Note di Matematica Vol. XI, 61-91 (1991)

SCALARIZATION OF VECTORIAL RELATIONS APPLIED TO CERTAIN OPTIMIZATION PROBLEMS

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *In this paper we consider certain optimization problems which are described by inequalities in partially ordered vector spaces. Using the scalarization procedure developed in [6, 7] we derive optimality conditions for optimization problems of maximum type and for vector optimization problems. As applications we obtain various optimality conditions including an alternation theorem for the Chebyshev approximation with certain side-conditions and a scalarization for vector optimization problems where efficiency is defined by a cone.*

1. INTRODUCTION

An important tool for the investigation of optimization problems with inequalities as side-conditions is the concept of an *active inequality* provided the set of feasible points of the problem is described by scalar inequalities. One of us developed a procedure for transforming a general vector inequality into an equivalent system of scalar inequalities, i.e. which describe the same feasible set, compare [6, 7]. As an application characterization and stability theorems were derived for optimization problems, for best approximation in normed and in certain metric vector spaces. In its simplest form, which will be used in this paper, the method consists of the following:

Let X be a locally $\mathbb{R}$-vector space partially ordered by a closed convex cone K such that int $(K) \neq \emptyset$. Then the dual cone K^* has a $\sigma(X^*, X)$-compact basis B^* and the set of extreme points of B^*

$$E^* := \text{ ext } (B^*)$$

is non-empty. Then an element $x \in X$ is contained in K if and only if

$$\underset{e^* \in E^*}{\forall} e^*(x) \geq 0 .$$

In this paper, we use this method for the investigation of optimization problems of *maximum type*, i.e. where the maximum of a set of objective functions is minimized, and for the reformulation of vector optimization problems, which are defined by a cone, by objective functions and their scalarization. In more detail we consider the following problems:

(*) Partially supported by Deutscher Akademischer Austauschdienst (F.R. Germany).

(**) Partially supported by CNPq Grant 300344/87.

Let X be a locally convex $\mathbb{R}$-vector space partially ordered by a closed convex cone $K \subset X$ such that $\operatorname{int}(K) \neq \emptyset$. Further let Y be a locally convex $\mathbb{R}$-vector space and let $\sigma := (A, \gamma, b)$, where

(i) $A : Y \to X$ is a continuous linear mapping,

(ii) γ is an element in $K \setminus \{0\}$,

(iii) b is an element in X.

For each parameter σ consider the minimization problem

MP (σ). Minimize the function $p : Y \times \mathbb{R} \to \mathbb{R}$ defined by $p(v, z) := z$ subject to the side-condition

$$A_v - \gamma z - b \in -K.$$

Using the method described in [6, 7] the minimization problem MP(σ) can be replaced by the equivalent minimization problem with scalar side-conditions:

MPS (σ). Minimize the function $p : Y \times \mathbb{R} \to \mathbb{R}$ defined by $p(v, z) := z$ subject to the side-conditions

$$\underset{e^* \in E^*}{\forall} e^*\left(A_v\right) - ze^*(\gamma) \leq e^*(b).$$

These optimization problems are equivalent in the sense that they have the same objective function and the same feasible set. They are called of maximum type since, under certain assumptions, they can be represented as:

Minimize on Y the function

$$\psi(v) := \max\left\{\frac{e^*\left(A_v - b\right)}{e^*(\gamma)} \in \mathbb{R} \,|\, e^* \in E^*\right\}.$$

We introduce some notation: we denote by $\mathfrak{P}$ the set of all parameters σ. For each $\sigma \in \mathfrak{P}$ and $u = (v, z)$ in $A(Y) \times \mathbb{R}$, we define the *set of active inequalities* by

$$\mathfrak{G}_{\sigma,u} := \{e^* \in E^* | - e^*(v) e^*(\gamma) z = e^*(b)\}.$$

Further we define the set

$$\Sigma_{\sigma,u} := \{x^* \in B^* | x^*(v) - x^*(\gamma) z = e^*(b)\},$$

which is equal to

$$\overline{\operatorname{con}}\left(\mathfrak{G}_{\sigma,u}\right).$$

It is easy to see, that $\Sigma_{\sigma,u}$ is an extremal, compact and convex subset of B^*. Thus, we have

$$\mathfrak{G}_{\sigma,u} := \operatorname{ext}\left(\mathfrak{G}_{\sigma,u}\right) = \Sigma_{\sigma,u} \cap E^*.$$

