}_{\leqslant 17}$ with $|G|$ and $D(G)$, it is easy to see that P can be isomorphic to $U_3(17)$. Therefore $N = 1$ and so $G = G_1$ is isomorphic to H, because $|G| = |H|$.

(5) $H \cong S_4(13)$. The proof is quite similar to the proof in part (4), so we avoid here full explanation of all details. Assume that G is a finite group such that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^4 \cdot 17$ and $D(G) = (3, 3, 1, 3, 3, 1)$. Then, the prime graph of G is one of the following graphs (as shown in Fig. 3), according to $\Gamma(G)$ is disconnected or connected.

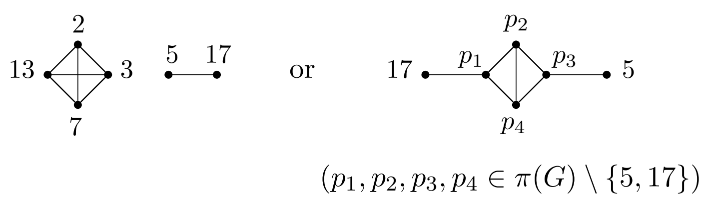

Fig. 3. All possibilities for the prime graph of G.

We will consider two cases separately.

(5.1) Suppose first that $\Gamma(G)$ is connected. In this case $5 \nsim 17$ in $\Gamma(G)$. Since $\{5, 17, p_2\}$ is an independent set, $t(G) \geqslant 3$ and so G is a non-solvable group. Moreover, since $\deg_G(2) = 3$ and $|\pi(G)| = 6$, $t(2, G) \geqslant 2$. Thus by Theorem 3 there exists a finite non-Abelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G. As before, one can show that K is a $\{2, 3, 13\}$-group. Since K and $\mathrm{Out}(S)$ are $\{2, 3, 13\}$-groups, $|S|$ is divisible by $5 \cdot 7^2 \cdot 17$. Comparing the orders of simple groups in $\mathcal{S}_{\leqslant 17}$ yields S is isomorphic to $S_4(13)$, and so $K = 1$ and G is isomorphic to $S_4(13)$, because $|G| = |S_4(13)|$. But then $\Gamma(G) = \Gamma(S_4(13))$ is disconnected, which is impossible.

(5.2) Suppose next that $\Gamma(G)$ is disconnected, which immediately implies that $\Gamma(G) = \Gamma(S_4(13))$. We now apply Theorem 2 to obtain a simple section of G. Similar to the previous case, G is neither Frobenius nor 2-Frobenius. Therefore G is an extension of a nilpotent normal $\{2, 3, 7, 13\}$-group N by a group G_1, where $P \leqslant G_1 \leqslant \mathrm{Aut}(P)$, P is a non-Abelian simple group and G_1/P is a $\{2, 3, 7, 13\}$-group. Hence $|P| = 2^\alpha \cdot 3^\beta \cdot 5 \cdot 7^\gamma \cdot 13^\lambda \cdot 17$, where $2 \leqslant \alpha \leqslant 6$, $0 \leqslant \beta \leqslant 2$, $0 \leqslant \gamma \leqslant 2$ and $0 \leqslant \lambda \leqslant 4$, and comparing the order and degree pattern of simple groups in $\mathcal{S}_{\leqslant 17}$ with the order and degree pattern of G, it is easy to see that P can be isomorphic to $S_4(13)$. Therefore $N = 1$ and so $G = G_1$ is isomorphic to $S_4(13)$, because $|G| = |S_4(13)|$.

(6) $H \cong L_3(16)$. In this case, we have $|G| = 2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$, $D(G) = (1, 1, 3, 1, 1, 1)$ and it is easy to see that $s(G) = 2$, $5 \in \pi_1(G)$ and G has no element of order 6 (Note that $\pi_2(G)$ is a clique (Remark 2.1 (b)) and hence $5 \notin \pi_2(G)$). Now, it follows from Theorem 3 that there exists a finite non-Abelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G. We claim now that K is a $\{2, 3\}$-group. In fact, if p_1, p_2, p_3 are the primes in $\pi(G) \setminus \{2, 3, 5\}$ and if there exists $p_j \in \pi(K)$, then with similar arguments as part (1), we can verify that for each $i \neq j$, $p_j \sim p_i$ in $\Gamma(G)$, and this contradicts the fact that $\deg_G(p_i) = 1$. Moreover, if $5 \in \pi(K)$, then $5 \sim p_i$ for each $i = 1, 2, 3$. But since $5 \in \pi_1(G)$, we also have $2 \sim 5$, against our hypothesis that $\deg_G(5) = 3$. Our claim follows. Therefore, since $\mathrm{Out}(S)$ is a $\{2, 3\}$-group, $|S|$ is divisible by $5^2 \cdot 7 \cdot 13 \cdot 17$. Comparing the orders of simple groups in $\mathcal{S}_{\leqslant 17}$ yields S is isomorphic to $L_3(16)$.

