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ON THE C^*-COMPARISON ALGEBRA OF A CLASS OF SINGULAR STURM-LIOUVILLE EXPRESSIONS ON THE REAL LINE

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

In this article we study a C^*-comparison algebra in the sense of [C2] with generators related to the ordinary differential expression H on the full real line $\mathbb{R}$ where, with constants $\alpha \geq 0$, $\beta \in \mathbb{R}$,

$$(1.1) \qquad H = -\partial_x \left(1 + x^2\right)^\beta \partial_x + \left(1 + x^2\right)^\alpha, \qquad x \in \mathbb{R}.$$

More precisely, the algebra, called $\mathbf{A}$, is generated by the multiplications $a(M) : u(x) \to \to a(x)u(x)$, by functions $a(x) \in C([-\infty, +\infty])$ and the (singular integral) operators $S_0 = (1 + x^2)^{\alpha/2}\Lambda$, $iS_1 = (1 + x^2)^{\beta/2}\partial_x\Lambda$, and their adjoints. Here $\Lambda = H^{-1/2}$, the inverse positive square root of the unique self-adjoint realization H of the expression (1.1), in the Hilbert space $\mathbf{H} = L^2(\mathbb{R})$. (We use the same notation for both, (1.1) and its realization.)

The case of $\beta < \alpha + 1$ was discussed earlier in [Tg1], even for all n-dimensional problem. The commutators are compact and the Fredholm properties of operators in $\mathbf{A}$ are determined by a complex-valued symbol on a symbol space homeomorphic to that of the usual Laplace comparison algebra on $\mathbb{R}^n$, although the symbol itself is calculated by different formulas.

The algebra, perhaps, is of interest because the singular Sturm-Liouville expression H of (1.1) suggests the existence of a «boundary» at $\pm\infty$, insofar as only finitely many powers H^m are in the limit point case of Weyl - i.e., have a unique self-adjoint realization. Actually, it was shown in [C2], V, Theorem 4.4 that for $\beta \leq 1$ all powers of the minimal operator are essentially self-adjoint, while for large β only H itself has a unique self-adjoint realization (cf. also [CA], Theorem 1.6).

Here we focus on the case $\beta \geq \alpha + 1$. In the special case $\beta = 1$, $\alpha = 0$ the algebra proves to be identical with a well known algebra of [CH] (Theorem 2.1). For all other $\beta \geq \alpha + 1$ we prove the following result:

Theorem 1.1. *The algebra* $\mathbf{A}$ *contains the ideal* $\mathbf{K}(\mathbf{H})$ *of compact operators. Commutators in* $\mathbf{A}$ *are compact; we have* $\mathbf{A}^\vee = \mathbf{A}/\mathbf{K}$ *commutative.*

Moreover, $\mathbf{A}^\vee = C(\mathbb{M})$, where the space $\mathbb{M}$ is a rectangle with sides $I_{-\infty} = \{-\infty \leq \leq t \leq \infty\}$, $I_\infty = \{-\infty \leq t \leq \infty\}$, $I_+ = \{-\infty \leq x \leq \infty\}$, $I_- = \{-\infty \leq x \leq \infty\}$, with endpoints identified as in Fig. 1.1.

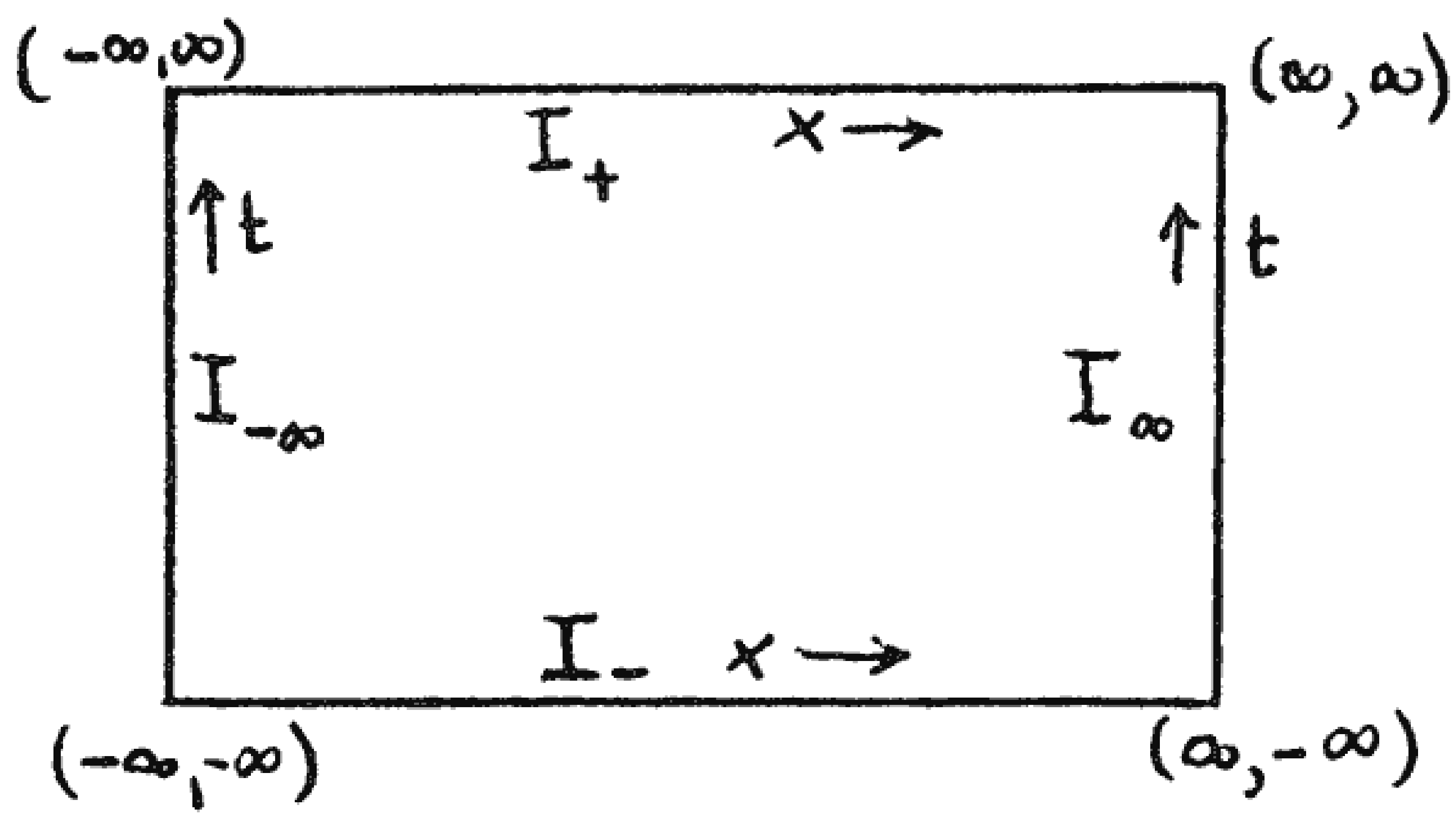

Fig. 1.1

The symbol (i.e., the map $\sigma : A \to A^\vee \to C(M)$) of the generators is given as follows. For $a \in C([-\infty,\infty])$,

$$\sigma_{a(M)} = a(x) \text{ on } I_+ \cup I_-, \tag{1.2}$$

continuous constant $= a(\pm\infty)$ on $I_{\pm\infty}$.

