Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **34** (2014) no. 2, 41–74. doi:10.1285/i15900932v34n2p41

On Certain New Modular Relations for the Rogers-Ramanujan Type Functions of Order Ten and Applications to Partitions

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Received: 10.7.2013; accepted: 20.3.2014.

Abstract. In this paper, we consider the Rogers-Ramanujan type functions $J(q)$ and $K(q)$ of order ten and establish several modular relations involving these identities, which are analogues to Ramanujan's forty identities for the Rogers-Ramanujan functions. Furthermore, we give partition theoretic interpretations of some of our modular relations.

Keywords: Rogers-Ramanujan functions, theta functions, partitions, colored partitions, modular relations.

MSC 2000 classification: 33D15, 33D90, 11P82, 11P83.

1 Introduction

Throughout the paper, we use the customary notation $(a;q)_0 := 1$,

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad n \geq 1,$$

$$(a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

The well-known Rogers-Ramanujan functions [19, 20, 23], are defined, for $|q|<1$, by

$$G(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{1}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{2}$$

In his Lost Notebook [21], Ramanujan recorded forty beautiful modular relations involving the Rogers-Ramanujan functions without proof. The forty identities were first brought before the mathematical world by B. J. Birch [10]. Many of these identities have been established by L. J. Rogers [25], G. N. Watson [29], D. Bressoud [12, 13], A. J. F. Biagioli [9]. Recently, B. C. Berndt et al. [8] offered proofs of 35 of the 40 identities. Most likely these proofs might have given by Ramanujan himself. A number of mathematicians tried to find new identities for the Rogers-Ramanujan functions similar to those which have been found by Ramanujan [21], including Berndt and H. Yesilyurt [7], Yesilyurt [31], S. Robins [22] and C. Gugg [15].

In view of the Ramanujan's forty identities, many of the Rogers-Ramanujan type functions were studied by many mathematicians. For example, S.-S. Huang [18] and S.-L. Chen and Huang [14] have derived a list of new modular relations for the Göllnitz-Gordon functions, N. D. Baruah, J. Bora, and N. Saikia [5] offered new proofs of many of these identities, and E. X. W. Xia and X. M. Yao [30] offered new proofs of some modular relations established by Huang [18] and Chen and Huang [14]. They also established some new relations which involve only Göllnitz-Gordon functions. H. Hahn [16, 17] has established several modular relations for the septic analogues of the Rogers-Ramanujan functions as well as relations that are connected with the Rogers-Ramanujan and Göllnitz-Gordon functions. Similarly Baruah and Bora [4] have obtained modular relations for the nonic analogues of the Rogers-Ramanujan functions. C. Adiga, K. R. Vasuki and N. Bhaskar [1] established several modular relations for the cubic functions. Vasuki, G. Sharat and K. R. Rajanna [28] studied two different cubic functions. Baruah and Bora [3] considered two functions of order twelve which are analogues of the Rogers-Ramanujan functions. Vasuki and P. S. Guruprasad [27] considered the Rogers-Ramanujan type functions of order twelve and established modular relations involving them. Adiga, Vasuki and B. R. Srivatsa Kumar [2] established modular relations involving two functions of Rogers-Ramanujan type. Almost all of these functions which have been studied so far are due to Rogers [24] and L. J. Slater [26]. Motivated by these, in Section 3 of this paper, we establish certain modular relations for the functions defined by

$$J(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{(-q;q)_\infty (q^3;q^{10})_\infty (q^7;q^{10})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty} \tag{3}$$

and

$$K(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+3)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{(-q;q)_\infty (q;q^{10})_\infty (q^9;q^{10})_\infty (q^{10};q^{10})_\infty}{(q;q)_\infty}, \quad (4)$$

which are analogous to the Rogers-Ramanujan functions. The identities (3) and (4) are due to Rogers [24]. In Section 4, we give partition theoretic interpretations of some of our modular relations.

2 Some Preliminary Results

Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (5)$$

The well-known Jacobi triple product identity [6, p. 35, Entry 19] in Ramanujan's notation is

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \quad (6)$$

The function $f(a,b)$ satisfies the following basic properties [6]:

$$f(a,b) = f(b,a), \quad (7)$$

$$f(1,a) = 2f(a,a^3), \quad (8)$$

$$f(-1,a) = 0, \quad (9)$$

and, if n is an integer,

$$f(a,b) = a^{n(n+1)/2} b^{n(n-1)/2} f(a(ab)^n, b(ab)^{-n}). \quad (10)$$

The three most interesting special cases of (5) are [6, p. 36, Entry 22]

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \quad (11)$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \quad (12)$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \quad (13)$$

Also, after Ramanujan, define

$$\chi(q) := (-q; q^2)_\infty.$$

For convenience, we define

$$f_n := f(-q^n) = (q^n; q^n)_\infty,$$

for a positive integer n.

In order to prove our modular relations involving $J(q)$ and $K(q)$, we first establish some lemmas.

Lemma 2.1. We have

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad f(q) = \frac{f_2^3}{f_1 f_4}, \qquad \chi(q) = \frac{f_2^2}{f_1 f_4},$$

$$\varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2} \qquad \text{and} \qquad \chi(-q) = \frac{f_1}{f_2}.$$

This lemma is a consequence of (6) and Entry 24 of [6, p. 39]. We shall use Lemma 2.1 many times in this paper.

It is easy to verify that

$$G(q) = \frac{f(-q^2, -q^3)}{f_1}, \qquad H(q) = \frac{f(-q, -q^4)}{f_1}, \tag{14}$$

$$J(q) = \frac{f(-q^3, -q^7)}{\varphi(-q)}, \qquad K(q) = \frac{f(-q, -q^9)}{\varphi(-q)}, \tag{15}$$

$$G(q)H(q) = \frac{f_5}{f_1} \qquad \text{and} \qquad J(q)K(q) = \frac{f_2 f_{10}^3}{f_1^3 f_5}. \tag{16}$$

Lemma 2.2. Let $m = [s/(s-r)]$, $l = m(s-r) - r$, $k = -m(s-r) + s$ and $h = mr - m(m-1)(s-r)/2$, $0 \le r < s$. Here $[x]$ denote the largest integer less than or equal to x. Then,

(*i*) $f(q^{-r}, q^s) = q^{-h} f(q^l, q^k)$,

(*ii*) $f(-q^{-r}, -q^s) = (-1)^m q^{-h} f(-q^l, -q^k)$.

Proof. Using (6), we have

$$\begin{aligned} f(q^{-r}, q^s) =& (-q^{-r}; q^{s-r})_\infty (-q^s; q^{s-r})_\infty (q^{s-r}; q^{s-r})_\infty \\ =& \Big\{ (1 + q^{-r})(1 + q^{-r+(s-r)}) \cdots \\ & \cdots (1 + q^{-r+(m-1)(s-r)})(1 + q^{-r+m(s-r)}) \cdots \Big\} \\ & \times (-q^s; q^{s-r})_\infty (q^{s-r}; q^{s-r})_\infty. \end{aligned}$$

Since $m = [s/(s-r)] = 1 + [r/(s-r)] > r/(s-r)$ and $m-1 = [r/(s-r)] \leq r/(s-r)$, we have $-r + m(s-r) > 0$ and $-r + (m-1)(s-r) \leq 0$. It is also clear that $-r + i(s-r) \leq 0,\ \forall\ i = 0, 1, ..., m-1$. Therefore, we can write

$$\begin{aligned} f(q^{-r}, q^{s}) =& q^{-r-r+(s-r)-r+2(s-r)-\cdots-r+(m-1)(s-r)} f(q^{r-(m-1)(s-r)}, q^{-r+m(s-r)}) \\ =& q^{-\left(mr-\frac{1}{2}m(m-1)(s-r)\right)} f(q^{-m(s-r)+s}, q^{m(s-r)-r}) \\ =& q^{-h} f(q^{l}, q^{k}). \end{aligned}$$

This completes the proof of (i). The proof of (ii) follows similarly. QED

Lemma 2.3. We have

$$\varphi(q^{1/5}) - \varphi(q^5) = 2q^{1/5} f(q^3, q^7) + 2q^{4/5} f(q, q^9), \tag{17}$$

$$32qf^5(q^3, q^7) + 32q^4 f^5(q, q^9) = \left(\frac{\varphi^2(q)}{\varphi(q^5)} - \varphi(q^5)\right)\{\varphi^4(q) - 4\varphi^2(q)\varphi^2(q^5) + 11\varphi^4(q^5)\}, \tag{18}$$

$$f(-q, -q^4) f(-q^2, -q^3) = f(-q) f(-q^5), \tag{19}$$

$$f(q, q^9) f(q^3, q^7) = \chi(q) f(-q^5) f(-q^{20}). \tag{20}$$

For the proof of the Lemma 2.3 see [6, Entries 9(vii) and 10(ii, vii), Chapter 19].

Lemma 2.4. [6, Entry 25(i) and (ii), p. 40]. We have

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4), \tag{21}$$

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8). \tag{22}$$

The following identity follows immediately from (21) and (22).

$$\varphi(q^4) + 2q\psi(q^8) = \varphi(q). \tag{23}$$

Also the following lemma is an easy consequence of (22).

Lemma 2.5. We have

$$\varphi(-q^a)\varphi(q^b) + \varphi(q^a)\varphi(-q^b) = 2\,\varphi(q^{4a})\varphi(q^{4b}) - 8\,q^{a+b}\psi(q^{8a})\psi(q^{8b}). \tag{24}$$

The function $f(a, b)$ satisfies a beautiful addition formula, which we need in proving some identities. For each positive integer k, let

$$U_k := a^{k(k+1)/2} b^{k(k-1)/2} \qquad \text{and} \qquad V_k := a^{k(k-1)/2} b^{k(k+1)/2}.$$

Then

$$f(a,b)=f(U_1,V_1)=\sum_{n=0}^{k-1}U_nf\left(\frac{U_{k+n}}{U_n},\frac{V_{k-n}}{U_n}\right). \tag{25}$$

For the proof of (25) see [6, p. 48, Entry 31]. The following two identities follow from (25) by setting $k=2$, $a=q$ and q^3 and $b=q^9$ and q^7, respectively:

$$f(q,q^9) = f(q^{12},q^{28})+qf(q^8,q^{32}), \tag{26}$$

$$f(q^3,q^7) = f(q^{16},q^{24})+q^3f(q^4,q^{36}). \tag{27}$$

Lemma 2.6. [6, Entry 29, p. 45]. If $ab=cd$, then

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)=2f(ac,bd)f(ad,bc), \tag{28}$$

$$f(a,b)f(c,d)-f(-a,-b)f(-c,-d)=2af\left(\frac{b}{c},ac^2d\right)f\left(\frac{b}{d},acd^2\right). \tag{29}$$

The following two identities follow immediately from (28) and (29), the first one by adding (28) and (29), and the second by setting $c=-a$ and $d=-b$ in (28).

$$f(a,b)f(c,d)=f(ac,bd)f(ad,bc)+af(b/c,ac^2d)f(b/d,acd^2), \tag{30}$$

$$f(a,b)f(-a,-b)=f(-a^2,-b^2)\varphi(-ab). \tag{31}$$

By induction, one can easily obtain the following interesting identities.