(7) $H \cong U_4(4)$. Here, we have $|G| = 2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$ and $D(G) = (2, 3, 3, 1, 1)$. We have to show that $G \cong U_4(5)$. First of all, from the structure of the degree pattern of G, it is easy to see that $13 \nsim 17$ in $\Gamma(G)$, since otherwise $\deg(3) \leqslant 2$, which is impossible. In fact, there are only two possibilities for the prime graph of G shown in Fig. 4.:

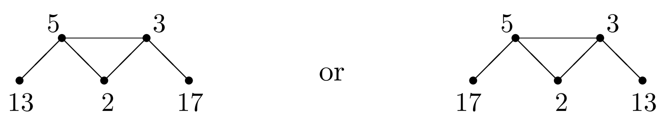

Fig. 4. All possibilities for the prime graph of G.

Clearly, in both cases, $t(G) \geqslant 3$ and $t(2, G) \geqslant 2$. Now, from Theorem 3, we conclude that there exists a finite non-Abelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G. As before, one can show that K is a $\{13, 17\}'$-group, and so $\{13, 17\} \subseteq \pi(S)$. Moreover, since $|S|$ divides $|G|$, we obtain $|S| = 2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma} \cdot 13 \cdot 17$, where $2 \leqslant \alpha \leqslant 12$, $0 \leqslant \beta \leqslant 2$ and $0 \leqslant \gamma \leqslant 3$. Comparing the orders of simple groups listed in Table 2, we observe that the only possibility for S is $U_4(4)$, and since $|G| = |U_4(4)|$, we obtain $|K| = 1$ and G is isomorphic to $U_4(4)$.

(8) $H \cong S_6(4)$ or $O_8^+(4)$. We only prove the first case and the second one

goes similarly. Suppose that $H \cong S_6(4)$. In this case, we have $|G| = 2^{18} \cdot 3^4 \cdot 5^3 \cdot 7 \cdot 13 \cdot 17$ and $D(G) = (3, 4, 4, 1, 1, 3)$. From the degree pattern of G, it is easy to see that $13 \nsim 7$ in $\Gamma(G)$. In fact, there are only two possibilities for the prime graph of G shown in Fig. 5.:

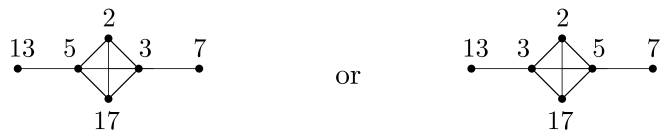

Fig. 5. All possibilities for the prime graph of G.

Clearly, in both cases, $t(G) \geqslant 3$ and $t(2, G) \geqslant 3$. Now, from Theorem 3, we conclude that there exists a finite non-Abelian simple group S such that $S \leqslant G/K \leqslant \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G. As before, one can show that K is a $\{7, 13, 17\}'$-group, and so $\{7, 13, 17\} \subseteq \pi(S)$. Moreover, since $|S|$ divides $|G|$, we obtain $|S| = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7 \cdot 13 \cdot 17$, where $2 \leqslant \alpha \leqslant 18$, $0 \leqslant \beta \leqslant 4$ and $0 \leqslant \gamma \leqslant 3$. Comparing the orders of simple groups listed in Table 2, we observe that the only possibilities for S are $L_3(16)$ and $S_6(4)$. If $S \cong L_3(16)$, then $7 \sim 13$ in $\Gamma(S)$ and so in $\Gamma(G)$, which is a contradiction. Therefore S is isomorphic to $S_6(4)$, and since $|G| = |S_6(4)|$, we obtain $|K| = 1$ and G is isomorphic to $S_6(4)$.