In case of $\alpha = \beta - 1 > 0$ we get

$$\sigma_{S_0} = \frac{\gamma(t)}{\beta - 1},\ \sigma_{S_1} = \frac{i + i\beta - t}{2}\gamma(t) \text{ on } I_{\pm\infty}, \tag{1.3}$$

continuous constant on $I_\pm$, where the function $\gamma(t) \in C^\infty(\mathbb{R}) \cap CO(\mathbb{R})$ is explicitly given by

$$\begin{aligned} \gamma(t) &= \frac{2}{\sqrt{\pi^3}} \int_0^\infty \mathrm{d}\sigma \cosh\left((1+it)\frac{\sigma}{2}\right) \int_0^\infty \mathrm{d}\tau \frac{\sinh^q \tau}{\sinh^{q+1}(\sigma+\tau)}, \\ &\text{with } q = \frac{1}{2}\left\{\frac{1}{\beta-1}\sqrt{(2\beta-1)^2+4} - 1\right\}. \end{aligned} \tag{1.4}$$

In case of $0 \le \alpha < \beta - 1$ we get

$$\sigma_{S_0} = 0 \text{ on } \mathbb{M},\quad \sigma_{S_1} = \frac{t - i}{2}\gamma(t) \text{ on } I_{\pm\infty}, \tag{1.5}$$

continuous constant on $I_\pm$, with the function $\gamma(t)$ of (1.4), where now q of the second line of (1.4) must be substituted by

$$q = \frac{1}{2}\frac{\beta}{\beta - 1} = \frac{1}{2}(1 + \eta). \tag{1.6}$$

In each case the symbols $\sigma_{S_j^*}$ are the complex conjugates of the symbols σ_{S_j}.

We shall discuss the proof of Theorem 1.1 in section 4, after extensive preparations. The function $\gamma(t)$ of (1.4) really is the Mellin transform of a certain limit of the integral kernel of the Sturm-Liouville transformed operator S_1. The appearance of the Mellin transform instead of the Fourier transform indeed seems to parallel the fact that boundary conditions are needed at $\pm\infty$.

We note the series of results of Sohrab [Sr] concerned with similar C^*-algebras of singular Sturm-Liouville or Schroedinger operators, where normally the condition $\nabla q = o(q)$ is imposed on the potential q. In case of $\beta < \alpha + 1$, discussed in [Tg1], our algebra transforms into an algebra of this type, even for higher dimensions. Our present case seems to be different. Especially, we expect noncompact commutators and an operator-valued symbol as in the algebras discussed by Arsenovic [Ar], Melo [Ml], Plamenewski [Plj], Rabinovic [Rb]. For discussion of other C^*-algebras and more general Banach algebras of similar nature see also [Go], [GKj], [Du], [Pw].

Our proof will be accomplished by relating the algebra $\mathbf{A}$ to the algebras of [Tg2] and [CA] on L^2 of the half-line $\mathbb{R}_+$ by a Sturm-Liouville transform, a perturbation of the comparison expression, and a technique of [C2], VIII, called algebra surgery.

Surgery is used to «take apart» a comparison algebra, and examine the closed ideals corresponding to different endpoints of the interval separately; also relate such ideals between algebras living on different manifolds. This was used extensively in [C2], and also in [CDg] and [Tg1]. In all cases it amounts to a standard procedure, repeating the same conclusions.

The perturbation technique also was introduced in [C2] and used in [CDg]. However, in the present case this technique seems possible only by a rather delicate argument, discussed in section 3.

In section 2 we perform a Sturm-Liouville transform, for a modified algebra $\mathbf{S}$, relating this algebra to an algebra on the half-line. In section 4 the half-line algebra is related to the algebras of [Tg1] and [CA], using the perturbation of section 3. Finally, a surgery argument leads to the full description of the structure of $\mathbf{A}$. Also, the case $\beta > \alpha + 1$ is reduced to $\beta = \alpha + 1$.

2. STURM-LIOUVILLE TRANSFORM

We first assume $\alpha = \beta + 1$. Instead of focusing on the expression H of (1.1) we first will deal with a modified expression M. Both, H and M are defined on $\mathbb{R}$. For $\alpha = \beta - 1 = 0$ we set $M = H$. For $\alpha > 0$ we let $M = H$ only for $x \geq 0$. For $x \leq -1$ we set $M = \frac{4}{5}H = \frac{4}{5}(1 - \partial_x \langle x\rangle^2 \partial_x)$, with the standard abbreviation $\langle x\rangle = \sqrt{1+x^2}$. In the interval $(-1, 0)$ we let M interpolate between both expressions. For example, we set

$$(2.1) \quad M = -\partial_x p(x)\partial_x + q(x), \quad p = \frac{4}{5}\omega_-\langle x\rangle^2 + \omega_+\langle x\rangle^{2\beta}, \quad q = \frac{4}{5}\omega_- + \omega_+\langle x\rangle^{2\beta-2},$$

with $\omega_\pm \in C^\infty(\mathbb{R})$, $0 \le \omega_\pm \le 1$, $\omega_+ + \omega_- = 1$, $\omega_\pm = 1$ near $(0,\infty)$ and $(-\infty,-1)$, resp.

In $H = L^2(\mathbb{R})$ again we generate a C^*-algebra $\mathbf{S}$, using the same multiplication as in section 1, but replacing S_j, respectively, by

(2.2)

$$U_0 = \left(\tau\omega_- + \omega_+\langle x\rangle^\alpha\right) M^{-1/2}, \quad U_1 = -i\left(\tau\omega_-\langle x\rangle + \omega_+\langle x\rangle^\beta\right)\partial_x M^{-1/2}, \quad \tau = \frac{2}{\sqrt{5}}.$$

Our point is that the algebras $\mathbf{A}$ and $\mathbf{S}$ «coincide» over the subinterval $(0,\infty)$ of $\mathbb{R}$, by algebra surgery of [C2], VIII. Similarly, for reason of symmetry, $\mathbf{A}$ «coincides» with $S_-\mathbf{S}S_-$ over $(-\infty,0)$, where $S_-u(x) = u(x)$. Thus our task is reduced to the discussion of the structure of the algebra $\mathbf{S}$, only «over the interval $(0,\infty)$ ». We will work out details in section 4.