For each $\sigma \in \mathfrak{P}$ we define the set

$$Z_\sigma := \bigcap_{e^* \in E^*} \left\{ (v,z) \in Y \times \mathbb{R} \,|\, e^*\left(A_v\right) - e^*(\gamma) z \leq e^*(b) \right\},$$

which is called *feasible set.* Further we introduce the *minimum value*

$$E_\sigma := \inf \left\{ z \in \mathbb{R} \,|\, (v,z) \in Z_\sigma \right\}.$$

The set of all solutions of $\mathrm{MP}(\sigma)$ in Z_σ is denoted by

$$P_\sigma := \left\{ (v,z) \in Z_\sigma | z = E_\sigma \right\}.$$

Further, we introduce the set

$$\mathfrak{M} := \left\{ \sigma \in \mathfrak{P} \,|Z_\sigma \neq \emptyset \right\}.$$

We say, that the parameter σ satisfies the *Slater condition*, if the set

$$Z_\sigma^< := \bigcap_{e^* \in E^*} \left\{ (v,z) \in Y \times \mathbb{R} \,|\, e^*\left(A_v\right) - z e^*(\gamma) < e^*(b) \right\}$$

is non-empty; we denote by

$$\mathfrak{M}_0 := \left\{ \sigma \in \mathfrak{p} \,|Z_\sigma^< \neq \emptyset \right\}$$

the set of all these parameters.

In § 2 we derive for the minimization problem $\mathrm{MP}(\sigma)$ two characterization theorems for a minimal solution which generalize the Kolmogoroff-Theorem (Theorem 2.2) and the Kuhn-Tucker-Theorem (Theorems 2.4 and 2.5) to the more general situation considered here. These criteria can be applied to Chebyshev-approximation with side-conditions. How this can be done is shown exemplarily in § 4 for the cases of one-sided approximation, restricted range approximation and for the approximation of a function and its derivative. For these particular cases various characterization theorems for a best Chebyshev-approximation are derived including a general alternation theorem, which seems to be not known, compare Theorem 4.6.

Further we characterize a set of minimal points of $\mathrm{MP}(\sigma)$. This result will be used in a forthcoming paper, to obtain conditions for the unique solvability of problem $\mathrm{MP}(\sigma)$.

In § 3 we investigate linear vector optimization problems as follows: As before let X be a locally convex $\mathbb{R}$-vector space partially ordered by a closed convex cone K such that $\mathrm{int}(K) \neq \emptyset$. Further let be given an element $b \in X$, a locally convex $\mathbb{R}$-vector space Y and a continuous linear mapping

$$A : Y \to X.$$

We denote by $\mathfrak{P}_0$ the set of all parameters $\sigma = (A, b)$. For each parameter $\sigma \in \mathfrak{P}_0$ we define the feasible set

$$Z_\sigma := \{v \in Y | A_v - b \in -K\}.$$

Let K_0 be a line-free closed convex cone contained in Y such that $\text{int}(K_0) \neq \emptyset$. In vector optimization one tries to determine efficient or weakly efficient points in the feasible set Z_σ (with respect to the cone K_0). The precise definition is as follows:

Definition 1.1. *(i) A point v in Z_σ is called K_0-efficient if and only if*

$$(v - K_0) \cap Z_\sigma = \{v\}.$$

(ii) A point v in Z_σ is called weakly K_0-efficient if and only if

$$(v - \text{ int } (K_0)) \cap Z_\sigma \neq \emptyset.$$

In § 3 we replace these definitions by scalar conditions. In this way we can reformulate the notions of efficient and weakly efficient points in terms of objective functions which is mostly used in the application, compare Theorems 3.1 and 3.2. Using these formulations we derive necessary and sufficient conditions for a point to be weakly efficient. This generalize results of Brosowski [1], who considered the case of dim $Y < \infty$, a finite number of objective functions and the space $C[T]$ with its natural order. These results are used in § 4 to extend the scalarization of Brosowski, Conci [2, 3] to the more general situation considered here. It should be mentioned that even in the case considered by Brosowski, Conci [3] our scalarization is more general since each objective function can be weighted and the space of the scalarization parameters can be chosen of dimension at most

$$\min (N - 1, \# E^* - 1).$$

This will reduce the computational efforts for the determination of suitable efficient points by interactive methods.