The proof is complete. $\square$

3 Appendix

As mentioned in the introduction, it was shown that many finite simple groups are OD-characterizable or 2-fold OD-characterizable. Table 3 lists finite simple groups which are currently known to be k-fold OD-characterizable for $k \in \{1, 2\}$. In this table q is a power of a prime number.

Among non-simple groups, there are many groups which are k-fold OD-characterizable for $k \geq 3$. In connection with such groups, Table 4 lists finite non-solvable groups which are currently known to be OD-characterizable or k-fold OD-characterizable with $k \geq 1$.

Table 3. Some simple groups S with $h_{\rm OD}(S) = 1$ or 2.

S	Conditions on S	$h_{\rm OD}$	Refs.
$\mathbb{A}_n$	$n = p, p+1, p+2$ (p a prime)	1	[15], [18]
	$5 \leqslant n \leqslant 100, n \neq 10$	1	[6], [9], [14], [16], [36]
	$n = 106,\ 112$	1	[26]
	$n = 10$	2	[17]
$L_2(q)$	$q \neq 2, 3$	1	[15], [18], [34]
$L_3(q)$	$\lvert\pi(\frac{q^2+q+1}{d})\rvert = 1,\ d = (3, q-1)$	1	[18]
$U_3(q)$	$\lvert\pi(\frac{q^2-q+1}{d})\rvert = 1,\ d = (3, q+1), q > 5$	1	[18]
$L_4(q)$	$q \leqslant 17$	1	[1], [4]
$L_3(9)$		1	[35]
$U_3(5)$		1	[33]
$U_4(5)$		1	[2]
$U_4(7)$		1	[4]
$L_6(3)$		1	[2]
$L_n(2)$	$n = p$ or $p+1,\ 2^p - 1$ Mersenne prime	1	[4]
$L_n(2)$	$n = 9, 10, 11$	1	[7], [13]
$R(q)$	$\lvert\pi(q \pm \sqrt{3q} + 1)\rvert = 1,\ q = 3^{2m+1},\ m \geq 1$	1	[18]
$\mathrm{Sz}(q)$	$q = 2^{2n+1} \geq 8$	1	[15], [18]
$B_m(q), C_m(q)$	$m = 2^f \geq 4,\ \lvert\pi\big((q^m+1)/2\big)\rvert = 1,$	2	[3]
$B_2(q) \cong C_2(q)$	$\lvert\pi\big((q^2+1)/2\big)\rvert = 1,\ q \neq 3$	1	[3]
$B_m(q) \cong C_m(q)$	$m = 2^f \geq 2,\ 2 \mid q,\ \lvert\pi\big(q^m+1\big)\rvert = 1,$ $(m, q) \neq (2, 2)$	1	[3]
$B_p(3), C_p(3)$	$\lvert\pi\big((3^p-1)/2\big)\rvert = 1,\ p$ is an odd prime	2	[3], [18]
$B_3(5), C_3(5)$		2	[3]
$C_3(4)$		1	[12]
S	A sporadic group	1	[18]
S	A group with $\lvert\pi(S)\rvert = 4,\ \ S \neq \mathbb{A}_{10}$	1	[32]
S	A group with $\lvert S\rvert \leqslant 10^8,\ S \neq \mathbb{A}_{10},\ U_4(2)$	1	[30]
S	A simple $C_{2,2}$- group	1	[15]

Table 4. Some non-solvable groups G with certain $h_{\mathrm{OD}}(G)$.

G	Conditions on G	$h_{\mathrm{OD}}(G)$	Refs.
$\mathrm{Aut}(M)$	M is a sporadic group $\neq J_2, M^cL$	1	[15]
$\mathbb{S}_n$	$n = p,\ p+1$ ($p \geq 5$ is a prime)	1	[15]
$\mathrm{PGL}(2,q)$		1	[29]
M	$M \in \mathcal{C}_1$	2	[17]
M	$M \in \mathcal{C}_2$	8	[17]
M	$M \in \mathcal{C}_3$	3	[6], [9], [14], [16], [26]
M	$M \in \mathcal{C}_4$	2	[17]
M	$M \in \mathcal{C}_5$	3	[17]
M	$M \in \mathcal{C}_6$	6	[14]
M	$M \in \mathcal{C}_7$	1	[31]
M	$M \in \mathcal{C}_8$	9	[31]
M	$M \in \mathcal{C}_9$	1	[33]
M	$M \in \mathcal{C}_{10}$	3	[33]
M	$M \in \mathcal{C}_{11}$	6	[33]
M	$M \in \mathcal{C}_{12}$	1	[27]
M	$M \in \mathcal{C}_{13}$	1	[13]