Generally, for a Sturm-Liouville expression of the form (2.1) one may change both dependent and independent variable by setting

$$u = \gamma v, \quad \gamma = p^{-1/4}, \quad s(x) = \int_x^\infty \frac{dt}{\sqrt{p(t)}}. \tag{2.3}$$

This change (from u and x to v and s, resp.) is meaningful only if the integral exists. For our expression we have $p = \langle x\rangle^{2\beta}$, $\frac{1}{\sqrt{p}} = \langle x\rangle^{-\beta}$, as $x \ge 0$. The latter is integrable at ∞ if $\beta = \alpha - 1 > 0$. For the left- over case $\beta = 1$, $\alpha = 0$ we replace the integral in (2.3) by $\int_0^x$.

For $\beta > 1$ the function $s(x)$ is decreasing from $+\infty$ to 0, as $-\infty < x < \infty$. Its inverse function decreases from ∞ to $-\infty$, providing a diffeomorphism $(-\infty,\infty) \leftrightarrow (0,\infty)$. Then (2.3) provides a unitary map $Uu = v$, where $U : \mathbf{H} \to \mathbf{H}_+ = L^2(\mathbb{R}_+)$, by $v(s) = u(x(s))/\gamma(x(s))$. Indeed, we have $\int_0^\infty |v(s)|^2 ds = \int_{-\infty}^\infty |u(x)|^2 dx$, by an integral substitution.

For $\beta = 1$ we explicitly get $s(x) = \log(x + \langle x\rangle)$, a diffeomorphism $\mathbb{R} \leftrightarrow \mathbb{R}$, and U, defined as above, is a unitary map $\mathbf{H} \to \mathbf{H}$. In either case executing the Sturm-Liouville transform amounts to conjugating all bounded or unbounded operators involved by U^*.

The principal feature of (2.3), called Sturm-Liouville transform, is that the differential expression M above goes into

$$M^\Delta = -\partial_s^2 + q^\Delta(s), \quad \text{where} \quad q^\Delta(s) = (H\gamma/\gamma)|_{x=x(s)}, \tag{2.4}$$

as confirmed by a calculation ([C2], V, (5.2)). In particular, speaking in terms of self-adjoint operators, $M^\Delta = UMU^*$ is the closure and the Friedrichs extension of the minimal operator of M^{Δ} of (2.4), since the minimal domains, self-adjointness and positivity all correspond to each other, under conjugation by U^*.

For $1 \leq \beta < \infty$ we have $p = \langle x \rangle^{2\beta}$, $q = \langle x \rangle^{2\beta-2}$, in $x \geq 0$, hence

$$q^{\Delta}(s(x)) = \left\{ \langle x \rangle^{2\beta-2} - \langle x \rangle^{\beta/2} \partial_x \left(\langle x \rangle^{2\beta} \partial_x \langle x \rangle^{-\beta/2} \right) \right\}, \quad x \geq 0. \tag{2.5}$$

Formulas (2.5) holds for $\beta = 1$ and $\beta > 1$, but $s(x)$ has different features in either case. Important for us will be the behaviour of q^{Δ} near the endpoint 0 (for $\beta > 0$) or ∞ (for $\beta = 0$).

In the following let $P(t) = 1 + a_1 t + \ldots$ denote any power series in t convergent near $t = 0$, and with constant term 1, where we will not distinguish between two such series. For example,

$$\langle t \rangle^{-\beta} = \left(1 + t^2\right)^{-\beta/2} = t^{-\beta} \left(1 + \frac{1}{t^2}\right)^{-\beta/2} = t^{-\beta} p \left(\frac{1}{t^2}\right) \quad \text{near} \quad t = \infty.$$

Integrating this between x and ∞ we get (in case of $\beta > 1$)

$$s(x) = \frac{1}{\beta - 1} x^{1-\beta} P\left(\frac{1}{x^2}\right), \tag{2.6}$$

for large x. To invert this relation let $z = 1/x$, $\eta = \frac{1}{\beta-1}$, so that (2.6) yields $s = \eta z^{1/\eta} P(z^2)$. Or, $\eta^{-\eta} s^{\eta} = z P(z^2)$, $z = \eta^{-\eta} s^{\eta} P(z^2)$. Or,

$$x(s) = \frac{1}{\varepsilon} s^{-\eta} P\left(s^{2\eta}\right), \quad \varepsilon = (\beta - 1)^{1/(\beta-1)}, \quad 0 < s < s_0, \tag{2.7}$$

for small s_0.

For the function q^{Δ} of (2.5) we get

$$\begin{aligned} q^{\Delta}(x(s)) &= x^{2\beta-2} P\left(z^2\right) - x^{\beta/2} P\left(z^2\right) \partial_x \left(x^{2\beta} P\left(z^2\right) \partial_x \left(x^{-\beta/2} P\left(z^2\right) \right) \right) = \\ &= \frac{1}{4}\left(3\beta^2 - 2\beta + 4\right) x^{2\beta-2} P\left(z^2\right), \end{aligned} \tag{2.8}$$

where we used the rule of differentiation $\partial_x(x^{\theta} P(z^2)) = \theta x^{\theta-1} P(z^2)$, as $\theta \neq 0$, and that $P(z^2) P(z^2) = P(z^2)$. Thus (2.7) and (2.8) imply

$$q^{\Delta}(s) = \frac{3\beta^2 - 2\beta + 4}{4(\beta - 1)^2} \frac{1}{s^2} P\left(s^{2\eta}\right), \quad \eta = \frac{1}{\beta - 1}, \quad 0 < s \leq s_0, \quad \text{as} \quad \beta > 1. \tag{2.9}$$

In case of $\beta = \alpha + 1 = 1$ we get

$$M^{\Delta} = H^{\Delta} = -\partial_s^2 + q^{\Delta}(s), \quad q^{\Delta}(s) = \frac{5}{4} P\left(\frac{1}{x(s)^2}\right), \quad |s| > s_0. \tag{2.10}$$

Here $e^{|s|} = |x|(1+\sqrt{1+z^2}) = 2|x|P(z^2)$. Or, $|z|P(z^2) = 2e^{-|s|}$, i.e., $z^2 = = 4e^{-2s}P(e^{-2s})$. In other words, with a constant b, we get

$$M^{\Delta} = H^{\Delta} = -\partial_s^2 + q^{\Delta}(s), \quad q^{\Delta}(s) = \frac{5}{4} + be^{-2s}P\left(e^{-2s}\right), \quad s \geq s_0. \tag{2.11}$$

This result at once gives a complete answer, regarding the structure of $\mathbf{A}$, in case of $\alpha = 0$, $\beta = 1$. A calculation shows that

$$US_0U^* = \Lambda^{\Delta} = H^{\Delta - 1/2}, \quad US_1U^* = -i\left(\partial_s + e(s)\right)\Lambda^{\Delta}, \tag{2.12}$$

where $e(s) = -\frac{1}{2}\frac{x}{\langle x\rangle}|_{x=x(s)} \in C([-\infty,\infty])$. Thus $\mathbf{A}^{\Delta} = U\mathbf{A}U^*$ is generated by Λ^{Δ}, $-i\partial_s\Lambda^{\Delta}$ (and the multipliers in $C[-\infty,\infty]$) as well.