2. OPTIMIZATION PROBLEMS OF MAXIMUM TYPE

Lemma 2.1. *If $u = (v, E_\sigma)$ in Z_σ is a solution of $MP(\sigma)$, then the set of active inequalities $\mathfrak{G}_{\sigma,u}$ is non-empty.*

Proof. If $\mathfrak{G}_{\sigma,u} = \emptyset$, then we have

$$\underset{e^* \in E^*}{\forall} e^*(A_v) - e^*(\gamma) E_\sigma - e^*(b) < 0,$$

which implies

$$\underset{e^* \in B^*}{\forall} x^*\left(A_v\right) - x^*(\gamma) E_\sigma - x^*(b) \le -\delta < 0,$$

with a suitable number $\delta > 0$. Let

$$C := \max\{|x^*(\gamma)| \in \mathbb{R} \,|\, x^* \in B^*\}$$

and choose $0 < \varepsilon < \delta/2C$. Then we have

$$\underset{e^* \in B^*}{\forall} x^*\left(A_v\right) - x^*(\gamma)\left(E_\sigma - \varepsilon\right) - x^*(b) \le -\delta + |x^*(\gamma)|\varepsilon < -\frac{\delta}{2} < 0,$$

i.e. the element $(v, E_\sigma - \varepsilon)$ is also contained in Z_σ, which is a contradiction. ■

Theorem 2.2. *Let σ in $\mathfrak{M}_0$. Then we have: an element $u_0 = (v_0, z_0)$ in Z_σ is a solution of $MP(\sigma)$ if and only if*

$$\underset{v \in Y}{\forall} \min_{e^* \in \mathfrak{G}_{\sigma,u_0}} e^*\left(A_v\right) \le 0.$$

Proof. $(\Leftarrow)$. Assume (v_0, z_0) is not a solution of $\mathrm{MP}(\sigma)$. Then there exists an element (v, z) in Z_σ such that $z < z_0$. Since $\overline{Z_\sigma^<} = Z_\sigma$ (compare [6, Satz 3.10]), we can assume $(v, z) \in Z_\sigma^<$. Thus, we have

$$\underset{e^* \in E^*}{\forall} e^*\left(A_v\right) - e^*(\gamma) z < e^*(b) = e^*\left(A_{v_0}\right) - e^*(\gamma) z_0$$

which implies

$$\underset{e^* \in \mathfrak{G}_{\sigma,u_0}}{\forall} 0 \le e^*(\gamma)\left(z_0 - z\right) < e^*\left(A_{v_0 - v}\right).$$

Since $v_0 - v \in Y$ we have a contradiction

$(\Rightarrow)$. Assume there exists an element $v \in Y$ such that

$$\underset{e^* \in \mathfrak{G}_{\sigma,u_0}}{\forall} e^*\left(A_v\right) > 0.$$

Then we have also

$$\underset{x^* \in \Sigma_{\sigma,u_0}}{\forall} x^*\left(A_v\right) \ge \alpha > 0$$

with a suitable real number $\alpha > 0$. Define the open set

$$W_\alpha := \left\{x^* \in B^* | x^*\left(A_v\right) > \frac{\alpha}{2}\right\}$$

which contains Σ_{σ,u_0} . Thus, there exists a real number $\delta > 0$ such that

$$\underset{x^* \in B^* \setminus W_\alpha}{\forall} x^*\left(A_{v_0}\right) - x^*(\gamma) E_\sigma - x^*(b) \leq -\delta < 0 .$$

Define the real numbers

$$C_0 := \max \{|x^*(\gamma)| \in \mathbf{R} \,|x^* \in B^*\},$$

$$C_1 := \max \left\{|x^*\left(A_{v_0}\right)| \in \mathbf{R} \,|x^* \in B^*\right\},$$

and choose a real number $\rho > 0$ such that

$$\rho < \min\left(\frac{\alpha}{2 C_0}, \frac{\delta}{2\left(C_0 + C_1\right)}, 1\right) .$$