Lemma 2.7. For any integers $m>1$ and $r\geq 1$, we have

$$G(q^{m^r})=G(q)\prod_{n=0}^{r-1}\frac{f^{m-1}(-q^{5m^n})}{\prod_{l=1}^{m-1}f(-\omega_m^lq^{m^n},-\omega_m^{m-l}q^{4m^n})}, \tag{32}$$

and

$$H(q^{m^r})=H(q)\prod_{n=0}^{r-1}\frac{f^{m-1}(-q^{5m^n})}{\prod_{l=1}^{m-1}f(-\omega_m^lq^{2m^n},-\omega_m^{m-l}q^{3m^n})}. \tag{33}$$

where $\omega_m=e^{2\pi i/m}$.

We use a theorem of R. Blecksmith, J. Brillhart, and I. Gerst [11], which provides a representation for a product of two theta functions as a sum of m products of pairs of theta functions, under certain conditions. This theorem generalizes formulas of H. Schröter which can be found in [6].

Define, for $\epsilon\in\{0,1\}$ and $|ab|<1$,

$$f_\epsilon(a,b)=\sum_{n=-\infty}^{\infty}(-1)^{\epsilon n}(ab)^{n^2/2}(a/b)^{n/2}.$$

Theorem 2.8. (Blecksmith, Brillhart, and Gerst [11]). Let a, b, c, and d denote positive numbers with $|ab|, |cd| < 1$. Suppose that there exist positive integers α, β, and m such that

$$(ab)^{\beta} = (cd)^{\alpha(m-\alpha\beta)}.$$

Let $\epsilon_1, \epsilon_2 \in \{0, 1\}$, and define $\delta_1, \delta_2 \in \{0, 1\}$ by

$$\delta_1 \equiv \epsilon_1 - \alpha\epsilon_2 \ (mod\ 2) \quad and \quad \delta_2 \equiv \beta\epsilon_1 + p\epsilon_2 \ (mod\ 2),$$

respectively, where $p = m - \alpha\beta$. Then, if R denotes any complete residue system modulo m,

$$\begin{aligned} f_{\epsilon_1}(a,b)f_{\epsilon_2}(c,d) &= \sum_{r\in R}(-1)^{\epsilon_2 r}c^{r(r+1)/2}d^{r(r-1)/2} \\ &\times f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^{\alpha}}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^{\alpha}}\right) \\ &\times f_{\delta_2}\left(\frac{(b/a)^{\beta/2}(cd)^{p(m+1-2r)/2}}{c^{p}}, \frac{(a/b)^{\beta/2}(cd)^{p(m+1+2r)/2}}{d^{p}}\right). \end{aligned} \tag{34}$$

To prove some of our results we need the following Schröter's formulas. We assume that μ and ν are integers such that $\mu > \nu \geq 0$.

Lemma 2.9. [6, p. 68, (36.3)]. We have

$$\begin{aligned} &\frac{1}{2}\{\varphi(q^{\mu+\nu})\varphi(q^{\mu-\nu}) + \varphi(-q^{\mu+\nu})\varphi(-q^{\mu-\nu})\} \\ &= \sum_{m=0}^{\mu-1} q^{2\mu m^2} f(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)}) f(q^{2\mu+4\nu m}, q^{2\mu-4\nu m}). \end{aligned} \tag{35}$$

Lemma 2.10. [6, p. 68, (36.6)]. We have

$$\begin{aligned} &\frac{1}{2}\{\varphi(q^{\mu+\nu})\varphi(q^{\mu-\nu}) + \varphi(-q^{\mu+\nu})\varphi(-q^{\mu-\nu})\} + 2q^{\mu/2}\psi(q^{2\mu+2\nu})\psi(q^{2\mu-2\nu}) \\ &= \sum_{m=0}^{\mu-1} q^{2\mu m^2} f(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)}) f(q^{2\nu m+\mu/2}, q^{-2\nu m+\mu/2}). \end{aligned} \tag{36}$$

Lemma 2.11. [6, p. 72, (36.14)]. Let μ be an even positive integer, and suppose that ω is an odd positive integer such that $(\mu, \omega) = 1$, and $2\mu - \omega^2 > 0$, then

$$\begin{aligned} &\frac{1}{2}\{\varphi(q^{2\mu-\omega^2})\varphi(q) + \varphi(-q^{2\mu-\omega^2})\varphi(-q)\} + 2q^{\mu/2-(\omega^2-1)/4}\psi(q^{4\mu-2\omega^2})\psi(q^2) \\ &= \sum_{m=0}^{\mu-1} q^{4m^2} f(q^{(2\mu-\omega^2)(2\mu+4m)}, q^{(2\mu-\omega^2)(2\mu-4m)}) f(q^{\mu/2-2\omega m}, q^{\mu/2+2\omega m}). \end{aligned} \tag{37}$$

Yesilyurt [31, Theorem 3.1] gave a generalization of Rogers's identity, which has been used to prove some of the Ramanujan's forty identities for the Rogers-Ramanujan functions, as well as new identities for Rogers-Ramanujan functions. To prove some of our results, we use Corollary 3.2 found in [31].

Following Yesilyurt [31], we define

$$f_k(a,b)=\begin{cases} f(a,b) & \text{if } k\equiv 0 \pmod 2,\\ f(-a,-b) & \text{if } k\equiv 1 \pmod 2.\end{cases} \tag{38}$$

Let m be an integer and α,β,p and λ be positive integers such that

$$\alpha m^2+\beta=p\lambda.$$

Let δ and ε be integers. Further let l and t be real and x and y be nonzero complex numbers. Recall that the general theta functions f, f_k are defined by (5) and (38). With the parameters defined this way, we set

$$\begin{aligned} &R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)\\ &:=\sum_{\substack{k=0\\ n=2k+t}}^{p-1}(-1)^{\varepsilon k}y^k q^{\{\lambda n^2+p\alpha l^2+2\alpha nml\}/4} f_\delta(xq^{(1+l)p\alpha+\alpha nm}, x^{-1}q^{(1-l)p\alpha-\alpha nm})\\ &\quad\times f_{\varepsilon p+m\delta}(x^{-m}y^p q^{p\beta+\beta n}, x^m y^{-p} q^{p\beta-\beta n}). \end{aligned}\tag{39}$$

Lemma 2.12. [31, Corollary 3.2]. We have

$$R(\varepsilon,\delta,l,t,\alpha,\beta,m,p,\lambda,x,y)=R(\delta,\varepsilon,t,l,1,\alpha\beta,\alpha m,\lambda,p\alpha,y,x).$$

Lemma 2.13. If $m=5r\pm1$, then

$$\begin{aligned} &R(0,1,0,0,\alpha,\beta,m,5,\lambda,1,1)\\ &=\varphi(-q^{5\alpha})\varphi((-1)^m q^{5\beta})+2(-1)^{m+1}q^{(\alpha+\beta)/5}\varphi(-q^\alpha)\varphi((-1)^m q^\beta)\\ &\quad\times\left\{J(q^\alpha)J((-1)^{m+1}q^\beta)+(-1)^{m+1}q^{3(\alpha+\beta)/5}K(q^\alpha)K((-1)^{m+1}q^\beta)\right\}, \end{aligned}\tag{40}$$

and if $m=5r\pm2$, then

$$\begin{aligned} &R(0,1,0,0,\alpha,\beta,m,5,\lambda,1,1)\\ &=\varphi(-q^{5\alpha})\varphi((-1)^m q^{5\beta})+2(-1)^{m}q^{(4\alpha+\beta)/5}\varphi(-q^\alpha)\varphi((-1)^m q^\beta)\\ &\quad\times\left\{K(q^\alpha)J((-1)^{m+1}q^\beta)+(-1)^{m+1}q^{3(-\alpha+\beta)/5}J(q^\alpha)K((-1)^{m+1}q^\beta)\right\}. \end{aligned}\tag{41}$$

Proof. Using (39), we have

$$R(0,1,0,0,\alpha,\beta,m,5,\lambda,1,1)=\sum_{k=0}^{4}q^{\lambda k^2}f(-q^{5\alpha+2\alpha mk},-q^{5\alpha-2\alpha mk})\times f_m(q^{5\beta+2\beta k},q^{5\beta-2\beta k}). \tag{42}$$

Now, assume that $m=5r\pm1$. Setting $n=-rk$, $a=-q^{5\alpha+2\alpha k(5r\pm1)}$ and $b=-q^{5\alpha-2\alpha k(5r\pm1)}$ in (10), we find that

$$f(-q^{5\alpha+2\alpha k(5r\pm1)},-q^{5\alpha-2\alpha k(5r\pm1)})=(-1)^{rk}q^{\alpha rk^2(-5r\mp2)}f(-q^{5\alpha\pm2\alpha k},-q^{5\alpha\mp2\alpha k}).$$

Since $f(a,b)$ is symmetric in a and b, we have

$$f(-q^{5\alpha\pm2\alpha k},-q^{5\alpha\mp2\alpha k})=f(-q^{5\alpha+2\alpha k},-q^{5\alpha-2\alpha k}).$$

Since $\lambda=(25\alpha r^2\pm10\alpha r+\alpha+\beta)/5$, we can write (42) in the form

$$R(0,1,0,0,\alpha,\beta,m,5,\lambda,1,1)=\sum_{k=0}^{4}(-1)^{rk}q^{k^2(\alpha+\beta)/5}f(-q^{5\alpha+2\alpha k},-q^{5\alpha-2\alpha k})\times f_m(q^{5\beta+2\beta k},q^{5\beta-2\beta k}). \tag{43}$$

On employing Lemma 2.2 in (43) and after some simplifications, we obtain (40). The identity (41) can be proved in a similar way with $m=5r\pm2$. QED

3 Main Results

In this section, we present some of modular relations involving $J(q)$ and $K(q)$. In most of these identities the functions $J(q)$ and $K(q)$ occur in combinations

$$J(q^r)J(q^s)+q^{3(r+s)/5}K(q^r)K(q^s),\qquad \text{where } r+s\equiv0(\text{mod } 5), \tag{44}$$

$$J(q^r)K(q^s)-q^{3(r-s)/5}K(q^r)J(q^s),\qquad \text{where } r-s\equiv0(\text{mod } 5), \tag{45}$$

or when one or both of q^r and q^s are replaced by $-q^r$ and $-q^s$, respectively in either (44) or (45).

Theorem 3.1. We have

(i) $K(q)J(-q)+K(-q)J(q)=2\dfrac{f_4^3f_{20}}{f_2^4}$,

(ii) $K(q)J(-q)-K(-q)J(q)=-2q\dfrac{f_4f_{20}^3}{f_2^3f_{10}}$.