Theorem 2.1. *In the case $\alpha = 0$, $\beta = 1$ the algebra UAU^* is identical with the algebra $\mathfrak{S}$ of [CH], theorem 36, in the special cse $n = 1$ (cf. also the problems of IV, section 1, where the algebra is called $\mathfrak{B}$).*

Proof. Cf. [CDg], Theorem 1.1, dealing again with the algebra $\mathfrak{S} = \mathfrak{B}$, where it is shown that, originally generated by $a(x) \in C([-\infty,\infty])$, $(1-\partial_x^2)^{-1/2}$, $\partial_x(1-\partial_x^2)^{-1/2}$, may also be generated replacing $1-\partial_x^2$ by a perturbed expression $1-\partial_x^2 + r(x)$. Perturbations allowed there include the above as a special case. (We will discuss a similar more sophisticated result in section 3, below.) Before applying Theorem 1.1, we must conjugate with a suitable dilation $u(x) \to u(\delta x)$, to make the constant term of the expansion of $q^{\Delta}(s)$ at $\pm\infty$ equal to 1. Such a dilation leaves the algebra $\mathfrak{S} = \mathfrak{B}$ invariant.

From now on we consider only the case $\beta > 1$ (right now, $\alpha = \beta - 1$). We then have (2.9) near $s = 0$. For large s we essentially will get the expression H^{Δ}, with $\alpha = 0$, by our construction of M. In detail, for $x < -1$ we have $s(x) = \frac{1}{\tau}\int_x^0 \mathrm{d}x/\langle x\rangle + c_0$, $c_0 = s(-1) - \frac{1}{\tau}\int_{-1}^0 \mathrm{d}x/\langle x\rangle$, $\tau = \frac{2}{\sqrt{5}}$. Or, $s(x) = c_0 + \frac{1}{\tau}\log(|x| + \langle x\rangle)$, $s < -1$. This function may be inverted as above. A calculation yields (with a constant b_0)

$$q^{\Delta}(s) = 1 + b_0 e^{-2\tau(s-c_0)} P\left(e^{-2\tau(s-c_0)}\right), \quad \text{as } s \geq s_0. \tag{2.13}$$

We summarize:

Proposition 2.2. *Let $0 < \alpha = \beta - 1$. Then the algebra $\mathbf{S}^\Delta = USU^* \subset \mathbf{L}(\mathbf{H}_+)$ is generated by all multiplications with functions $b(x) \in C([0,\infty])$, and the operators*

$$V_0 = \left(1 + \frac{1}{x}\right) M^{\Delta - 1/2}, \quad V_1 = -i\partial_x M^{\Delta-1/2}, \tag{2.14}$$

(and their adjoints), where we call the variable x again and where

$$M^\Delta = -\partial_x^2 + q^\Delta(x), \quad q^\Delta(x) \in C^\infty(\mathbb{R}_+), \tag{2.15}$$

with the asymptotic behaviour of q^Δ near 0 and ∞ determined by (2.9) and (2.13), respectively.

Indeed, we get

$$\begin{gathered} U_0^\Delta = \lambda(s) M^{\Delta-1/2}, \quad U_1^\Delta = -i\left(\mu(s)\partial_s + \nu(s)\right) M^{\Delta-1/2}, \quad \text{with} \\ \lambda(s) = \left(\tau\omega_- + \omega_+\langle x\rangle^{\beta-1}\right)|_{x=x(s)}, \quad \mu(s) = -\frac{\tau\omega_-\langle x\rangle + \omega_+\langle x\rangle^\beta}{\sqrt{\tau^2\omega_-\langle x\rangle^2 + \omega_+\langle x\rangle^{2\beta}}}|_{x=x(s)}. \end{gathered} \tag{2.16}$$

$$\nu(s) = -\frac{1}{4}\left(\tau\omega_-\langle x\rangle + \omega_+\langle x\rangle^\beta\right)\left(\frac{p'}{p}\right)|_{x=x(s)}, \quad p(x) \text{ as in } (2.1).$$

Clearly $\lambda(s)$, $-\mu(s)$ are positive and $C^\infty(\mathbb{R}_+)$. We get $\mu(s) = -1$ near 0 and $\mu = -1$ near ∞, and $\lambda = \tau$ near ∞. On the other hand,

$$U\langle x\rangle^\theta U^* = \langle x\rangle^\theta|_{x=x(s)} = \varepsilon^{-\theta}s^{-\theta\eta}P\left(s^{2\eta}\right), \quad \varepsilon,\ \eta \text{ of } (2.7), \tag{2.17}$$

applied for $\theta = \beta - 1$, shows that we have $\lambda(s) = \frac{1}{\beta-1}\frac{1}{s}P(s^{2\eta})$ for small s. We have $\nu(s) = -\frac{\tau}{4}\langle x\rangle\frac{p'}{p} = -\frac{\tau}{4}\frac{x}{\langle x\rangle}$ near $x = -\infty$, so $\nu(s) \approx \frac{\tau}{2}$ near $s = \infty$. Also, $\nu(s) = -\frac{1}{4}\langle x\rangle^\beta\frac{((1+x^2)^\beta)'}{(1+x^2)^\beta} = -\frac{\beta}{2}x\langle x\rangle^{\beta-2} = -\frac{\beta}{2}x^{\beta-1}P(z^2) = -\frac{\beta}{2}\frac{1}{s}P(\sigma^{2\eta})$, near $s = 0$. In view of the fact that all multiplications by functions in $C([0,\infty])$ are among the generators, it is then evident that we may replace the generators U_j^Δ by the operators (2.15).

We will need later that, near $x = 0$, we have

$$\begin{gathered} U_0^\Delta = \left(\frac{1}{\beta-1} + o(1)\right)\left(\frac{1}{x}M^{\Delta-1/2}\right), \\ U_1^\Delta = (-1 + o(1))\left(\frac{1}{i}\partial_x M^{\Delta-1/2}\right) + \left(i\frac{\beta}{2} + o(1)\right)\left(\frac{1}{x}M^{\Delta-1/2}\right), \end{gathered} \tag{2.18}$$

as was just verified.