Then we have

$$\underset{x^* \in W_\alpha}{\forall} x^*\left(A_{v_0 - \rho v}\right) - x^*(\gamma)\left(z_0 - \rho^2\right) - x^*(b) =$$

$$= x^*\left(A_{v_0}\right) - x^*(\gamma) z_0 - x^*(b) - \rho x^*\left(A_v\right) + x^*(\gamma)\rho^2 \leq$$

$$\leq \rho\left[-\frac{\alpha}{2} + \rho C_0\right] < 0$$

and

$$\underset{x^* \in B^* \setminus W_\alpha}{\forall} x^*\left(A_{v_0 - \rho v}\right) - x^*(\gamma)\left(z_0 - \rho^2\right) - x^*(b) =$$

$$= x^*\left(A_{v_0}\right) - x^*(\gamma) z_0 - x^*(b) - \rho x^*\left(A_v\right) + x^*(\gamma)\rho^2 \leq$$

$$\leq -\delta - \rho x^*\left(A_v\right) + x^*(\gamma)\rho^2 \leq$$

$$\leq -\delta + \rho\left(C_0 + C_1\right) < -\frac{\delta}{2},$$

i.e. the element $(v_0 - \rho v, z_0 - \rho^2)$ is contained in Z_σ . Thus, (v_0, z_0) is not a solution of $\mathrm{MP}(\sigma)$, which is a contradiction. ∎

Remark. For the second part of the proof we did not use the Slater condition.

Corollary 2.3. *Let σ in $\mathfrak{M}_0$. If $u_0 = (v_0, E_\sigma)$ in Z_σ is a solution of $MP(\sigma)$, then there exists an element e^* in $\mathfrak{G}_{\sigma,u_0}$ such that $e^*(\gamma) > 0$.*

Proof. Assume $e^*(\gamma) = 0$ for each e^* in $\mathfrak{G}_{\sigma,u_0}$. Choose a Slater element (v, z) in Z_σ. Then we have for each e^* in $\mathfrak{G}_{\sigma,u_0}$

$$e^*\left(A_{v_0}\right) - e^*(b) = 0$$

and

$$e^*\left(A_v\right) - e^*(b) < 0,$$

which implies

$$e^*\left(A_{v_0 - v}\right) > 0.$$

By Theorem 2.2 (v_0, E_σ) does not solve $\mathrm{MP}(\sigma)$, which is a contradiction. ∎

Theorem 2.4. *Let σ in $\mathfrak{M}_0$. Then we have: an element $u_0 = (v_0, z_0)$ in Z_σ is a solution of $MP(\sigma)$ if and only if there exists an element x^* in Σ_{σ,u_0} such that*

(*i*) $$x^*(\gamma) > 0,$$

(*ii*) $$\underset{v \in Y}{\forall}\; x^*\left(A_v\right) = 0.$$

Proof. $(\Leftarrow)$. Assume there exists a functional x^* in Σ_{σ,u_0} with the properties (i) and (ii). Choose an arbitrary element (v, z) in Z_σ. Then we have the estimate

$$z \geq \frac{x^*\left(A_v - b\right)}{x^*(\gamma)} = \frac{x^*\left(A_{v_0} - b\right)}{x^*(\gamma)} = z_0,$$

which implies (v_0, z_0) is a solution of $\mathrm{MP}(\sigma)$.
$(\Rightarrow)$. The restriction mapping $\pi^* : X^* \to A(Y)$ is a weak* continuous linear mapping, if we endow X^* and $A(Y)^*$ with the weak* topologies. Thus, the set

$$\Sigma' := \pi^*\left(\Sigma_{\sigma,u_0}\right)$$

is a weak* compact subset of $A(Y)^*$. If the zero functional of $A(Y)^*$ is not contained in Σ', then one can strictly separate the zero functional from the set Σ', i.e. there exists an element $w = A_v$ in $A(Y)$ such that

$$\forall_{x^* \in \Sigma'} x^*(w) > 0$$

which implies also

$$\forall_{x^* \in \pi^{*-1}(\Sigma')} x^*(w) > 0.$$

In particular, we have

$$\forall_{e^* \in \mathfrak{G}_{\sigma,u_0}} e^*(w) = e^*\left(A_v\right) > 0.$$

But this is a contradiction to Theorem 2.2.

We claim that for each functional x^* in Σ_{σ,u_0} such that $x^*(A(Y)) = 0$ we have $x^*(\gamma) > 0$. In fact, assume $x^*(\gamma) = 0$. Then we have for all (v, z) in Z_σ

$$x^*\left(A_v\right) - x^*(\gamma)z - x^*(b) = 0$$

which contradicts the Slater condition. ∎

For mappings $A : Y \to X$ such that $A(Y)$ has finite dimension, there is a representation for the zero functional. Choose a basis

$$v_1, v_2, \ldots, v_N$$

of the linear space $A(Y)$ and define for each $x^* \in X^*$ the vector

$$G\left(x^*\right) := \left(x^*\left(v_1\right), x^*\left(v_2\right), \ldots, x^*\left(v_N\right)\right).$$