Proof. To prove (i), we set $a = -q,\ b = -q^9,\ c = q^3$ and $d = q^7$ in (28), to get

$$f(-q,-q^9)f(q^3,q^7) + f(q,q^9)f(-q^3,-q^7) = 2f(-q^4,-q^{16})f(-q^8,-q^{12}).$$

Now, dividing the above identity by $\varphi(-q)\varphi(q)$, and employing (15) and (19) and then using Lemma 2.1, we obtain (i).
Setting $a = -q,\ b = -q^9,\ c = q^3$ and $d = q^7$ in (2.25), we get

$$f(-q,-q^9)f(q^3,q^7) - f(q,q^9)f(-q^3,-q^7) = -2qf(-q^6,-q^{14})f(-q^2,-q^{18}).$$

Dividing the above identity by $\varphi(-q)\varphi(q)$ and employing (15), (16) and Lemma 2.1, we get (ii). This completes the proof of the theorem. QED

Theorem 3.2. We have

$$J(q)J(-q^4) + q^3K(q)K(-q^4) = \frac{f_2f_4^3f_{16}^2f_{20}}{f_1^2f_8^5}. \tag{46}$$

Proof. Recall the following identity which is due to Ramanujan and proved by Rogers [25] and Berndt et al. [8, Entry 3.5, p.8]:

$$G(q^{16})H(q) - q^3G(q)H(q^{16}) = \chi(q^2). \tag{47}$$

Identity (47) can be written in the form

$$\frac{G(q^{16})H(q)}{G(q)H(q^{16})} = \frac{\chi(q^2)}{G(q)H(q^{16})} + q^3. \tag{48}$$

Putting $r = 4$ and $m = 2$ in (32) and (33), and then multiplying the resulting identities by $H(q)$ and $G(q)$, respectively, we get

$$G(q^{16})H(q) = G(q)H(q)\prod_{n=0}^{3}\frac{f(-q^{5\cdot 2^n})}{f(q^{2^n},q^{4\cdot 2^n})}, \tag{49}$$

$$G(q)H(q^{16}) = G(q)H(q)\prod_{n=0}^{3}\frac{f(-q^{5\cdot 2^n})}{f(q^{2\cdot 2^n},q^{3\cdot 2^n})}. \tag{50}$$

Dividing (49) by (50), we find

$$\frac{G(q^{16})H(q)}{G(q)H(q^{16})} = \frac{f(q^2,q^3)f(q^4,q^6)f(q^8,q^{12})f(q^{16},q^{24})}{f(q,q^4)f(q^2,q^8)f(q^4,q^{16})f(q^8,q^{32})}. \tag{51}$$

Now, we show that

$$\frac{f(q^2,q^3)f(q^4,q^6)}{f(q,q^4)f(q^2,q^8)} = \frac{J(-q)}{K(-q)}. \tag{52}$$

By (6), we have

$$\begin{aligned}
&\frac{f(q^2,q^3)f(q^4,q^6)}{f(q,q^4)f(q^2,q^8)}\\
=&\frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty(q^5;q^5)_\infty(-q^4;q^{10})_\infty(-q^6;q^{10})_\infty(q^{10};q^{10})_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty(-q^2;q^{10})_\infty(-q^8;q^{10})_\infty(q^{10};q^{10})_\infty}\\
=&\frac{(-q;q^{10})_\infty(-q^2;q^{10})_\infty(-q^3;q^{10})_\infty(-q^4;q^{10})_\infty(-q^5;q^{10})_\infty}{(-q;q^{10})_\infty(-q^2;q^{10})_\infty(-q^3;q^{10})_\infty(-q^4;q^{10})_\infty(-q^5;q^{10})_\infty}\\
&\times\frac{(-q^6;q^{10})_\infty(-q^7;q^{10})_\infty(-q^8;q^{10})_\infty(-q^9;q^{10})_\infty(-q^{10};q^{10})_\infty}{(-q^6;q^{10})_\infty(-q^7;q^{10})_\infty(-q^8;q^{10})_\infty(-q^9;q^{10})_\infty(-q^{10};q^{10})_\infty}\\
&\times\frac{(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty}{(-q;q^{10})_\infty(-q^9;q^{10})_\infty}\\
=&\frac{f(q^3,q^7)}{f(q,q^9)}=\frac{J(-q)}{K(-q)}.
\end{aligned}$$

Using (52), we can write (51) in the form

$$\frac{G(q^{16})H(q)}{G(q)H(q^{16})}=\frac{J(-q)J(-q^4)}{K(-q)K(-q^4)}. \tag{53}$$

From (48) and (53), we see that

$$J(-q)J(-q^4)-q^3K(-q)K(-q^4)=\frac{\chi(q^2)K(-q)K(-q^4)}{G(q)H(q^{16})}. \tag{54}$$

We show that

$$\frac{\chi^2(q)H(q)K(-q)}{H(q^4)}=\frac{f_5}{f_1}=G(q)H(q). \tag{55}$$

Using (2), (15), (6) and Lemma 2.1, we see that

$$\begin{aligned}
&\frac{\chi^2(q)H(q)K(-q)}{H(q^4)}\\
&=\frac{f_{10}}{f_2}\frac{(q^8;q^{20})_\infty(q^{12};q^{20})_\infty(-q;q^{10})_\infty(-q^9;q^{10})_\infty}{(q;q^{10})_\infty(-q;q^{10})_\infty(q^{12};q^{20})_\infty(q^7;q^{10})_\infty(q^3;q^{10})_\infty(q^8;q^{20})_\infty(q^9;q^{10})_\infty(-q^9;q^{10})_\infty}\\
&=\frac{f_{10}\chi(-q^5)}{f_2\chi(-q)}=\frac{f_5}{f_1}=G(q)H(q).
\end{aligned}$$

Applying (55) and (16) in (54) and then using Lemma 2.1, we obtain (46).

QED

Theorem 3.3. We have

$$J(-q)J(-q^4)+q^3K(-q)K(-q^4)=\frac{1}{2q}\left\{1-\frac{f_1^2f_4^4f_{10}^5f_{16}^2f_{40}^5}{f_2^5f_5^2f_8^5f_{20}^4f_{80}^2}\right\}, \qquad (56)$$

$$J(-q^2)J(-q^3)+q^3K(-q^2)K(-q^3)=\frac{f_2^2f_3^2f_8^2f_{12}^2}{2qf_4^5f_6^5}\left\{\frac{f_2^5f_{12}^5}{f_1^2f_4^2f_6^2f_{24}^2}-\frac{f_{20}^5f_{30}^5}{f_{10}^2f_{15}^2f_{40}^2f_{60}^2}\right\}, \qquad (57)$$

$$K(-q)J(-q^6)+q^3J(-q)K(-q^6)=\frac{f_1^2f_4^2f_6^2f_{24}^2}{2q^2f_2^5f_{12}^5}\left\{\frac{f_4^5f_6^5}{f_2^2f_3^2f_8^2f_{12}^2}-\frac{f_{10}^5f_{60}^5}{f_5^2f_{20}^2f_{30}^2f_{120}^2}\right\}. \qquad (58)$$

Proof. By using (15) and Lemma 2.1, we see that (56) is equivalent to

$$f(q^3,q^7)f(q^{12},q^{28})+q^3f(q,q^9)f(q^4,q^{36})=\frac{1}{2q}\left(\varphi(q)\varphi(q^4)-\varphi(q^5)\varphi(q^{20})\right). \qquad (59)$$

Setting $\epsilon_1=\epsilon_2=0$, $a=b=q$, $c=d=q^4$, $\alpha=1$, $\beta=4$ and $m=5$ in Theorem 2.8 and using Lemma 2.2, we obtain

$$\begin{aligned}\varphi(q)\varphi(q^4)=&\varphi(q^5)\varphi(q^{20})+q^4f(q^{-3},q^{13})f(q^{12},q^{28})\\&+q^{16}f(q^{-11},q^{21})f(q^4,q^{36})+q^{36}f(q^{-19},q^{29})f(q^{-4},q^{44})\\&+q^{64}f(q^{-27},q^{37})f(q^{-12},q^{52})\\=&\varphi(q^5)\varphi(q^{20})+2qf(q^3,q^7)f(q^{12},q^{28})+2q^4f(q,q^9)f(q^4,q^{36}),\end{aligned}$$

which is same as (59). This completes the proof of (56). Proofs of (57) and (58) are similar to that of (56). QED

Theorem 3.4. We have

$$K(q^5)J(q^{10})+q^3J(q^5)K(q^{10})=\frac{f_{20}f_{50}^6}{2qf_5f_{10}^2f_{25}^3f_{100}}\left\{\frac{f_2^5f_{25}^2f_{100}^2}{f_1^2f_4^2f_{50}^5}-1\right\}, \qquad (60)$$

$$\begin{aligned}&J(q^5)J(-q^5)-q^6K(q^5)K(-q^5)\\&=\frac{f_2^4f_{20}^2}{4q^2f_4^2f_{10}^4}\left(\frac{f_2f_{25}^2}{f_1^2f_{50}}-1\right)\left(1-\frac{f_1^2f_4^2f_{50}^5}{f_2^5f_{25}^2f_{100}^2}\right)-2q^8\frac{f_{20}f_{100}^3}{f_{10}^3f_{50}},\end{aligned} \qquad (61)$$

$$\begin{aligned}&K^5(q)J^5(q^2)+q^3J^5(q)K^5(q^2)\\&=\frac{f_2^{10}f_{10}^{10}f_{20}^3}{32qf_1^{13}f_4^3f_5^7}\left\{\frac{f_2^{10}f_5^4f_{20}^4}{f_1^4f_4^4f_{10}^{10}}-1\right\}\left\{1-4\frac{f_1^4f_4^4f_{10}^{10}}{f_2^{10}f_5^4f_{20}^4}+11\frac{f_1^8f_4^8f_{10}^{20}}{f_2^{20}f_5^8f_{20}^8}\right\}.\end{aligned} \qquad (62)$$

Proof. We proceed to prove (60). We apply (17) with replacing q by q^5, we obtain

$$f(q^{15},q^{35})+q^3f(q^5,q^{45})=\frac{\varphi(q)-\varphi(q^{25})}{2q}.$$

Now, setting $a=q^5$ and q^{15} and $b=q^{45}$ and q^{35}, respectively, in (31) and substituting the resulting identities in the above equation, we deduce

$$\begin{aligned}f(-q^5,-q^{45})f(-q^{30},-q^{70})&+q^3f(-q^{15},-q^{35})f(-q^{10},-q^{90})\\&=f(-q^5,-q^{45})f(-q^{15},-q^{35})\frac{\varphi(q)-\varphi(q^{25})}{2q\varphi(-q^{50})}.\end{aligned}\tag{63}$$

Change q to $-q^5$ in (20), employ it in (63) and then use (15) and Lemma 2.1 to get (60).

Use (17) twice with replacing q by q^5 and $-q^5$, and then multiplying the resulting equations, we obtain

$$\begin{aligned}&f(-q^{15},-q^{35})f(q^{15},q^{35})-q^6f(-q^5,-q^{45})f(q^5,q^{45})\\&+q^3\{f(-q^{15},-q^{35})f(q^5,q^{45})-f(q^{15},q^{35})f(-q^5,-q^{45})\}\\&=-\frac{1}{4q^2}\left(\varphi(q)-\varphi(q^{25})\right)\left(\varphi(-q)-\varphi(-q^{25})\right).\end{aligned}\tag{64}$$

Divide both sides of (64) by $\varphi(-q^5)\varphi(q^5)$, then employ Theorem 3.1(ii) in the resulting equation, and then use Lemma 2.1, to obtain (61).

Employing (31) in (18), we obtain

$$\begin{aligned}f^5(-q,-q^9)f^5(-q^6,-q^{14})&+q^3f^5(-q^3,-q^7)f^5(-q^2,-q^{18})\\&=\frac{f^5(-q,-q^9)f^5(-q^3,-q^7)}{32q\varphi^5(-q^{10})}\left(\frac{\varphi^2(q)}{\varphi(q^5)}-\varphi(q^5)\right)\\&\qquad\left\{\varphi^4(q)-4\varphi^2(q)\varphi^2(q^5)+11\varphi^4(q^5)\right\}.\end{aligned}\tag{65}$$

Now, employ (15) and (20) in (65), and use Lemma 2.1, to get (62). This completes the proof of the theorem. QED

We prove the following two theorems using ideas similar to those of Watson [29]. In all proofs, one expresses the left sides of the identities in terms of theta functions by using (15). After clearing fractions, we see that the right side can be expressed as a product of two theta functions, say with summations indices m and n. One then tries to find a change of indices of the form

$$\alpha m+\beta n=5M+a\quad\text{and}\quad\gamma m+\delta n=5N+b,$$

or

$$\alpha m + \beta n = 10M + a \quad \text{and} \quad \gamma m + \delta n = 10N + b,$$

so that the product on the right side decomposes into the requisite sum of two products of theta functions on the left side.