We finally indicate the changes to be made in our above discussion, if $0 \leq \alpha < \beta - 1$. Clearly the Sturm-Liouville transform is independent of the choice of α; it only involves β. Hence M^Δ is of the form (2.11) again, on the halfline $\mathbb{R}_+$. However, the potential q^Δ near $s = 0$ now is of the form

(2.19)
$$q^\Delta(s) = \frac{(3\beta-2)\beta}{4(\beta-1)^2}\frac{1}{s^2}P\left(s^{2\eta}\right) + \frac{1}{(\beta-1)^{2-2\iota}}\frac{1}{s^{2-2\iota}}P\left(s^{2\eta}\right), \quad \iota = \frac{\beta-1-\alpha}{\beta-1},$$
$$\frac{(3\beta-2)\beta}{4(\beta-1)^2}\frac{1}{s^2}\left\{1 + \left(bs^{2\eta} + cs^{2\iota}\right)P\left(s^{2\eta}\right)\right\},$$

with constants b, c, while there is no change in q^Δ near ∞.

Also, the generator V_0 of (2.14) now must be replaced by

$$W_0 = \left(1 + x^{\iota-1}\right)M^{\Delta-1/2}, \quad \iota \text{ of } (2.19). \tag{2.20}$$

3. A PERTURBATION OF THE COMPARISON EXPRESSION

Let us again work in $\mathbf{H}_+ = L^2(\mathbb{R}_+)$. Consider the two expressions

$$H = -\partial_x^2 + \frac{\kappa}{x^2} + 1, \quad K = H + p(x), \quad 0 < x < \infty, \tag{3.1}$$

where $p \in C^\infty(\mathbb{R}_+)$ with $p^{(k)}(x) = x^{\varepsilon-2-k}\chi_k(x)$, $\chi_k = 0(1)$, $k = 1, 2, \ldots$. Assume that $\kappa > 0$ is a given constant, and that the «perturbation» $p(x)$ does not destroy the positivity property of H. That is, we assume that still $K \geq c_0 > 0$, or,

$$(u, Ku) \geq c_0(u, u), \quad \text{as} \quad u \in C_0^\infty\left(\mathbb{R}_+\right). \tag{3.2}$$

Denote by $\mathbf{T}_H$ and $\mathbf{T}_K$ the comparison algebras, generated as C^*-subalgebras of $\mathbf{L}(\mathbf{H}_+)$ by the multipliers $a(M)$, $a \in C([0,\infty])$, and the pair of operators $\frac{1}{x}H^{-1/2}$, $-i\partial_x H^{-1/2}$ (for the algebra $\mathbf{T}_H$), and $\frac{1}{x}K^{-1/2}$, $-i\partial_x K^{-1/2}$ (for $\mathbf{T}_K$), respectively.

Theorem 3.1. *Assume that* $\nu^2 = \kappa + \frac{1}{4} > 1$. *Then we have* $\mathbf{T}_H = \mathbf{T}_K$. *Moreover, the operators* $S = K^{1/2}H^{-1/2}$ *and* $S^{-1} = H^{1/2}K^{-1/2}$ *are in this algebra* $\mathbf{T}$, *and are both of the form* $1 + C$, $C \in K(H_+)$.

Proof. Let us first give a survey of this proof. We start with examining the operator $KH^{-1} = 1 + pH^{-1}$, and (i) will show that $pH^{-1} \in \mathbf{K}(\mathbf{H}_+)$. It is trivial that

$$\| K^{1/2}u \|^2 = (u, Ku) \leq c(u, Hu) = c\| H^{1/2}u \|^2, \quad u \in C_0^\infty, \tag{3.3}$$

which implies that $S = K^{1/2}H^{-1/2} \in \mathbf{L}(\mathbf{H}_+)$. In fact, the well known result of E. Heinz and K. Loewner (cf. [C2], 1.5) implies that $K^s H^{-s} \in \mathbf{L}(\mathbf{H}_+)$ for all $0 \leq s \leq \frac{1}{2}$. We then will show that (ii) $K^{1/2}H^{-1/2}$ is hermitian mod $\mathbf{K}(\mathbf{H}_+)$, and (iii) $(K^{1/4}H^{-1/4})^2 = K^{1/2}H^{-1/2}$ (mod $\mathbf{K}$), where $K^{1/4}H^{-1/4}$ also is hermitian (mod $\mathbf{K}$), so that the coset mod K of $K^{1/2}H^{-1/2}$ is positive hermitian. Finally (iv) we show that $S^2 = (K^{1/2} H^{-1/2})^2 = KH^{-1}$ (mod $\mathbf{K}$). All together it then follows that the coset $S^\vee = S + \mathbf{K}$ must be the positive square root of $(KH^{-1})^\vee = I^\vee$. By uniqueness of the positive square root it then follows that $S = 1 + C$, $C \in \mathbf{K}$. Next (v) we conclude that S^{-1} exists. Then, of course, it also must be of the form $1 + C$, $C \in \mathbf{K}$. However, then it follows that the generators of $\mathbf{T}_H$ are contained in $\mathbf{T}_K$ (and viceversa), since every comparison algebra contains $\mathbf{K}(\mathbf{H})$ and since $\frac{1}{x}H^{-1/2} = \frac{1}{x}K^{-1/2}S = (\frac{1}{x}K^{-1/2})(1+C)$, $C \in \mathbf{K}$, etc. Q.E.D.

To complete this program we start with (v): This follows if the converse of (3.3) can be established. However, (3.2) implies $(N+1)(u,Ku) \geq Nc_0(u,u) + \| u' \|^2 + \kappa \| \frac{u}{x} \|^2 - (|p|u,u)$, where we may use the estimate $|p| \leq \frac{\delta}{x^2} + c(\delta)$, valid for all $\delta > 0$ with suitable $c(\delta)$, for $(N+1)(u,Ku) \geq (Nc_0 - c(\kappa + \frac{1}{4})) \| u \|^2 \geq \| u \|^2$ for $N = (1 + c(\kappa + \frac{1}{4}))/c_0$. Thus,

$$(3.4) \qquad (u,Hu) \leq c(u,Ku), \text{ for all } u \in C_0^\infty(\mathbb{R}_+), \quad c = \frac{1}{N+1},$$

and this implies existence of $S^{-1} = H^{1/2}K^{-1/2}$. Next we prove (i).

Proposition 3.2. *The operators $U = \frac{1}{x^2}H^{-1}$ and $V = \frac{1}{x}\partial_x H^{-1}$ are bounded. Moreover, the same is true for $\frac{1}{x^2}R(\lambda)$ and $\frac{1}{x}\partial_x R(\lambda)$ whenever $\lambda > 0$, where we have set $R(\lambda) = (H+\lambda)^{-1}$, and then we have the estimates*

$$(3.5) \qquad \| \frac{1}{x^2}R(\lambda) \| \leq c, \quad \| \frac{1}{x}\partial_x R(\lambda) \| \leq c, \quad 0 \leq \lambda < \infty,$$

with c independent of λ.

Also, the operators $b(M)U$ and $b(M)V$ are compact whenever $b \in C(\mathbb{R}_+)$, $b(0) = b(\infty) = 0$.