With this notation we have:

Theorem 2.5. *Let σ in $\mathfrak{M}_0$ and assume $\dim A(Y) = N$. Then we have: An element $u_0 = (v_0, z_0)$ in Z_σ is a solution $MP(\sigma)$ if and only if*

$$0 \in \operatorname{con}\left(\left\{G\left(e^*\right) \in \mathbb{R}^N | e^* \in \mathfrak{G}_{\sigma,u_0}\right\}\right).$$

Proof. The mapping $G : X^* \to \mathbb{R}^N$ is continuous and linear. Consequently, we have

$$\operatorname{ext}\left(G\left(\Sigma_{\sigma,u_0}\right)\right) \subset G\left(\operatorname{ext}\left(\Sigma_{\sigma,u_0}\right)\right) = G\left(\mathfrak{G}_{\sigma,u_0}\right).$$

Since $G(\Sigma_{\sigma,u_0})$ is a compact set in a finite dimensional space, it follows that

$$G\left(\Sigma_{\sigma,u_0}\right) = \text{con } G\left(\Sigma_{\sigma,u_0}\right).$$

$(\Leftarrow)$. Assume 0 in con $(G(\mathfrak{G}_{\sigma,u_0})) = (G(\Sigma_{\sigma,u_0}))$. Thus, the set

$$G^{-1}(0) \cap \Sigma_{\sigma,u_0}$$

contains a functional x_0^* such that $x_0^*(A(Y)) = 0$. For each such functional we have $x_0^*(\gamma) > 0$, compare the end of the proof of Theorem 2.4. By Theorem 2.4, u_0 is a solution of MP(σ).

$(\Rightarrow)$. Assume u_0 is a solution of MP(σ). By Theorem 2.4, there exists a functional $x_0^* \in \in \Sigma_{\sigma,u_0}$ such that $x_0^*(A(Y)) = 0$. Then we have

$$0 \in G\left(\Sigma_{\sigma,u_0}\right) = \text{con } G\left(\mathfrak{G}_{\sigma,u_0}\right)$$ ∎

Let $\sigma \in \mathfrak{M}$ and a subset $U \subset Z_\sigma$ be given. Then we define the *common set of active inequalities* by

$$\mathfrak{G}_{\sigma,U} := \bigcap_{u \in U} \mathfrak{G}_{\sigma,u}.$$

Theorem 2.6. *Let σ in $\mathfrak{M}_0$. Then a subset $U \subset Z_\sigma$ is a set of minimal points for MP(σ) if and only if*

$$\forall_{v \in Y} \min_{e^* \in \mathfrak{G}_{\sigma,U}} e^*\left(A_v\right) \leq 0.$$

Proof. $(\Leftarrow)$. For each u in U we have $\mathfrak{G}_{\sigma,U} \subset \mathfrak{G}_{\sigma,u}$. Thus, we have also

$$\forall_{v \in Y} \min_{e^* \in \mathfrak{G}_{\sigma,u}} e^*\left(A_v\right) \leq 0.$$

By Theorem 2.2, u is a solution of MP(σ).

$(\Rightarrow)$. Case 1. $\# U < \infty$. Then each element in the finite dimensional set con (U) is also a solution of MP(σ). Choose a point $u_0 = (v_0, E_\sigma)$ in relint (con U) and let $u = (v, E_\sigma)$ be an arbitrary point in U. Then there exists an element $u_1 = (v_1, E_\sigma)$ in con U and a real number $0 < \rho < 1$ such that

$$u_0 = \rho u + (1 - \rho) u_1.$$

We claim $\mathfrak{G}_{\sigma,u_0} \subset \mathfrak{G}_{\sigma,u}$. In fact, choose $e^* \in \mathfrak{G}_{\sigma,u_0}$. Then we have

$$\begin{aligned} 0 &= e^*\left(A_{v_0}\right) - e^*(\gamma)E_\sigma - e^*(b) = \\ &= \rho\left[e^*\left(A_v\right) - e^*(\gamma)E_\sigma - e^*(b)\right] + \\ &+ (1-\rho)\left[e^*\left(A_{v_1}\right) - e^*(\gamma)E_\sigma - e^*(b)\right] \le \\ &\le \rho\left[e^*\left(A_v\right) - e^*(\gamma)E_\sigma - e^*(b)\right] \le 0, \end{aligned}$$

which implies

$$e^*\left(A_v\right) - e^*(\gamma)E_\sigma - e^*(b) = 0,$$

i.e. $e^* \in \mathfrak{G}_{\sigma,u}$. Thus, we have

$$\mathfrak{G}_{\sigma,u} \supset \mathfrak{G}_{\sigma,u_0}.$$

Since

$$\underset{v\in Y}{\forall} \min_{e^*\in\mathfrak{G}_{\sigma,u}} e^*\left(A_v\right) \le \min_{e^*\in\mathfrak{G}_{\sigma,u_0}} e^*\left(A_v\right) \le 0,$$

the result follows for U finite.