Theorem 3.5. We have

$$K(q)J(q^6) + q^3 J(q)K(q^6) = \frac{f_2 f_{12}}{2q^2 f_1^2 f_6^2}\left\{\frac{f_5^2 f_{30}^2}{f_{10} f_{60}} - \frac{f_2^2 f_6^5}{f_4 f_3^2 f_{12}^2}\right\}, \tag{66}$$

$$J(q)J(q^4) + q^3 K(q)K(q^4) = \frac{f_2 f_8}{2q f_1^2 f_4^2}\left\{\frac{f_2^5}{f_1^2 f_8} - \frac{f_5^2 f_{20}^2}{f_{10} f_{40}}\right\}, \tag{67}$$

$$J(q^2)J(q^3) + q^3 K(q^2)K(q^3) = \frac{f_6 f_4}{2q f_3^2 f_2^2}\left\{\frac{f_2^5 f_6^2}{f_1^2 f_4^2 f_{12}} - \frac{f_{10}^2 f_{15}^2}{f_{20} f_{30}}\right\}. \tag{68}$$

Proof. Using (15) and Lemma 2.1, we can write (66) in the form

$$f(-q,-q^9)f(-q^{18},-q^{42}) + q^3 f(-q^3,-q^7)f(-q^6,-q^{54}) = \frac{1}{2q^2}\left\{\varphi(-q^5)\varphi(-q^{30}) - \varphi(-q^2)\varphi(q^3)\right\}. \tag{69}$$

Now, we shall start with the product $\varphi(q^3)\varphi(-q^2)$, which can be written as

$$\varphi(q^3)\varphi(-q^2) = f(q^3,q^3)f(-q^2,-q^2) = \sum_{m,n=-\infty}^{\infty} (-1)^n q^{3m^2+2n^2}.$$

In this representation, we make the change of indices by setting

$$3m - 2n = 5M + a \quad \text{and} \quad m + n = 5N + b,$$

where a and b have values selected from the set $\{0, \pm 1, \pm 2\}$. Then

$$m = M + 2N + (a+2b)/5 \quad \text{and} \quad n = -M + 3N + (3b - a)/5.$$

It follows that values of a and b are associated as in the following table:

a	0	± 1	± 2
b	0	± 2	∓ 1

When a assumes the values $-2,\ -1,\ 0,\ 1,\ 2$ in succession, it is easy to see that the corresponding values of $3m^2 + 2n^2$ are, respectively,

$$5M^2 - 4M + 30N^2 + 12N + 2,$$

$$5M^2 - 2M + 30N^2 - 24N + 5,$$

$$5M^2 + 30N^2,$$

$$5M^2+2M+30N^2+24N+5,$$
$$5M^2+4M+30N^2-12N+2.$$

It is evident, from the equations connecting m and n with M and N that, there is a one-one correspondence between all pairs of integers (m,n) and all sets of integers (M,N,a). From this correspondence, we deduce that

$$\begin{aligned}\varphi(q^3)\varphi(-q^2) =&f(q^3,q^3)f(-q^2,-q^2)=\sum_{m,n=-\infty}^{\infty}(-1)^n q^{3m^2+2n^2}\\ =&-q^2\sum_{M,N=-\infty}^{\infty}(-1)^{M+N}q^{5M^2-4M+30N^2+12N}\\ &-q^5\sum_{M,N=-\infty}^{\infty}(-1)^{M+N}q^{5M^2-2M+30N^2-24N}\\ &+\sum_{M,N=-\infty}^{\infty}(-1)^{M+N}q^{5M^2+30N^2}\\ &-q^5\sum_{M,N=-\infty}^{\infty}(-1)^{M+N}q^{5M^2+2M+30N^2+24N}\\ &-q^2\sum_{M,N=-\infty}^{\infty}(-1)^{M+N}q^{5M^2+4M+30N^2-12N}\\ =&-q^2f(-q,-q^9)f(-q^{18},-q^{42})-q^5f(-q^3,-q^7)f(-q^6,-q^{54})\\ &+f(-q^5,-q^5)f(-q^{30},-q^{30})-q^5f(-q^3,-q^7)f(-q^6,-q^{54})\\ &-q^2f(-q,-q^9)f(-q^{18},-q^{42}).\end{aligned}$$

After some simplifications, we arrive at (69). This completes the proof of (66). Using (15) and Lemma 2.1, we see that (67) is equivalent to

$$\begin{aligned}f(-q^3,-q^7)f(-q^{12},-q^{28})+&q^3f(-q,-q^9)f(-q^4,-q^{36})\\ &=\frac{1}{2q}\left\{\varphi(q)\varphi(-q^4)-\varphi(-q^5)\varphi(-q^{20})\right\}. \qquad (70)\end{aligned}$$

We have

$$\varphi(q)\varphi(-q^4)=f(q,q)f(-q^4,-q^4)=\sum_{m,n=-\infty}^{\infty}(-1)^n q^{m^2+4n^2}. \qquad (71)$$

In this representation, we make the change of indices by setting

$$m+n=5M+a \quad \text{and} \quad m-4n=5N+b,$$

where a and b have values selected from the set $\{0,\pm1,\pm2\}$. Then

$$m=4M+N+(4a+b)/5 \quad \text{and} \quad n=M-N+(a-b)/5.$$

It follows easily that $a=b$, and so $m=4M+N+a$ and $n=M-N$, where $-2\leq a\leq 2$. Thus, there is one-to-one correspondence between the set of all pairs of integers (m,n), $-\infty<m,n<\infty$, and triples of integers (M,N,a), $-\infty<M,N<\infty$, $-2\leq a\leq 2$. From (71), we find that

$$\begin{aligned}\varphi(q)\varphi(-q^4)&=\sum_{m,n=-\infty}^{\infty}(-1)^n q^{m^2+4n^2}\\&=\sum_{a=-2}^{2}q^{a^2}\sum_{M=-\infty}^{\infty}(-1)^M q^{20M^2+8aM}\\&\quad\times\sum_{N=-\infty}^{\infty}(-1)^N q^{5N^2+2aN}\\&=\sum_{a=-2}^{2}q^{a^2}f(-q^{20+8a},-q^{20-8a})f(-q^{5+2a},-q^{5-2a})\\&=\varphi(-q^5)\varphi(-q^{20})+2qf(-q^3,-q^7)f(-q^{12},-q^{28})\\&\quad+2q^4f(-q,-q^9)f(-q^4,-q^{36}),\end{aligned}$$

which is same as (70).
Using (15) and Lemma 2.1, we may write (68) in the form

$$\begin{aligned}f(-q^6,-q^{14})f(-q^9,-q^{21})&+q^3f(-q^2,-q^{18})f(-q^3,-q^{27})\\&=\frac{1}{2q}\left(\varphi(-q^6)\varphi(q)-\varphi(-q^{10})\varphi(-q^{15})\right). \qquad (72)\end{aligned}$$

We have

$$\varphi(-q^6)\varphi(q)=\sum_{m,n=-\infty}^{\infty}(-1)^m q^{6m^2+n^2}.$$

In this representation, we make the change of indices by setting

$$2m+n=5M+a \quad \text{and} \quad -3m+n=5N+b,$$

where a and b will have values from the set $\{0,\pm1,\pm2\}$. Then

$$m=M-N+(a-b)/5 \quad \text{and} \quad n=3M+2N+(3a+2b)/5.$$

It follows easily that $a = b$, and so $m = M - N$ and $n = 3M + 2N + a$, where $-2 \leq a \leq 2$. Thus, there is a one-one correspondence between all pairs of integers (m, n) and all sets of integers (M, N, a) as given above. Thus

$$\begin{aligned}\varphi(-q^6)\varphi(q) &= \sum_{m,n=-\infty}^{\infty} (-1)^m q^{6m^2+n^2} \\ &= \sum_{a=-2}^{2} q^{a^2} f(-q^{15+6a}, -q^{15-6a}) f(-q^{10+4a}, -q^{10-4a}) \\ &= \varphi(-q^{15})\varphi(-q^{10}) + 2q f(-q^9, -q^{21}) f(-q^6, -q^{14}) \\ &\quad + 2q^4 f(-q^3, -q^{27}) f(-q^2, -q^{18}),\end{aligned}$$

which is nothing but (72). This completes the proof of the theorem. $\boxed{QED}$

Theorem 3.6. We have

$$J(q)J(q^9) + q^6 K(q)K(q^9) = \frac{f_2 f_{18}}{2\,q^2 f_1^2 f_9^2} \left\{ \frac{f_4^5 f_{36}^5}{f_2^2 f_8^2 f_{18}^2 f_{72}^2} - 4q^5 \frac{f_8^2 f_{72}^2}{f_4 f_{36}} - \frac{f_5^2 f_{45}^2}{f_{10} f_{90}} \right\}, \quad (73)$$

$$K(q)J(q^{21}) + q^{12} J(q)K(q^{21}) = \frac{f_2 f_{42}}{2q^5 f_1^2 f_{21}^2} \left\{ \frac{f_5^2 f_{105}^2}{f_{10} f_{210}} + 4\,q^5 \frac{f_{24}^2 f_{56}^2}{f_{12} f_{28}} - \frac{f_{12}^5 f_{28}^5}{f_6^2 f_{14}^2 f_{24}^2 f_{56}^2} \right\}, \quad (74)$$

$$J(q^3)J(q^7) + q^6 K(q^3)K(q^7) = \frac{f_6 f_{14}}{2q^2 f_3^2 f_7^2} \left\{ \frac{f_4^5 f_{84}^5}{f_2^2 f_8^2 f_{42}^2 f_{168}^2} - 4\,q^{11} \frac{f_8^2 f_{168}^2}{f_4 f_{84}} - \frac{f_{15}^2 f_{35}^2}{f_{30} f_{70}} \right\}. \quad (75)$$

Proof. Using (15) and Lemma 2.1, we find that (73) is equivalent to

$$\begin{aligned}&f(-q^3, -q^7) f(-q^{27}, -q^{63}) + q^6 f(-q, -q^9) f(-q^9, -q^{81}) \\ &\quad = \frac{1}{2q^2} \{\varphi(q^2)\varphi(q^{18}) - 4\,q^5 \psi(q^4)\psi(q^{36}) - \varphi(-q^{45})\varphi(-q^5)\}. \quad (76)\end{aligned}$$

Changing q to q^2 in (76) and then using (24) with $a = 1$ and $b = 9$, we obtain

$$\begin{aligned}&f(-q^6, -q^{14}) f(-q^{54}, -q^{126}) + q^{12} f(-q^2, -q^{18}) f(-q^{18}, -q^{162}) \\ &\quad = \frac{1}{4q^4} \left\{ \varphi(q)\varphi(-q^9) + \varphi(-q)\varphi(q^9) - 2\varphi(-q^{90})\varphi(-q^{10}) \right\}. \quad (77)\end{aligned}$$

Thus, it suffices to establish (77).