Proof. We recall from [CA], (3.7), (3.8) that the resolvent $R(1) = (H+\lambda)^{-1}$ is an integral operator with kernel $G_\lambda(x,y)$ expressible in terms of Bessel functions, i.e.,

$$(3.6) \qquad G_\lambda(x,y) = G_\lambda(y,x) = -\sqrt{xy}K_\nu(x\sqrt{1+\lambda})I_\nu(y\sqrt{1+\lambda}), \text{ as } y < x.$$

First set $\lambda = 0$. It suffices to show boundedness of the integral operator with kernel $\frac{1}{x^2}G_\lambda(x,y)$ in $L^2((0,1))$. Indeed, for any partition $1 = \chi + \omega$, $\chi \in C_0^\infty([0,\infty))$, $\chi = 1$ near 0, write $Z = \frac{1}{x^2}R(\lambda) = \chi Z \chi + \omega Z \omega + \chi \omega^0 Z \omega + \omega Z \omega^0 \chi + C$, $C \in \mathbf{K}(\mathbf{H})$, where $\omega\omega^0 = \omega$, $\omega^0 = 0$ near 0, and where we used Proposition 1.2 to commute (mod $\mathbf{K}(\mathbf{H})$) ω^0 and $R(\lambda)$. Only the first term $\chi Z \chi$ needs consideration, since all others are trivially bounded, the functions $\frac{\omega}{x^2}$ or, ω^0/x^2 being bounded at 0. Since χ has compact support, the operator $\chi Z \chi$ only involves a bounded interval $[0,a]$, where we may assume $a = 1$.

In $0 < x \leq 1$ we may estimate

$$K_\nu(x) = o\left(|x|^{-\nu}\right), I_\nu(x) = o\left(|x|^{\nu}\right), \tag{3.7}$$

hence

$$\frac{1}{x^2}G_0(x,y) = \frac{\sqrt{y}}{\sqrt{x^3}}O\left(\text{Min}\left\{\left|\frac{y}{x}^\nu\right|, \left|\frac{x}{y}^\nu\right|\right\}\right). \tag{3.8}$$

Thus we get $\| \frac{1}{x^2}\int_0^1 \mathrm{d}y G_0(x,y)u(y)\mathrm{d}y \|^2_{L^2((0,1))} \leq c \| T_\psi u \|^2$, with the Mellin convolution

$$T_\psi u(x) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \psi\left(\frac{x}{y}\right)u(y)\frac{\mathrm{d}y}{y}, \quad \psi(t) = t^{-3/2}\text{ Min }\left\{t^\nu, t^{-\nu}\right\}. \tag{3.9}$$

The function ψ is $L^1(\mathbf{R}_+, \frac{\mathrm{d}x}{\sqrt{x}})$, as $\nu > 1$, since $\int_0^\infty \frac{\mathrm{d}t}{\sqrt{t}}\psi(t) = \int_0^1 t^{\nu-2}\mathrm{d}t + \int_1^\infty t^{-2-\nu} <$ $< \infty$. Thus we indeed have $\frac{1}{x^2}H^{-1} = \frac{1}{x^2}R(0) \in \mathbf{L}(\mathbf{H})$, as $\nu > 1$.

Remark. It is interesting to note that this operator is no longer bounded for $\nu \leq 1$.

The estimates (3.7) remain true if differentiated, assuming $\nu > 1$ again. Therefore the same conclusion leads to boundedness of the operator $\frac{1}{x}\partial_x H^{-1}$.

For general λ we conclude a bound independent of λ by a scaling argument: For $\eta > 0$ define the (unitary) dilation operator $J_\eta : H_+ \to H_+$ by setting $J_\eta u(x) = \sqrt{\eta}u(\eta x)$. From (3.1) we conclude that

$$J_\eta^* H J_\eta u = \eta^2(H + \lambda)u, \quad \lambda = \frac{1}{\eta^2} - 1, \quad u \in C_0^\infty([0,\infty)), \tag{3.10}$$

and this relation holds for our realization H as well. For any $\lambda \geq 0$ and $\eta = \frac{1}{\sqrt{\lambda+1}}$ we thus get

$$R(\lambda) = \frac{1}{\lambda+1}J_\eta^* R(0) J_\eta, \qquad \lambda \geq 0. \tag{3.11}$$

On the other hand, we have $J^*_\eta \frac{1}{x^2} J_\eta = \frac{1}{\lambda+1}\frac{1}{x^2}$, $J^*_\eta \frac{1}{x}\partial_x J_\eta = \frac{1}{\lambda+1}\frac{1}{x}\partial_x$, so that

(3.12) $$\frac{1}{x^2}R(\lambda) = J^*_\eta \frac{1}{x^2}R(0)J_\eta, \quad \frac{1}{x}\partial_x R(\lambda) = J^*_\eta \frac{1}{x}\partial_x R(0)J_\eta.$$

Since J_η is unitary and we have proven boundedness of $\frac{1}{x^2}H^{-1}$, and $\frac{1}{x}\partial_x H^{-1}$ the uniform boundedness (3.5) follows.

Finally, regarding compactness of $\frac{b(x)}{x^2}H^{-1} = Y$, write $Y = b(x)(\frac{1}{x^2}H^{-1})$, where the second factor is bounded, as just seen, while $b(x)$ vanishes near 0 and ∞, hence is uniform limit of a sequence $b_j(x) \in C_0^\infty((0,\infty))$. Notice that $b_j(\frac{1}{x^2}H^{-1}) = (b_j/x^2 1)H^{-1}$ is compact, since $b_k/x^2 \in C_0$. Also $\| Y - b_j(\frac{1}{x^2}H^{-1}) \| \to 0$, so that compactness of Y follows. Similarly one proves compactness of $\frac{b(x)}{x}\partial_x H^{-1}$. Q.E.D.

Note that (i) is proven as well.

We are left with proving (ii), (iii) and (iv). These involve the operators $V_s = K^s H^{-s}$ and their cosets $U_s = V_s + \mathbf{K}(\mathbf{H})$, for various s, notably $s = \frac{1}{2}$ and $s = \frac{1}{4}$. We need $U_s^* = U_s$, and the semi-group property $U_s^2 = U_{2s}$, for $s = \frac{1}{4}$ and $s = \frac{1}{2}$. The first amounts to compactness of

(3.13) $$K^s H^{-s} - H^{-s}K^s = K^s\left[H^{-s}, K^{-s}\right]K^s.$$

The second amounts to compactness of

(3.14) $$K^{2s}H^{-2s} - K^s H^{-s} K^s H^{-s} = \left\{K^{2s}\left[H^{-s}, K^{-s}\right]\right\}\left(K^s H^{-s}\right).$$

To control the commutator $[H^{-s}, K^{-s}]$ we involve the resolvent integrals

(3.15)
$$H^{-\sigma} = \frac{\sin\pi\sigma}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^\sigma}R(\lambda), \quad K^{-\sigma} = \frac{\sin\pi\sigma}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^\sigma}S(\lambda), \quad S(\lambda) = (K+\lambda)^{-1}.$$