Case 2. $\# U = \infty$. Assume there exists an element $u = (v, E_\sigma)$ in U such that

$$\underset{e^*\in\mathfrak{G}_{\sigma,U}}{\forall} e^*\left(A_v\right) > 0.$$

Since the set $\Sigma_{\sigma,U} := \bigcap_{u\in U} \Sigma_{\sigma,u}$ is an extremal subset of B^* we have also

$$\underset{x^*\in\Sigma_{\sigma,U}}{\forall} x^*\left(A_v\right) > 0.$$

This inequality implies

$$\begin{aligned} \emptyset &= \Sigma_{\sigma,U} \cap \left\{x^* \in B^* | x^*\left(A_v\right) \le 0\right\} = \\ &= \bigcap_{u\in U} \left(\Sigma_{\sigma,u} \cap \left\{x^* \in B^* | x^*\left(A_v\right) \le 0\right\}\right). \end{aligned}$$

By compactness of the involved sets there exists a finite subset

$$U_0 := \{u_1, u_2, \ldots, u_n\} \subset U$$

such that

$$\Sigma_{\sigma,U_0} \cap \left\{x^* \in B^* | x^*\left(A_v\right) \le 0\right\} = \emptyset,$$

which is impossible by case 1. ∎

3. LINEAR VECTOR OPTIMIZATION

Theorem 3.1. *Let K_0 be a line-free cone. A point v_0 in Z_σ is K_0-efficient if and only if*

$$\underset{e^* \in E_0^*}{\forall} e^*(v) \le e^*(v_0) \Rightarrow \underset{e^* \in E_0^*}{\forall} e^*(v) = e^*(v_0),$$

for each $v \in Z_\sigma$.

Proof. The condition of the theorem is equivalent to

$$\underset{v \in Z_\sigma}{\forall} v_0 - v \in K_0 \Rightarrow v = v_0.$$

$(\Rightarrow)$. Assume v_0 is K_0-efficient. Then one has

$$(v_0 - K_0) \cap Z_\sigma = \{v_0\}.$$

Let v be in Z_σ such that $v_0 - v \in K_0$. Then we have $v \in v_0 - K_0$ and consequently

$$v \text{ in } (v_0 - K_0) \cap Z_\sigma = \{v_0\},$$

i.e. $v = v_0$.
$(\Leftarrow)$. Now assume

$$\underset{v \in Z_\sigma}{\forall} v_0 - v \in K_0 \cap Z_\sigma.$$

Let v be in $(v_0 - K_0) \cap Z_\sigma$. Then v in Z_σ and $v_0 - v \in K_0$ and, consequently, $v = v_0$, i.e. v_0 is K_0-efficient.

Theorem 3.2. *A point v_0 in Z_σ is weakly K_0-efficient if and only if*

$$\underset{e^* \in E_0^*}{\forall} e^*(v) \le e^*(v_0) \Rightarrow \underset{e^* \in E_0^*}{\exists} e^*(v) = e^*(v_0),$$

for each $v \in Z_\sigma$.

Proof. $(\Rightarrow)$. Assume v_0 is weakly K_0-efficient. Then we have

$$(v_0 - \operatorname{int}(K_0)) \cap Z_\sigma = \emptyset.$$

Let $v \in Z_\sigma$ be such that

$$\underset{e^* \in E_0^*}{\forall} e^*(v) \le e^*(v_0).$$

Then $v - v_0 \notin \operatorname{int}(K_0)$. By Satz 1.1 of [6], there exists an element $e_0^* \in E_0^*$ such that $e_0^*(v - v_0) = 0$ or $e_0^*(v) = e_0^*(v_0)$.
($\Leftarrow$). Now assume $v_0 \in Z_\sigma$ satisfies the relation of the theorem, and there exists an element $v \in Z_\sigma$ such that