$$
\begin{aligned}
\mathcal{C}_1 &= \{\mathbb{A}_{10}, J_2 \times \mathbb{Z}_3\} \\
\mathcal{C}_2 &= \{\mathbb{S}_{10},\ \mathbb{Z}_2 \times \mathbb{A}_{10},\ \mathbb{Z}_2 \cdot \mathbb{A}_{10},\ \mathbb{Z}_6 \times J_2,\ \mathbb{S}_3 \times J_2,\ \mathbb{Z}_3 \times (\mathbb{Z}_2 \cdot J_2), \\
&\quad (\mathbb{Z}_3 \times J_2) \cdot \mathbb{Z}_2,\ \mathbb{Z}_3 \times \mathrm{Aut}(J_2)\}. \\
\mathcal{C}_3 &= \{\mathbb{S}_n,\ \mathbb{Z}_2 \cdot \mathbb{A}_n,\ \mathbb{Z}_2 \times \mathbb{A}_n\},\ \text{ where } 9 \leqslant n \leqslant 100 \\
&\quad \text{with } n \neq 10, 27, p, p+1 \ (p \text{ a prime}) \text{ or } n = 106,\ 112. \\
\mathcal{C}_4 &= \{\mathrm{Aut}(M^cL),\ \mathbb{Z}_2 \times M^cL\}. \\
\mathcal{C}_5 &= \{\mathrm{Aut}(J_2),\ \mathbb{Z}_2 \times J_2,\ \mathbb{Z}_2 \cdot J_2\}. \\
\mathcal{C}_6 &= \{\mathrm{Aut}(S_6(3)),\ \mathbb{Z}_2 \times S_6(3),\ \mathbb{Z}_2 \cdot S_6(3),\ \mathbb{Z}_2 \times O_7(3), \\
&\quad \mathbb{Z}_2 \cdot O_7(3),\ \ \mathrm{Aut}(O_7(3))\}. \\
\mathcal{C}_7 &= \{L_2(49) : 2_1,\ L_2(49) : 2_2,\ L_2(49) : 2_3\}. \\
\mathcal{C}_8 &= \{L \cdot 2^2,\ \mathbb{Z}_2 \times (L : 2_1),\ \mathbb{Z}_2 \times (L : 2_2),\ \mathbb{Z}_2 \times (L \cdot 2_3),\ \mathbb{Z}_2 \cdot (L : 2_1), \\
&\quad \mathbb{Z}_2 \cdot (L : 2_2),\ \mathbb{Z}_2 \cdot (L \cdot 2_3),\ \mathbb{Z}_4 \times L,\ (\mathbb{Z}_2 \times \mathbb{Z}_2) \times L\} \\
&\quad \text{where } L = L_2(49). \\
\mathcal{C}_9 &= \{U_3(5),\ U_3(5) : 2\} \\
\mathcal{C}_{10} &= \{U_3(5) : 3,\ \mathbb{Z}_3 \times U_3(5),\ \mathbb{Z}_3 \cdot U_3(5)\} \\
\mathcal{C}_{11} &= \{L : \mathbb{S}_3,\ \mathbb{Z}_2 \cdot (L : 3),\ \mathbb{Z}_3 \times (L : 2),\ \mathbb{Z}_3 \cdot (L : 2),\ (\mathbb{Z}_2 \times L) \cdot \mathbb{Z}_2, \\
&\quad (\mathbb{Z}_3 \cdot L) \cdot \mathbb{Z}_2\},\ \text{where } L = U_3(5). \\
\mathcal{C}_{12} &= \{\mathrm{Aut}(O_{10}^{+}(2), \mathrm{Aut}(O_{10}^{-}(2)\}, \\
\mathcal{C}_{13} &= \{\mathrm{Aut}(L_p(2)), \mathrm{Aut}(L_{p+1}(2))\},\ \text{where } 2^p - 1 \text{ is a prime.}
\end{aligned}
$$

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