We have

$$\varphi(q)\varphi(-q^9) = \sum_{m,n=-\infty}^{\infty} (-1)^n q^{m^2+9n^2}.$$

In this representation, we make the change of indices by setting

$$m+n = 10M + a \qquad \text{and} \qquad -9m+n = 10N+b,$$

where a and b will have values from the set $\{0, \pm 1, \pm 2, \pm 3, \pm 4, 5\}$. Then

$$m = M - N + (a-b)/10 \qquad \text{and} \qquad n = 9M + N + (9a+b)/10.$$

It follows easily that $a = b$, and so $m = M - N$ and $n = 9M + N + a$, where $-4 \leq a \leq 5$. Thus, there is a one-one correspondence between all pairs of integers (m, n) and all sets of integers (M, N, a) as given above. We therefore deduce from (11),

$$\begin{aligned}\varphi(q)\varphi(-q^9) = & \sum_{m,n=-\infty}^{\infty} (-1)^n q^{m^2+9n^2} \\ = & \sum_{a=-4}^{5} q^{a^2} \sum_{M=-\infty}^{\infty} (-1)^M q^{90M^2+18aM} \sum_{N=-\infty}^{\infty} (-1)^N q^{10N^2+2aN} \\ = & \varphi(-q^{90})\varphi(-q^{10}) + 2q f(-q^{72}, -q^{108}) f(-q^8, -q^{12}) \\ & + 2q^4 f(-q^{126}, -q^{54}) f(-q^6, -q^{14}) \\ & + 2q^9 f(-q^{36}, -q^{144}) f(-q^4, -q^{16}) \\ & + 2q^{16} f(-q^{18}, -q^{162}) f(-q^2, -q^{18}). \end{aligned} \tag{78}$$

Changing q to $-q$ in (78) and then adding the resulting identity with (78), we obtain (77). This completes the proof of (73).
Using (15) and Lemma 2.1, identity (74) can be written as

$$\begin{aligned} & f(-q, -q^9) f(-q^{63}, -q^{147}) + q^{12} f(-q^3, -q^7) f(-q^{21}, -q^{189}) \\ & = \frac{1}{2q^5} \{\varphi(-q^5)\varphi(-q^{105}) + 4\, q^5 \psi(q^{12})\psi(q^{28}) - \varphi(q^6)\varphi(q^{14})\}. \end{aligned} \tag{79}$$

Changing q to q^2 in (76) and then using (24) with $a = 3$ and $b = 7$, we obtain

$$\begin{aligned} & f(-q^2, -q^{18}) f(-q^{126}, -q^{294}) + q^{24} f(-q^6, -q^{14}) f(-q^{42}, -q^{378}) \\ & = \frac{1}{4q^{10}} \left\{2\, \varphi(-q^{10})\varphi(-q^{210}) - \varphi(q^3)\varphi(-q^7) - \varphi(-q^3)\varphi(q^7)\right\}. \end{aligned} \tag{80}$$

Thus, it suffices to establish (80).

We have

$$\varphi(q^3)\varphi(-q^7) = f(q^3, q^3)f(-q^7, -q^7) = \sum_{m,n=-\infty}^{\infty} (-1)^n q^{3m^2+7n^2}.$$

In this representation, we make the change of indices by setting

$$3m + 7n = 10M + a \quad \text{and} \quad -m + n = 10N + b,$$

where a and b have values selected from the set $\{0, \pm1, \pm2, \pm3, \pm4, 5\}$. Then

$$m = M - 7N + (a - 7b)/10 \quad \text{and} \quad n = M + 3N + (3b + a)/10.$$

It follows that values of a and b are associated as in the following table:

a	0	±1	±2	±3	±4	5
b	0	±3	∓4	∓1	±2	5

When a assumes the values $0, \pm1, \pm2, \pm3, \pm4, 5$ in succession, it is easy to see that the corresponding values of $3m^2 + 7n^2$ are, respectively,

$$\begin{gathered} 10M^2 + 210N^2 \\ 10M^2 \pm 2M + 210N^2 \pm 126N + 19, \\ 10M^2 \pm 4M + 210N^2 \mp 168N + 34, \\ 10M^2 \pm 6M + 210N^2 \mp 42N + 3, \\ 10M^2 \pm 8M + 210N^2 \pm 84N + 10, \\ 10M^2 + 10M + 210N^2 + 210N + 55. \end{gathered}$$

As before, it is evident from the equations connecting m and n with M and N that there is a one-one correspondence between all pairs of integers (m, n) and all sets of integers (M, N, a). From this correspondence, we deduce that

$$\begin{aligned} \varphi(q^3)\varphi(-q^7) =& \varphi(-q^{10})\varphi(-q^{210}) - 2q^{19} f(-q^8, -q^{12}) f(-q^{84}, -q^{336}) \\ & - 2q^{34} f(-q^6, -q^{14}) f(-q^{42}, -q^{378}) \\ & + 2q^3 f(-q^4, -q^{16}) f(-q^{168}, -q^{252}) \\ & - 2q^{10} f(-q^2, -q^{18}) f(-q^{126}, -q^{294}). \end{aligned} \tag{81}$$

Changing q to $-q$ in (81) and then adding the resulting identity with (81), we obtain (80). This completes the proof of (74).

The proof of (75) follows similarly, where we start with the product $\varphi(q)\varphi(-q^{21})$, and we make the change of indices by setting

$$m+7n=10M+a \quad \text{and} \quad -m+3n=10N+b.$$

This completes the proof of the theorem. QED

Theorem 3.7. We have

$$J(q)J(-q^{24})-q^{15}K(q)K(-q^{24}) \\ =\frac{f_2f_{24}^2f_{96}^2}{4q^5f_1^2f_{48}^5}\left\{2\frac{f_5^2f_{240}^5}{f_{10}f_{120}^2f_{480}^2}-\frac{f_3^2f_4^5}{f_2^2f_6f_8^2}-\frac{f_2^2f_6^5}{f_3^2f_4f_{12}^2}\right\}, \tag{82}$$

$$K(q)J(-q^{16})-q^9J(q)K(-q^{16}) \\ =\frac{f_2f_{16}^2f_{64}^2}{4q^4f_1^2f_{32}^5}\left\{\frac{f_1^2f_8^5}{f_2f_4^2f_{16}^2}+\frac{f_2^5}{f_1^2f_8}-2\frac{f_5^2f_{160}^5}{f_{10}f_{80}^2f_{320}^2}\right\}. \tag{83}$$

Proof. Setting $\mu=5$ and $\nu=1$ in (35), we find that

$$\frac{1}{2}\{\varphi(q^6)\varphi(q^4)+\varphi(-q^6)\varphi(-q^4)\}=f(q^{240},q^{240})f(q^{10},q^{10}) \\ +q^{10}f(q^{336},q^{144})f(q^{14},q^6)+q^{40}f(q^{432},q^{48})f(q^2,q^{18}) \\ +q^{90}f(q^{528},q^{-48})f(q^{-2},q^{22})+q^{160}f(q^{-144},q^{624})f(q^{-6},q^{26}). \tag{84}$$

Using Lemma 2.2 in (84), and then changing q^2 by $-q$, we obtain

$$f(-q^3,-q^7)f(q^{72},q^{168})-q^{15}f(-q,-q^9)f(q^{24},q^{216}) \\ =\frac{1}{4q^5}\left\{2\varphi(-q^5)\varphi(q^{120})-\varphi(-q^3)\varphi(q^2)-\varphi(q^3)\varphi(-q^2)\right\}. \tag{85}$$

Now, dividing (85) throughout by $\varphi(-q)\varphi(q^{24})$, and employing (15), and using Lemma 2.1, we obtain (82). Similarly, we obtain (83) by setting $\mu=5$ and $\nu=3$ in (35). This completes the proof of the theorem. QED

Theorem 3.8. We have

$$J(q)J(q^{99})+q^{60}K(q)K(q^{99}) \\ =\frac{f_2f_{198}}{4q^{20}f_1^2f_{99}^2}\left\{\frac{f_9^2f_{11}^2}{f_{18}f_{22}}+\frac{f_{18}^5f_{22}^5}{f_9^2f_{11}^2f_{36}^2f_{44}^2}-4q^5\frac{f_{36}^2f_{44}^2}{f_{18}f_{22}}-2\frac{f_5^2f_{495}^2}{f_{10}f_{990}}\right\}, \tag{86}$$

$$K(q)J(-q^{96})-q^{57}J(q)K(-q^{96}) \\ =\frac{f_2f_{96}^2f_{384}^2}{4q^{20}f_1^2f_{192}^5}\left\{\frac{f_{16}^5f_{24}^5}{f_8^2f_{12}^2f_{32}^2f_{48}^2}+\frac{f_8^2f_{12}^2}{f_{16}f_{24}}-4q^5\frac{f_{32}^2f_{48}^2}{f_{16}f_{24}}-2\frac{f_5^2f_{960}^5}{f_{10}f_{480}^2f_{1920}^2}\right\}, \tag{87}$$

$$K(q)J(q^{91})+q^{54}J(q)K(q^{91})$$
$$=-\frac{f_2f_{182}}{4q^{19}f_1^2f_{91}^2}\left\{\frac{f_7^2f_{13}^2}{f_{14}f_{26}}+\frac{f_{14}^5f_{26}^5}{f_7^2f_{13}^2f_{28}^2f_{52}^2}-4q^5\frac{f_{28}^2f_{52}^2}{f_{14}f_{26}}-2\frac{f_5^2f_{455}^2}{f_{10}f_{910}}\right\}, \quad (88)$$

$$J(q)J(-q^{84})-q^{51}K(q)K(-q^{84})$$
$$=-\frac{f_2f_{84}^2f_{336}^2}{4q^{17}f_1^2f_{168}^5}\left\{\frac{f_{12}^5f_{28}^5}{f_6^2f_{14}^2f_{24}^2f_{56}^2}+\frac{f_6^2f_{14}^2}{f_{12}f_{28}}-4q^5\frac{f_{24}^2f_{56}^2}{f_{12}f_{28}}-2\frac{f_5^2f_{840}^5}{f_{10}f_{420}^2f_{1680}^2}\right\}, \quad (89)$$

$$J(q)J(-q^{64})-q^{39}K(q)K(-q^{64})$$
$$=-\frac{f_2f_{64}^2f_{256}^2}{4q^{13}f_1^2f_{128}^5}\left\{\frac{f_8^5f_{32}^5}{f_4^2f_{16}^4f_{64}^2}+\frac{f_4^2f_{16}^2}{f_8f_{32}}-4q^5\frac{f_{16}^2f_{64}^2}{f_8f_{32}}-2\frac{f_5^2f_{640}^5}{f_{10}f_{320}^2f_{1280}^2}\right\}, \quad (90)$$

$$K(q)J(q^{51})+q^{30}J(q)K(q^{51})$$
$$=-\frac{f_2f_{102}}{4q^{11}f_1^2f_{51}^2}\left\{\frac{f_3^2f_{17}^2}{f_6f_{34}}+\frac{f_6^5f_{34}^5}{f_3^2f_{12}^2f_{17}^2f_{68}^2}-4q^5\frac{f_{12}^2f_{68}^2}{f_6f_{34}}-2\frac{f_5^2f_{255}^2}{f_{10}f_{510}}\right\}, \quad (91)$$

$$K(q)J(-q^{36})-q^{21}J(q)K(-q^{36})$$
$$=\frac{f_2f_{36}^2f_{144}^2}{4q^8f_1^2f_{72}^5}\left\{\frac{f_4^5f_{36}^5}{f_2^2f_8^2f_{18}^2f_{72}^2}+\frac{f_2^2f_{18}^2}{f_4f_{36}}-4q^5\frac{f_8^2f_{72}^2}{f_4f_{36}}-2\frac{f_5^2f_{360}^5}{f_{10}f_{180}^2f_{720}^2}\right\}, \quad (92)$$

$$J(q)J(q^{19})+q^{12}K(q)K(q^{19})$$
$$=\frac{f_2f_{38}}{4q^4f_1^2f_{19}^2}\left\{\frac{f_1^2f_{19}^2}{f_2f_{38}}+\frac{f_2^5f_{38}^5}{f_1^2f_4^2f_{19}^2f_{76}^2}-4q^5\frac{f_4^2f_{76}^2}{f_2f_{38}}-2\frac{f_5^2f_{95}^2}{f_{10}f_{190}}\right\}, \quad (93)$$

$$K(q)J(q^{11})+q^6J(q)K(q^{11})$$
$$=-\frac{f_2f_{22}}{4q^3f_1^2f_{11}^2}\left\{\frac{f_1^2f_{11}^2}{f_2f_{22}}+\frac{f_2^5f_{22}^5}{f_1^2f_4^2f_{11}^2f_{44}^2}-4q^3\frac{f_4^2f_{44}^2}{f_2f_{22}}-2\frac{f_5^2f_{55}^2}{f_{10}f_{110}}\right\}. \quad (94)$$