We get

(3.16) $$\left[H^{-s}, K^{-s}\right] = \frac{\sin^2\pi s}{\pi^2}\int_0^\infty\int_0^\infty \frac{d\lambda d u}{(\lambda\mu)^s}S(\mu)R(\lambda)[H,K]R(\lambda)S(\mu).$$

Actually, we first verify (3.15) for all $u \in \tilde{\mathbf{D}} = H_+ \cap C^\infty(\mathbf{R}_+)$, noting the fact that $\tilde{\mathbf{D}}$ is dense in H_+ while $f = R(\lambda)S(\mu)u \in \tilde{\mathbf{D}}$ has $f \in \text{dom } H = \text{dom } K$, $Hf \in \text{dom } K =$

dom H, so that the expression $[H,K]f$ is well defined as an element in $\widetilde{\mathbf{D}}$ again. The fact that $f = RSu \in C^\infty$ again is a matter of hypo-ellipticity of $(K+\mu)(H+\lambda)$. It of course may be proven directly, using the Greens function (36) of $H+\lambda$ (which turns out to be a parametrix of $K+\lambda$ as well).

Let us look at the commutator

$$(3.17) \qquad [H,K] = [H,(H+p)] = [H,p] = -\left[\partial_x^2, p\right] = -2p'\partial_x - p''.$$

Examining p' and p'', for $p = \chi(x)x^{\varepsilon-2}$, we find that expressions of the form $\frac{\chi(x)}{x^{\varepsilon-4}}$ and $\frac{\chi(x)}{x^{\varepsilon-3}}\partial_x$ occur, where χ has the properties of χ above, but need not be the same function. In view of Proposition 3.2 it is clear that the expression $R(\lambda)[H,K]R(\lambda) = I(\lambda)$ is not only bounded, but even compact, as a map $\mathbf{H}_+ \to \mathbf{H}_+$. For (3.13) and (3.14) we need compactness of $K^sS(\mu)I(\lambda)S(\mu)K^s = X$ and $K^{2s}S(\mu)I(\lambda)S(\mu) = Y$ respectively. Also the functions $X(\lambda,\mu)$, $Y(\lambda,\mu)$ should be seen norm continuous and integrable (after division with $(\lambda\mu)^s$).

Clearly $S(\mu)$ and $K^sS(\mu)$ and $K^{2s}S(\mu)$ are bounded so that X and Y are compact, due to compactness of $I(\lambda)$. Norm continuity of the two functions of λ and μ is no problem: write $X(\lambda,\mu) = (K^{2s}S(\mu))(HR(\lambda))(H^{-1}[H,K]H^{-1})(HR(\lambda))(K^sS(\mu))$ each of these factors is either bounded and constant or norm continuous, in view of the resolvent formula $HR(\lambda) - HR(\lambda') = (\lambda'-\lambda)HR(\lambda)R(\lambda')$. Similarly for $Y(\lambda,\mu)$.

Regarding integrability of $Y(\lambda,\mu)$: we need only $s = \frac{1}{2}$ and $s = \frac{1}{4}$. For $s = \frac{1}{2}$ write $Y = (KS(\mu))(R(\lambda)[H,K]R(\lambda)R(\mu))(1+pR(\mu)^{-1})^{-1} = F_1F_2F_3$. Here $F_1 = O(1)$, and $F_3 = O(1)$ as well. Indeed, we get $\| pR(\lambda)u \| = \| x^{\varepsilon-2}\chi(x)R(\lambda)u \| \le \frac{c}{1+\lambda} \| x^{\varepsilon-2}\chi J_\eta^* H^{-1}J_\eta u \| = \frac{c}{(1+\lambda)^\zeta} \| J_\eta^*(x^\varepsilon\chi(\eta x)\frac{1}{x^2}H^{-1}J_\eta u \| \le \frac{c}{(1+\lambda)^\zeta} \| u \|$, with $\zeta = \varepsilon/2$, hence $\| pR(\lambda) \| \to 0$, as $\lambda \to \infty$. Or,

$$(3.18) \qquad \| pR(\lambda) \| = o\left((1+\lambda)^{-\varepsilon/2}\right), \text{ as } 0 \le \lambda < \infty.$$

In particular we get $\| pR(\lambda) \| < 1$ for large λ so that the inverse $(1+pR(\mu))^{-1}$ exists and is bounded for $\mu \ge \mu_0$, for some μ_0. Thus $F_3 = o(1)$, since we may restrict the argument to large μ. Regarding the factor F_2 with (3.18) we now get

$$(3.19) \qquad F_2 = O\left(\mathrm{Min}\left\{\frac{(1+\lambda)^{-\zeta}}{1+\mu}, \frac{(1+\mu)^{-\zeta}}{1+\lambda}\right\}\right).$$

One checks easily that $\frac{Y}{(\lambda\mu)^{1/2}}$ is integrable on $\mathbb{R}_+ \times \mathbb{R}_+$.

For $s = \frac{1}{4}$ we get $F_1 = O(1+\mu)^{-1/2})$ and the same estimates as before for F_2 and F_3. It follows that

$$\frac{Y}{(\lambda\mu)^{1/4}} = \frac{1}{(\lambda\mu)^{1/4}}\frac{1}{1+\mu}O\left(\operatorname{Min}\left\{\frac{(1+\lambda)^{-\varsigma}}{1+\mu}, \frac{(1+\mu)^{-\varsigma}}{1+\lambda}\right\}\right), \tag{3.20}$$

which is integrable as well.

Now we write

(3.21)

$$X = G_1G_2G_3G_4G_4, \quad \text{with} \quad G_1 = G_5 = H^sR(\mu)^{1/2},$$

$$G_2 = G_4 = R(\mu)^{-1/2}S(\mu)R(\mu)^{-1/2}, \quad G_3 = R(\mu)^{1/2}R(\lambda)[H,K]R(\lambda)R(\mu)^{1/2}.$$

Clearly $G_1 = G_5$ are bounded, and $G_2 = G_4$ is bounded as well: combining (3.3) and (3.4) we get $c(u,Hu) \leq (u,Ku) \leq C(u,Hu)$, where one may assume $c \leq 1$, $C \leq 1$, so that also

$$c(u,(H+\lambda)u) \leq (u,(K+\lambda)u) \leq C(u,(H+\lambda)u), \quad u \in C_0^\infty(\mathbb{R}_+). \tag{3.22}$$

This implies $(H+\lambda)^{1/2}(K+\lambda)^{-1/2} = o(1)$, $(K+\lambda)^{1/2}(H+\lambda)^{-1/2} = o(1)$, so that $G_2 = G_4$ are bounded.