$$v \in v_0 - \operatorname{int}(K_0) .$$

Then we have

$$\forall_{e^* \in E_0^*} e^*(v) < e^*\left(v_0\right),$$

which contradicts the relation of the theorem. ∎

Theorem 3.3. *Let $\sigma \in \mathfrak{M}_0$, i.e. σ satisfies the Slater-condition. Then a point v_0 in Z_σ is weakly K_0-efficient if and only if*

$$\forall_{v \in Y} \min_{e^* \in \mathfrak{G}^0_{\sigma,v_0} \cup E_0^*} e^*(v) \leq 0,$$

where

$$\mathfrak{G}^0_{\sigma,v_0} := \left\{e^* \circ A \in Y^* | e^* \in \mathfrak{G}_{\sigma,v_0}\right\}.$$

Proof. ($\Leftarrow$). Assume the condition is satisfied and v_0 is not weakly K_0-efficient. Then, by Theorem 3.2, there exists an element $v \in Z_\sigma$ such that

$$\forall_{e^* \in E_0^*} e^*(v) < e^*\left(v_0\right),$$

which implies also

$$\forall_{x^* \in B_0^*} x^*(v) < x^*\left(v_0\right).$$

By compactness of B_0^*, there exists a real number $\delta > 0$ such that

$$\forall_{x^* \in B_0^*} x^*(v) - x^*\left(v_0\right) \leq -\delta < 0 .$$

Since $\overline{Z_\sigma^<} = Z_\sigma$ (compare Satz 3.10 of [6], in each neighborhood of v there exists an element $\overline{v}$ in $Z_\sigma^<$. Thus, we can assume

$$\forall_{x^* \in B_0^*} x^*(\overline{v}) - x^*\left(v_0\right) \leq -\frac{\delta}{2} < 0$$

and consequently,

$$\forall_{e^* \in E_0^*} e^*(\overline{v}) < e^*\left(v_0\right).$$

Further we have

$$\forall_{e^* \in \mathfrak{G}^0_{\sigma,v_0}} e^*(\overline{v}) < e^*(b) = e^*\left(v_0\right).$$

Thus, we have

$$\forall_{e^* \in \mathfrak{G}^0_{\sigma,v_0} \cup E_0^*} e^*\left(v_0 - \overline{v}\right) > 0,$$

which contradicts the assumed condition of the theorem.
$(\Rightarrow)$. Now assume $v_0 \in Z_\sigma$ is weakly K_0-efficient and suppose, by contradiction, that there exists an element $v \in Y$ such that

$$\forall_{e^* \in \mathfrak{G}^0_{\sigma,v_0} \cup E_0^*} e^*(v) > 0.$$

These inequalities imply

$$\forall_{x^* \in \Sigma_{\sigma,v_0}} x^*\left(A_v\right) > 0$$

and

$$\forall_{x^* \in B_0^*} x^*(v) > 0.$$

Since the set Σ_{σ,v_0} is compact, there exists a real number $\alpha > 0$ such that

$$\forall_{x^* \in \Sigma_{\sigma,v_0}} x^*\left(A_v\right) \geq \alpha > 0.$$

The open set

$$W_\alpha := \left\{x^* \in B^* | x^*\left(A_v\right) > \frac{\alpha}{2}\right\}$$

contains Σ_{σ,v_0}. Consequently, there exists a real number $\delta > 0$ such that

$$\forall_{x^* \in B^* \setminus W_\alpha} x^*\left(A_{v_0} - b\right) \leq -\delta < 0.$$

Define the element $\overline{v} := v_0 - \rho v$, where ρ satisfies the inequality

$$0 < \rho < \frac{\delta}{2C},$$

with $C := \max\{|x^*(A_v)| \in \mathbf{R} \,|\, x^* \in B^*\}$.

Then we have:

(i)
$$\forall_{x^* \in B_0^*} x^*(\overline{v}) = x^*(v_0) - \rho x^*(v) < x^*(v_0)$$

(ii)
$$\begin{aligned} \forall_{x^* \in W_\alpha} x^*(A_{\overline{v}}) - x^*(b) &= \\ &= x^*(A_{v_0}) - \rho x^*(A_v) - x^*(b) < \\ &< x^*(A_{v_0}) - x^*(b) \le 0 \end{aligned}$$

(iii)
$$\begin{aligned} \forall_{x^* \in B^* \setminus W_\alpha} x^*(A_{\overline{v}}) - x^*(b) &= \\ &= x^*(A_{v_0} - b) - \rho x^*(A_v) \le \\ &\le -\delta + \rho C < \frac{\delta}{2} < 0. \end{aligned}$$