Proof. We proceed to prove (86). Setting $\mu=10$ and $\nu=1$ in (36), and employing Lemma 2.2 several times, we find that

$$\begin{aligned}
&\frac{1}{2}\{\varphi(q^{11})\varphi(q^{9})+\varphi(-q^{11})\varphi(-q^{9})\}+2q^{5}\psi(q^{22})\psi(q^{18})\\
=&f(q^{1980},q^{1980})f(q^{5},q^{5})+q^{20}f(q^{2376},q^{1584})f(q^{7},q^{3})\\
&+q^{80}f(q^{2772},q^{1188})f(q^{9},q)+q^{180}f(q^{3168},q^{792})f(q^{11},q^{-1})\\
&+q^{320}f(q^{3564},q^{396})f(q^{13},q^{-3})+q^{500}f(q^{3960},1)f(q^{15},q^{-5})\\
&+q^{720}f(q^{4356},q^{-396})f(q^{17},q^{-7})+q^{980}f(q^{4752},q^{-792})f(q^{19},q^{-9})\\
&+q^{1280}f(q^{5148},q^{-1188})f(q^{21},q^{-11})+q^{1620}f(q^{5544},q^{-1584})f(q^{23},q^{-13})\\
=&\varphi(q^{1980})\varphi(q^{5})+2q^{20}f(q^{1584},q^{2376})f(q^{3},q^{7})\\
&+2q^{80}f(q^{1188},q^{2772})f(q,q^{9})+2q^{179}f(q^{792},q^{3168})f(q,q^{9})\\
&+2q^{317}f(q^{396},q^{3564})f(q^{3},q^{7})+2q^{495}\psi(q^{3960})\varphi(q^{5}). \qquad (95)
\end{aligned}$$

We can write (95) in the form

$$\begin{aligned}
&\frac{1}{2}\left\{\varphi(q^{11})\varphi(q^{9})+\varphi\ (-q^{11})\varphi(-q^{9})\right\}+2q^{5}\psi(q^{22})\psi(q^{18})-2q^{495}\psi(q^{3960})\varphi(q^{5})\\
&-\varphi(q^{1980})\varphi(q^{5})=q^{20}f(q^{7},q^{3})\left\{f(q^{1584},q^{2376})+q^{297}f(q^{396},q^{3564})\right\}\\
&\qquad+2q^{80}f(q,q^{9})\left\{f(q^{1188},q^{2772})+q^{99}f(q^{792},q^{3168})\right\}. \qquad (96)
\end{aligned}$$

Using (26) and (27) with replacing q by q^{99}, and then using (23) with replacing q by q^{495}, we deduce that

$$\begin{aligned}
&f(q^{3},q^{7})f(q^{297},q^{693})+q^{60}f(q,q^{9})f(q^{99},q^{891})\\
=&\frac{1}{4q^{20}}\left\{\varphi(q^{11})\varphi(q^{9})+\varphi(-q^{11})\varphi(-q^{9})+4q^{5}\psi(q^{22})\psi(q^{18})-2\varphi(q^{5})\varphi(q^{495})\right\}. \qquad (97)
\end{aligned}$$

Replacing q by $-q$ in (97), then dividing the resulting identity throughout by $\varphi(-q)\varphi(-q^{99})$, employing (15), and using Lemma 2.1, we obtain (86). The proofs of (87)-(93) follow similarly, by setting $\mu = 10$ and $\nu = 2, 3, 4, 6, 7, 8, 9$ in (36), respectively. The proof of (94) follows similarly, by setting $\mu = 10$ and $\omega = 3$ in (37). This completes the proof of the theorem. QED

Theorem 3.9. We have

$$\begin{aligned}
&J(q^{3})J(q^{17})+q^{12}K(q^{3})K(q^{17})\\
&\quad=\frac{f_6f_{34}}{2q^4f_3^2f_{17}^2}\left\{\frac{f_8^5f_{408}^5}{f_4^2f_{16}^2f_{204}^2f_{816}^2}-\frac{f_{15}^2f_{85}^2}{f_{30}f_{170}}-2q^{13}\frac{f_4^2f_{204}^2}{f_2f_{102}}+4q^{52}\frac{f_{16}^2f_{816}^2}{f_8f_{408}}\right\}, \qquad (98)
\end{aligned}$$

$$J(q^7)J(q^{13}) + q^{12}K(q^7)K(q^{13})$$
$$= \frac{f_{14}f_{26}}{2q^4 f_7^2 f_{13}^2} \left\{ \frac{f_8^5 f_{728}^5}{f_4^2 f_{16}^2 f_{364}^2 f_{1456}^2} - \frac{f_{35}^2 f_{65}^2}{f_{70}f_{130}} - 2q^{23}\frac{f_4^2 f_{364}^2}{f_2 f_{182}} + 4q^{92}\frac{f_{16}^2 f_{1456}^2}{f_8 f_{728}} \right\}, \tag{99}$$

$$J(q^9)J(q^{11}) + q^{12}K(q^9)K(q^{11})$$
$$= \frac{f_{18}f_{22}}{2q^4 f_9^2 f_{11}^2} \left\{ \frac{f_8^5 f_{792}^5}{f_4^2 f_{16}^2 f_{396}^2 f_{1584}^2} - \frac{f_{45}^2 f_{55}^2}{f_{90}f_{110}} - 2q^{25}\frac{f_4^2 f_{396}^2}{f_2 f_{198}} + 4q^{100}\frac{f_{16}^2 f_{1584}^2}{f_8 f_{792}} \right\}, \tag{100}$$

$$J(q^2)J(q^{13}) + q^9K(q^2)K(q^{13})$$
$$= \frac{f_4 f_{26}}{2q^3 f_2^2 f_{13}^2} \left\{ \frac{f_6^5 f_{78}^2}{f_3^2 f_{12}^2 f_{156}} - \frac{f_{10}^2 f_{65}^2}{f_{20}f_{130}} - 2q^9 \frac{f_2^2 f_3 f_{12} f_{26} f_{156}^2}{f_1 f_4 f_6 f_{52} f_{78}} \right\}, \tag{101}$$

$$K(q)J(-q^{26}) - q^{15}J(q)K(-q^{26})$$
$$= \frac{f_2 f_{26}^2 f_{104}^2}{2q^6 f_1^2 f_{52}^5} \left\{ \frac{f_{12}^5 f_{39}^2}{f_6^2 f_{24}^2 f_{78}} - \frac{f_5^2 f_{260}^5}{f_{10} f_{130}^2 f_{520}^2} - 2q^5 \frac{f_4^2 f_6 f_{13} f_{24} f_{78}^2}{f_2 f_8 f_{12} f_{26} f_{39}} \right\}. \tag{102}$$

Proof. From Lemma 2.12, we find that

$$R(0,1,0,1,3,17,1,5,4,q^{-3},q^{-4}) = R(1,0,1,0,1,51,3,4,15,q^{-4},q^{-3}).$$

By (39) and Lemma 2.2, we have

$$\begin{aligned}R(0,1,0,1,3,17,1,5,4,q^{-3},q^{-4}) &= qf(-q^{15},-q^{15})f(-q^{85},-q^{85})\\ &+ q^5 f(-q^{21},-q^9)f(-q^{119},-q^{51}) + q^{17}f(-q^{27},-q^3)f(-q^{153},-q^{17})\\ &+ q^{37}f(-q^{33},-q^{-3})f(-q^{187},-q^{-17}) + q^{65}f(-q^{39},-q^{-9})f(-q^{221},-q^{-51})\\ =&q\varphi(-q^{15})\varphi(-q^{85}) + 2q^5 f(-q^9,-q^{21})f(-q^{51},-q^{119})\\ &+ 2q^{17}f(-q^3,-q^{27})f(-q^{17},-q^{153}).\end{aligned} \tag{103}$$

Similarly, we have

$$\begin{aligned}R(1,0,1,0,1,51,3,4,15,q^{-4},q^{-3})\\ =&qf(q^4,q^4)f(q^{204},q^{204}) - q^{16}f(q^{10},q^{-2})f(q^{306},q^{102})\\ &+ q^{61}f(q^{16},q^{-8})f(q^{408},1) - q^{136}f(q^{22},q^{-14})f(q^{510},q^{-102})\\ =&q\varphi(q^4)\varphi(q^{204}) - 2q^{14}\psi(q^2)\psi(q^{102}) + 4q^{53}\psi(q^8)\psi(q^{408}).\end{aligned} \tag{104}$$

Combining (103) with (104), we obtain

$$f(-q^9,-q^{21})f(-q^{51},-q^{119}) + q^{12}f(-q^3,-q^{27})f(-q^{17},-q^{153})$$
$$= \frac{1}{2q^4}\{\varphi(q^4)\varphi(q^{204}) - \varphi(-q^{15})\varphi(-q^{85}) - 2q^{13}\psi(q^2)\psi(q^{102}) + 4q^{52}\psi(q^8)\psi(q^{408})\}. \tag{105}$$

Dividing (105) throughout by $\varphi(-q^3)\varphi(-q^{17})$, employing (15), and using Lemma 2.1, we obtain (98). The proofs of (99)-(102) follow similarly. $\boxed{QED}$

Theorem 3.10. We have

$$K(q)J(q^{36})+q^{21}J(q)K(q^{36}) = \frac{f_2f_{72}}{2q^8f_1^2f_{36}^2}\left\{\frac{f_5^2f_{180}^2}{f_{10}f_{360}} - \frac{f_8^2f_{72}^2}{f_{16}f_{144}} + 2q^5\frac{f_4f_{16}f_{36}f_{144}}{f_8f_{72}}\right\}, \tag{106}$$

$$K(q^3)J(q^{28})+q^{15}J(q^3)K(q^{28}) = \frac{f_6f_{56}}{2q^8f_3^2f_{28}^2}\left\{\frac{f_{15}^2f_{140}^2}{f_{30}f_{280}} - \frac{f_8^2f_{168}^2}{f_{16}f_{336}} - 2q^{11}\frac{f_4f_{16}f_{84}f_{336}}{f_8f_{168}}\right\}, \tag{107}$$

$$K(q^7)J(q^{12})+q^3J(q^7)K(q^{12}) = \frac{f_{14}f_{24}}{2q^8f_7^2f_{12}^2}\left\{\frac{f_{35}^2f_{60}^2}{f_{70}f_{120}} - \frac{f_8^2f_{168}^2}{f_{16}f_{336}} + 2q^{11}\frac{f_4f_{16}f_{84}f_{336}}{f_8f_{168}}\right\}. \tag{108}$$