To control G_3 we write $(H+\lambda)(H+\mu) = L^2 + (2+\lambda+\mu)L + (1+\lambda)(1+\mu) \geq$ $\geq (L + \sqrt{(1+\lambda)(1+\mu)})^2$, where $L = H - 1 = -\partial_x^2 + \frac{\kappa}{x^2} \geq 0$. It follows that

$$(R(\lambda)R(\mu))^{1/2} \leq R(v), \quad \text{where} \quad v = \sqrt{(1+\lambda)(1+\mu)} - 1. \tag{3.23}$$

Now we use the same estimates as for F_2:

$$G_3 = O\left(\operatorname{Min}\left\{\frac{(1+\lambda)(1+\mu)^{-\varsigma}}{1+\lambda}, \frac{(1+\lambda)^{-\varsigma}}{1+\mu}\right\}\right). \tag{3.24}$$

Substituting this we again get convergence of the integral for $s = \frac{1}{2}$, and for $s = \frac{1}{4}$ as well if we still observe that then $G_1 = G_2 = O((1+\mu)^{-1/4})$.

4. S^Δ and the half-line algebra; proof of Theorem 1.1

In this section we first will prove

Theorem 4.1. *For every $\beta = \alpha + 1 > 1$ the algebra $\mathbf{S}^{\Delta} = USU^*$ coincides with the half-line's Laplace comparison algebra $\mathbf{P}$ of [C1], V, 4, i.e., with the algebra $\mathbf{C}$ of [Tg2], or with $\mathbf{A}$ of [CA], section 3. Moreover, in the representation of the symbol space $\mathbb{M}(\mathbf{P})$ used in [CA] and described in detail in [C1], V, 7 (fig. 7.5) the symbols of the generators $b(x)$, and the operators (2.14) are given by*

$$\sigma_{b(x)} = b(x) \text{ on } I_3 \cup I_4, \text{ constant } = b(0) \text{ on } I_1, = b(\infty) \text{ on } I_2;$$

$$(4.1)\qquad \sigma_{V_0} = \gamma(t),\ \sigma_{V_1} = \frac{t-i}{2}\gamma(t) \text{ on } I_1, \text{ constant continuous on } I_3 \cup I_4,$$

$$\text{where } \gamma(t) = \frac{2}{\sqrt{\pi^3}} \int_0^\infty d\sigma \cosh\left((1+it)\frac{\sigma}{2}\right) \int_0^\infty d\tau \frac{\sinh^q \tau}{\sinh^{q+1}(\sigma+\tau)}.$$

Here $q = \sqrt{\kappa + 1/4} - \frac{1}{2} = \frac{1}{2}\{\frac{1}{\beta-1}\sqrt{(2\beta-1)^2+4} - 1\}$. The values of σ_{V_j} at I_4 are unimportant in the following (cf. [CA], (5.10)).

The proof of Theorem 4.1 is an immediate consequence of Theorem 3.1, combined with the statements of [CA], (5.10) concerning symbols. Just note that M^{Δ} of section 2 can be written as

$$(4.2)\qquad M^{\Delta} = K = -\partial_x^2 + \frac{\kappa}{x^2} + 1 + p(x), \quad p(x) = q^{\Delta}(x) - 1 - \frac{\kappa}{x^2},$$

where we let $\kappa = \frac{3\beta^2 - 2\beta + 4}{4(\beta-1)^2}$, and where $p(x)$ satisfies the assumptions of (3.1) (we denote s by x again). Indeed, near $x = 0$ we use (2.9) for $p(x) = \frac{\kappa}{x^2}\{(P(x^{2\eta}) - 1) - \frac{1}{\kappa}x^2\} = = bx^{2\eta-2}P(x^{2\eta}) - 1$, implying the estimates of (3.1) with $\varepsilon = \mathrm{Min}\{2\eta, 2\}$. Near $x = \infty$ we use (2.13) for $p(x) = be^{-2\tau(x-c)}P(e^{-2\tau(x-c)}) - \frac{\kappa}{x^2}$. Thus we again get (3.1), for any $\varepsilon > 0$. Thus the assumptions of Theorem 3.1 hold. (Observe that also $\kappa + \frac{1}{4} = \frac{4\beta^2 - 4\beta + 5}{4(\beta-1)^2} > 1$.) As a consequence of Theorem 3.1 and of [CA], Theorem 5.7 (together with the remark at the end of [CA]) it follows that P equals the algebra with generators $\frac{1}{x}M^{\Delta - 1/2}$, $-i\partial_x M^{\Delta-1/2}$ (and the multipliers). It also follows at once that $\frac{1}{x}M^{\Delta-1/2}$ may be replaced by $(1 + \frac{1}{x}M^{\Delta-1/2}$, since the algebra clearly contains $(1 + \frac{1}{x})M^{\Delta-1/2}$, and by inspection of symbols. This proves Theorem 4.1.

The proof of Theorem 1.1 now follows as a straight application of [C2], VIII, Theorem 3.3. Indeed, first of all, if we conjugate $\mathbf{S}^{\Delta} = \mathbf{P}$ with the unitary map U of section 2 we get $\mathbf{S} = U^*\mathbf{P}U$. Both algebras $\mathbf{A}$ and $\mathbf{S}$ live on $\mathbb{R}$, in $\mathbf{H} = L^2(\mathbb{R})$. The generating differential expressions U_0, U_1 of (2.2) coincides with those of $\mathbf{A}$ of section 1 near the subinterval $[0,\infty) \subset \mathbb{R}$. Thus by [C2], VIII, Theorem 3.3 symbol space and symbol of both algebras agree over $[0,\infty)$. Similarly for the other interval $[-\infty,0)$, as indicated in section 2. This established Theorem 1.1, as far as the case $\beta = \alpha + 1 > 1$ is concerned. In the case $\beta > \alpha + 1$ we will follow a similar course. Note first that now the asymptotic behaviour of q^{Δ} near $x = 0$ is slightly different, insofar as the first term at right of (2.5) now reads $\langle x\rangle^{2\alpha} = \langle x\rangle^{2(\alpha+1-\beta)}\langle x\rangle^{2\beta-2}$, where the exponent $\beta - 1 - \alpha > 0$. This term now is of lower order, compared to the other term. This leads to formula (2.19) instead of (2.13), near 0, accounting for the amended $\gamma(x)$ of (1.5), (1.6). secondly, we get $S_0 = \langle x\rangle^{\alpha-1-\beta}(\langle x\rangle^{\beta-1}H^{-1/2})$, so we must deal with a multiplication $\langle x\rangle^{\alpha-1-\beta} \in C([0,\infty))$. As a consequence, the algebra **A** now is a subalgebra of **A** with parameters α_0, $\beta_0 = \alpha_0 + 1$, where β_0 is chosen such that the factors κ in (3.1) coincide. The generators of this subalgebra are $a(M) : a \in C([-\infty,\infty])$, S_1 and $\langle x\rangle^{\alpha-1-\beta}S_0$. However, the subalgebra coincides with the entire algebra. To see this we require an application of the Stone-Weierstrass theorem similar to that at the end of [CA] which we will not discuss in detail.

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