Proof. Using (15) and Lemma 2.1, we can write (106) in the following form:

$$f(-q,-q^9)f(-q^{108},-q^{252}) + q^{21}f(-q^3,-q^7)f(-q^{36},-q^{324}) = \frac{1}{2q^8}\left\{\varphi(-q^5)\varphi(-q^{180}) - \varphi(-q^8)\varphi(-q^{72}) + 2q^5\psi(-q^4)\psi(-q^{36})\right\}. \tag{109}$$

Setting $k=2$, $a=-q^{36}$ and $b=-q^{108}$ in (25), changing q to $-q^{72}$ in (23) and then using the resulting identities in (109), we obtain

$$\begin{aligned}\varphi(-q^5)\varphi(-q^{180}) - 2q^8f(-q,-q^9)f(-q^{108},-q^{252}) &\\ - 2q^{29}f(-q^3,-q^7)f(-q^{36},-q^{324}) &\\ = \varphi(-q^8)\{\varphi(q^{288}) - 2q^{72}\psi(q^{576})\} &\\ - 2q^5\psi(-q^4)\{f(q^{216},q^{360}) - 2q^{36}f(q^{72},q^{504})\}. &\end{aligned} \tag{110}$$

Thus (106) is equivalent to (110). But identity (110) can be verified easily using (39), Lemma 2.12 and Lemma 2.2, with the following sets of choice of parameters: $\varepsilon=\delta=\alpha=x=y=1$, $l=t=0$, $\beta=36$, $m=2$, $p=5$ and $\lambda=8$. This completes the proof of (106). The proofs of (107) and (108) follow similarly. $\boxed{QED}$

Theorem 3.11. Define

$$\begin{aligned}
U(\alpha,\beta) &:= \varphi(-q^{\alpha})\varphi(-q^{\beta})\left\{J(q^{\alpha})J(q^{\beta}) + q^{3(\alpha+\beta)/5}K(q^{\alpha})K(q^{\beta})\right\},\\
U^*(\alpha,\beta) &:= \varphi(q^{\alpha})\varphi(-q^{\beta})\left\{J(-q^{\alpha})J(q^{\beta}) - q^{3(\alpha+\beta)/5}K(-q^{\alpha})K(q^{\beta})\right\},\\
V(\alpha,\beta) &:= \varphi(-q^{\alpha})\varphi(-q^{\beta})\left\{K(q^{\alpha})J(q^{\beta}) + q^{3(\beta-\alpha)/5}J(q^{\alpha})K(q^{\beta})\right\},\\
V^*(\alpha,\beta) &:= \varphi(q^{\alpha})\varphi(-q^{\beta})\left\{K(-q^{\alpha})J(q^{\beta}) - q^{3(\beta-\alpha)/5}J(-q^{\alpha})K(q^{\beta})\right\}.
\end{aligned}$$

Then

$$U(4,21) + q^{12}U^*(1,84) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{420}) - \varphi(-q^{20})\varphi(-q^{105})\}, \tag{111}$$

$$U(3,22) + q^{9}V^*(1,66) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{330}) - \varphi(-q^{15})\varphi(-q^{110})\}, \tag{112}$$

$$U(2,23) + q^{5}V^*(1,46) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{230}) - \varphi(-q^{10})\varphi(-q^{115})\}, \tag{113}$$

$$U(7,18) + q^{21}V^*(1,126) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{630}) - \varphi(-q^{35})\varphi(-q^{90})\}, \tag{114}$$

$$U^*(1,14) - V(2,7) = \frac{1}{2q^3}\{\varphi(q^5)\varphi(-q^{70}) - \varphi(-q^{10})\varphi(-q^{35})\}, \tag{115}$$

$$V(2,17) + q^{2}U(1,34) = \frac{1}{2q^5}\{\varphi(-q^{10})\varphi(-q^{85}) - \varphi(-q^{5})\varphi(-q^{170})\}, \tag{116}$$

$$U(8,17) + q^{23}V^*(1,136) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{680}) - \varphi(-q^{40})\varphi(-q^{85})\}, \tag{117}$$

$$U(6,19) + q^{18}U^*(1,114) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{570}) - \varphi(-q^{30})\varphi(-q^{95})\}, \tag{118}$$

$$V(4,9) - q^{3}V(1,36) = \frac{1}{2q^5}\{\varphi(-q^{20})\varphi(-q^{45}) - \varphi(-q^{5})\varphi(-q^{180})\}, \tag{119}$$

$$U(9,16) + q^{24}U^*(1,144) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{720}) - \varphi(-q^{45})\varphi(-q^{80})\}, \tag{120}$$

$$U(11,14) + q^{26}U^*(1,154) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{770}) - \varphi(-q^{55})\varphi(-q^{70})\}, \tag{121}$$

$$U(12,13) + q^{27}V^*(1,156) = \frac{1}{2q^5}\{\varphi(q^5)\varphi(-q^{780}) - \varphi(-q^{60})\varphi(-q^{65})\}, \tag{122}$$

$$V(3,13) + q^{3}U(1,39) = \frac{1}{2q^5}\{\varphi(-q^{15})\varphi(-q^{65}) - \varphi(-q^{5})\varphi(-q^{195})\}. \tag{123}$$

Proof. The proof of the theorem follows form Lemma 2.13 and Lemma 2.12. QED

4 Applications to the theory of partitions

Some of our modular relations yield theorems in the theory of partitions. In this section, we present partition interpretations of Theorem 3.1 and the identities (46) and (67).

Definition 4.1. A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called "colored partitions".

It is easy to see that

$$\frac{1}{(q^u;q^v)_\infty^k}$$

is the generating function for the number of partitions of n where all the parts are congruent to $u \pmod v$ and have k colors. For simplicity, we use the notation

$$(a_1,a_2,\ldots,a_n;q)_\infty := \prod_{i=1}^{n}(a_i;q)_\infty.$$

Also, we define

$$(q^{r\pm};q^s)_\infty := (q^r,q^{s-r};q^s)_\infty,$$

where r and s are positive integers with $r < s$.

Theorem 4.2. Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 4, ± 7, ± 8 (mod 20) with ± 3, ± 7 (mod 20) having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 4, ± 6, ± 8, ± 9 (mod 20) with ± 1, ± 9 (mod 20) having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 2, ± 3, ± 6, ± 7, ± 9, 10 (mod 20) with 10 (mod 20) having two colors. Then, for any positive integer $n \geq 1$,

$$p_1(n) + p_2(n) = 2p_3(n).$$

Proof. Using (15) and (31) in Theorem 3.1(i), we find that

$$f^2(-q,-q^9)f(-q^6,-q^{14})+f^2(-q^3,-q^7)f(-q^2,-q^{18}) = 2\frac{f_4^3 f_{20}}{f_2^4}\frac{f(-q,-q^9)f(-q^3,-q^7)\varphi^2(-q^2)}{\varphi(-q^{10})}. \quad (124)$$

Identity (124) is equivalent to

$$\begin{aligned}&\frac{1}{(q^3,q^7,q^{10};q^{10})_\infty^2(q^2,q^{18},q^{20};q^{20})_\infty}\\&+\frac{1}{(q,q^9,q^{10};q^{10})_\infty^2(q^6,q^{14},q^{20};q^{20})_\infty}\\=&2\frac{(q^4;q^4)_\infty(q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^2(q,q^3,q^7,q^9,q^{10},q^{10};q^{10})_\infty(q^2,q^6,q^{14},q^{18},q^{20},q^{20};q^{20})_\infty}.\end{aligned}\tag{125}$$

Now, rewrite all the products on both sides of (125) subject to the common base q^{20} to obtain

$$\begin{aligned}&\frac{1}{(q^{2\pm},q^{3\pm},q^{3\pm},q^{4\pm},q^{7\pm},q^{7\pm},q^{8\pm};q^{20})_\infty}\\&+\frac{1}{(q^{1\pm},q^{1\pm},q^{4\pm},q^{6\pm},q^{8\pm},q^{9\pm},q^{9\pm};q^{20})_\infty}\\&=\frac{2}{(q^{1\pm},q^{2\pm},q^{3\pm},q^{6\pm},q^{7\pm},q^{9\pm},q^{10},q^{10};q^{20})_\infty}.\end{aligned}\tag{126}$$

The three quotients of (126) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (126) is equivalent to

$$\sum_{n=0}^{\infty}p_1(n)q^n+\sum_{n=0}^{\infty}p_2(n)q^n=2\sum_{n=0}^{\infty}p_3(n)q^n,$$

where we set $p_1(0)=p_2(0)=p_3(0)=1$. Equating coefficients of q^n $(n\geq 1)$ on both sides yields the desired result. QED

Example 4.3. The following table illustrates the case $n=5$ in Theorem 4.2:

$p_1(5)=2$	$p_2(5)=8$	$p_3(5)=5$
3_r+2	$4+1_r$	$3+2$
3_g+2	$4+1_g$	$3+1+1$
	$1_r+1_r+1_r+1_r+1_r$	$2+2+1$
	$1_r+1_r+1_r+1_r+1_g$	$2+1+1+1$
	$1_r+1_r+1_r+1_g+1_g$	$1+1+1+1+1$
	$1_r+1_r+1_g+1_g+1_g$	
	$1_r+1_g+1_g+1_g+1_g$	
	$1_g+1_g+1_g+1_g+1_g$	

Theorem 4.4. Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 2, ± 3, ± 7 $(mod\ 20)$ with ± 3, ± 7 $(mod\ 20)$ having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 6, ± 9 $(mod\ 20)$ with ± 1, ± 9 $(mod\ 20)$ having two colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 3, ± 7, ± 9, 10 $(mod\ 20)$ with 10 $(mod\ 20)$ having two colors. Then, for any positive integer $n \geq 1$,

$$p_1(n) + 2p_3(n-1) = p_2(n).$$

Proof. Using (15) and (31) in Theorem 3.1(ii), we find that

$$\frac{1}{(q^{2\pm}, q^{3\pm}, q^{3\pm}, q^{7\pm}, q^{7\pm}; q^{20})_\infty} - \frac{1}{(q^{1\pm}, q^{1\pm}, q^{6\pm}, q^{9\pm}, q^{9\pm}; q^{20})_\infty}$$
$$= \frac{-2q}{(q^{1\pm}, q^{3\pm}, q^{7\pm}, q^{9\pm}, q^{10}, q^{10}; q^{20})}. \tag{127}$$

The three quotients of (127) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (127) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n)q^n - \sum_{n=0}^{\infty} p_2(n)q^n = -2q \sum_{n=0}^{\infty} p_3(n)q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating coefficients of q^n $(n \geq 1)$ on both sides yields the desired result. QED

Example 4.5. The following table illustrates the case $n = 6$ in Theorem 4.4:

$p_1(6) = 4$	$p_2(6) = 8$	$p_3(5) = 2$
$3_r + 3_r$	6	$3 + 1 + 1$
$3_r + 3_g$	$1_r + 1_r + 1_r + 1_r + 1_r + 1_r$	$1 + 1 + 1 + 1 + 1$
$3_g + 3_g$	$1_r + 1_r + 1_r + 1_r + 1_r + 1_g$	
$2 + 2 + 2$	$1_r + 1_r + 1_r + 1_r + 1_g + 1_g$	
	$1_r + 1_r + 1_r + 1_g + 1_g + 1_g$	
	$1_r + 1_r + 1_g + 1_g + 1_g + 1_g$	
	$1_r + 1_g + 1_g + 1_g + 1_g + 1_g$	
	$1_g + 1_g + 1_g + 1_g + 1_g + 1_g$	

Theorem 4.6. Let $p_1(n)$ denote the number of partitions of n into parts congruent to ± 1, ± 4, ± 8, ± 9, ± 11, ± 12, ± 16, ± 19, ± 20, ± 21, ± 28, ± 29, ± 31, ± 32, ± 36, ± 39 $(mod\ 80)$ with ± 12, ± 28 $(mod\ 80)$ having two colors. Let $p_2(n)$ denote the number of partitions of n into parts congruent to ± 3, ± 4, ± 7, ± 12, ± 13, ± 16, ± 17, ± 20, ± 23 ± 24, ± 27, ± 28, ± 32, ± 33, ± 36, ± 37 $(mod\ 80)$ with

± 4, ± 36 $(mod\ 80)$ having two colors. Let $p_3(n)$ denote the number of partitions of n into odd parts not congruent to ± 5, ± 15, ± 25, ± 35 $(mod\ 80)$ and parts congruent to ± 10, ± 30 $(mod\ 80)$. Then, for any positive integer $n \geq 3$, we have

$$p_1(n) + p_2(n-3) = p_3(n).$$

Proof. Using (15), (33) and Lemma 2.1, identity (46) can be written as

$$\begin{aligned} &\frac{1}{f_2 f_4 f_{20}\ \varphi(-q) f(-q,-q^9) f(-q^8,-q^{72}) f(-q^{12},-q^{28})} \\ &+ \frac{q^3}{f_2 f_4 f_{20}\ \varphi(-q) f(-q^3,-q^7) f(-q^{24},-q^{56}) f(-q^4,-q^{36})} \\ =&\frac{1}{f_1^2\ \varphi(-q^{40}) f(-q,-q^9) f(-q^3,-q^7) f(-q^8,-q^{72}) f(-q^{24},-q^{56})}. \end{aligned} \tag{128}$$

Using (6), (11) and (13) in (128) and rewriting all the products subject to the common base q^{80}, we deduce

$$\begin{aligned} &\frac{1}{(q^{1\pm}, q^{4\pm}, q^{8\pm}, q^{9\pm}, q^{11\pm}, q^{12\pm}, q^{12\pm}, q^{16\pm}, q^{19\pm}, q^{20\pm}, q^{21\pm}; q^{80})_\infty} \\ &\times \frac{1}{(q^{28\pm}, q^{28\pm}, q^{29\pm}, q^{31\pm}, q^{32\pm}, q^{36\pm}, q^{39\pm}; q^{80})_\infty} \\ &+ \frac{q^3}{(q^{3\pm}, q^{4\pm}, q^{4\pm}, q^{7\pm}, q^{12\pm}, q^{13\pm}, q^{16\pm}, q^{17\pm}, q^{20\pm}, q^{23\pm}, q^{24\pm}; q^{80})_\infty} \\ &\times \frac{1}{(q^{27\pm}, q^{28\pm}, q^{32\pm}, q^{33\pm}, q^{36\pm}, q^{36\pm}, q^{37\pm}; q^{80})_\infty} \\ =&\frac{1}{(q^{1\pm}, q^{3\pm}, q^{7\pm}, q^{9\pm}, q^{10\pm}, q^{11\pm}, q^{13\pm}, q^{17\pm}, q^{19\pm}, q^{21\pm}, q^{23\pm}; q^{80})_\infty} \\ &\times \frac{1}{(q^{27\pm}, q^{29\pm}, q^{30\pm}, q^{31\pm}, q^{33\pm}, q^{37\pm}, q^{39\pm}; q^{80})_\infty}. \end{aligned} \tag{129}$$

The three quotients of (129) represent the generating functions for $p_1(n)$, $p_2(n)$, and $p_3(n)$, respectively. Hence, (129) is equivalent to

$$\sum_{n=0}^{\infty} p_1(n) q^n + q^3 \sum_{n=0}^{\infty} p_2(n) q^n = \sum_{n=0}^{\infty} p_3(n) q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = 1$. Equating coefficients of q^n $(n \geq 3)$ on both sides yields the desired result. QED

Example 4.7. The following table illustrates the case $n = 9$ in Theorem 4.6:

$p_1(9)=5$	$p_2(6)=1$	$p_3(9)=6$
9	$3+3$	9
$8+1$		$7+1+1$
$4+4+1$		$3+3+3$
$4+1+1+1+1+1$		$3+3+1+1+1$
$1+1+1+1+1+1+1+1+1$		$3+1+1+1+1+1+1$
		$1+1+1+1+1+1+1+1$ $+1$

Theorem 4.8. Let $p_1(n)$ denote the number of partitions of n into parts not congruent to ±3, ±7, ±13, ±17, 40 (*mod* 40), parts congruent to ±5, ±8, ±10, ±12, ±15, ±16, 20 (*mod* 40) with two colors, and parts congruent to ±2, ±4, ±6, ±14, ±18 (*mod* 40) with three colors. Let $p_2(n)$ denote the number of partitions of n into parts not congruent to ±1, ±9, ±11, ±19, 40 (*mod* 40), parts congruent to ±4, ±5, ±8, ±10, ±15, ±16, 20 (*mod* 40) with two colors, and parts congruent to ±2, ±6, ±12, ±14, ±18 (*mod* 40) with three colors. Let $p_3(n)$ denote the number of partitions of n into parts congruent to ±1, ±3, ±7, ±9, ±11, ±13, ±17, ±19 (*mod* 40) having three colors and parts congruent to ±5, ±15 (*mod* 40) having four colors. Let $p_4(n)$ denote the number of partitions of n into parts not congruent to ±5, ±15, 20, 40 (*mod* 40), parts congruent to ±8, ±10, ±16 (*mod* 40) having two colors and parts congruent to ±2, ±4, ±6, ±12, ±14, ±18 (*mod* 40) having three colors. Then, for any positive integer $n \geq 4$, we have

$$2p_1(n-1)+2p_2(n-4)=p_3(n)-p_4(n).$$

Proof. Using (15) and Lemma 2.1, identity (67) can be written as

$$\begin{aligned}&\frac{2q}{f_2^5\ \varphi(-q^4)\varphi(-q^5)\varphi(-q^{20})f(-q,-q^9)f(-q^4,-q^{36})}\\&\quad+\frac{2q^4}{f_2^5\ \varphi(-q^4)\varphi(-q^5)\varphi(-q^{20})f(-q^3,-q^7)f(-q^{12},-q^{28})}\\&=\frac{1}{f_1^2f_4^2\ \varphi(-q^5)\varphi(-q^{20})f(-q,-q^9)f(-q^3,-q^7)f(-q^4,-q^{36})f(-q^{12},-q^{28})}\\&\quad-\frac{1}{f_2^5\ \varphi(-q^4)f(-q,-q^9)f(-q^3,-q^7)f(-q^4,-q^{36})f(-q^{12},-q^{28})}.\end{aligned}\tag{130}$$

Using (6), (11) and (13) in (130) and rewriting all the products subject to the common base q^{40}, we deduce

$$\frac{2q}{(q^{1\pm},q^{9\pm},q^{11\pm},q^{19\pm};q^{40})_\infty(q^{5\pm},q^{8\pm},q^{10\pm},q^{12\pm},q^{15\pm},q^{16\pm},q^{20};q^{40})_\infty^2}$$

$$\times \frac{1}{(q^{2\pm},q^{4\pm},q^{6\pm},q^{14\pm},q^{18\pm};q^{40})_\infty^3}$$
$$+ \frac{2q^4}{(q^{3\pm},q^{7\pm},q^{13\pm},q^{17\pm};q^{40})_\infty(q^{4\pm},q^{5\pm},q^{8\pm},q^{10\pm},q^{15\pm},q^{16\pm},q^{20};q^{40})_\infty^2}$$
$$\times \frac{1}{(q^{2\pm},q^{6\pm},q^{12\pm},q^{14\pm},q^{18\pm};q^{40})_\infty^3}$$
$$= \frac{1}{(q^{1\pm},q^{3\pm},q^{7\pm},q^{9\pm},q^{11\pm},q^{13\pm},q^{17\pm},q^{19\pm};q^{40})_\infty^3(q^{5\pm},q^{15\pm};q^{40})_\infty^4}$$
$$- \frac{1}{(q^{1\pm},q^{3\pm},q^{7\pm},q^{9\pm},q^{11\pm},q^{13\pm},q^{17\pm},q^{19\pm};q^{40})_\infty(q^{8\pm},q^{10\pm},q^{16\pm};q^{40})_\infty^2}$$
$$\times \frac{1}{(q^{2\pm},q^{4\pm},q^{6\pm},q^{12\pm},q^{14\pm},q^{18\pm};q^{40})_\infty^3}. \tag{131}$$

The four quotients of (131) represent the generating functions for $p_1(n)$, $p_2(n)$, $p_3(n)$, and $p_4(n)$, respectively. Hence, (131) is equivalent to

$$2q\sum_{n=0}^{\infty} p_1(n)q^n + 2q^4\sum_{n=0}^{\infty} p_2(n)q^n = \sum_{n=0}^{\infty} p_3(n)q^n - \sum_{n=0}^{\infty} p_4(n)q^n,$$

where we set $p_1(0) = p_2(0) = p_3(0) = p_4(0) = 1$. Equating coefficients of q^n $(n \geq 4)$ on both sides yields the desired result. QED

Example 4.9. Figure 1 illustrates the case $n = 4$ in Theorem 4.8.

Similarly, one can also establish partition theoretic interpretation of some of others modular relations proved in Section 3.

Acknowledgements. The authors thank the referee for several helpful comments and suggestions.

$p_1(3)=4$	$p_2(0)=1$	$p_3(4)=24$	$p_4(4)=14$
2_r+1		$1_r+1_r+1_r+1_r$	4_r
2_g+1		$1_r+1_r+1_r+1_g$	4_g
2_w+1		$1_r+1_r+1_g+1_g$	4_w
$1+1+1$		$1_r+1_g+1_g+1_g$	$3+1$
		$1_g+1_g+1_g+1_g$	2_r+2_r
		$1_g+1_w+1_w+1_w$	2_r+2_g
		$1_g+1_g+1_w+1_w$	2_g+2_g
		$1_g+1_g+1_g+1_w$	2_g+2_w
		$1_w+1_w+1_w+1_w$	2_r+2_w
		$1_w+1_w+1_w+1_r$	2_w+2_w
		$1_w+1_w+1_r+1_r$	2_r+1+1
		$1_w+1_r+1_r+1_r$	2_g+1+1
		$1_r+1_r+1_g+1_w$	2_w+1+1
		$1_g+1_g+1_r+1_w$	$1+1+1+1$
		$1_w+1_w+1_r+1_g$	
		$3_r+1_r, 3_r+1_g, 3_g+1_r$	
		$3_g+1_g, 3_g+1_w, 3_w+1_g$	
		$3_w+1_w, 3_r+1_w, 3_w+1_r$	

Figure 1.

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