These estimates imply that $\overline{v}$ is contained in Z_σ and satisfies for each $e^* \in E_0^*$ the inequalities

$$e^*(\overline{v}) < e^*(v_0),$$

i.e. v_0 is not weakly K_0-efficient. ■

Remark. For the proof of the necessity the Slater-condition was not used. Since each K_0-efficient point is also weakly K_0-efficient, we have the

Corollary 3.4. *Let $\sigma \in \mathfrak{M}$. If v_0 in Z_σ is K_0-efficient, then*

$$\forall_{v \in V} \min_{e^* \in \mathfrak{G}^0_{\sigma,v_0} \cup E_0^*} e^*(v) \le 0.$$

Theorem 3.5. *Let $\sigma \in \mathfrak{M}_0$. Then a point $v_0 \in Z_\sigma$ is weakly K_0-efficient if and only if the set*

$$F^* := \text{cone}\left(\Sigma^0_{\sigma,v_0}\right) + B_0^*$$

contains a linear functional x_0^* such that

(∗)
$$x_0^*(B) = 0,$$

where Σ^0_{σ,v_0} *denotes the weakly compact and convex set*

$$\overline{con}\ \mathfrak{G}^{\,0}_{\sigma,v_0}.$$

Proof. ($\Leftarrow$). Assume the set F^* contains a functional x_0^* with property (*). Such a functional has a representation

$$x_0^* = \lambda x_1^* + v_1^*$$

such that

$$x_1^* \in \Sigma^0_{\sigma,v_0}\ \&\ \lambda \geqq 0\ \&\ v_1^* \in B_0^*.$$

Choose an arbitrary element $v \in Y$. Then we have

$$0 = x_0^*(v) = \lambda x_1^*(v) + v_1^*(v),$$

which implies

$$\lambda x_1^*(v) \leq 0 \text{ or } v_1^*(v) \leq 0.$$

Case 1. $v_1^*(v) \leq 0$.

Then the set $\{x^*(v) \in \mathbb{R}\,|x^* \in B_0^*\}$ is an interval $[\alpha,\beta]$ with $\alpha \leq 0$. The extreme point α is the image of an extreme point B_0^*, i.e. we have

$$e^*(v) \leq 0$$

for at least one element of E_0^*.

Case 2. $\lambda = 0$.

Then we have $v_1^*(v) = 0$ and we can apply case 1.

Case 3. $\lambda > 0\ \&\ v_1^*(v) > 0$.

Then we have $x_1^*(v) \leq 0$ and the set

$$\left\{x^*(v) \in \mathbb{R}\,|x^* \in \Sigma^0_{\sigma,v_0}\right\}$$

is an interval $[\alpha,\beta]$ with $\alpha \leq 0$. The extreme point α is the image of an extreme point of Σ^0_{σ,v_0}, i.e. we have

$$e^*(v) \leq 0$$

for at least one element in $\mathfrak{G}^{\,0}_{\sigma,v_0}$.

Thus, we have for an arbitrary element $v \in V$ the inequality

$$\min_{e^* \in \mathfrak{G}^{\,0}_{\sigma,v_0} \cup E_0^*} e^*(v) \leq 0.$$

By Theorem 3.3, v_0 is weakly K_0-efficient.

$(\Rightarrow)$. Assume v_0 is weakly K_0-efficient and that set F^* does not contain an element x_0^* with the property (*). Then we claim that such a functional is not contained in the compact set

$$D := \operatorname{con}\left(\Sigma_{\sigma,v_0}^0 \cup B_0^*\right).$$

In fact, if x_0^* is contained in D, then we must have $x_0^* \in \Sigma_{\sigma,v_0}^0$ since otherwise x_0^* would be contained in F^*. Choose an element $\overline{v}$ in $Z_\sigma^<$. Then we have

$$0 = x_0^*\left(\overline{v} - v_0\right) = x_0^*(\overline{v}) - x_0^*\left(v_0\right) <$$
$$< x_0^*(b) - x_0^*\left(v_0\right) = 0,$$

which is a contradiction. Hence x_0^* is not contained in D.

The set D is a weak* compact subset of Y^*, which does not contain the zero functional. Consequently there exists an element v in Y such that

$$\forall_{x^* \in D} \; x^*(v) > 0.$$

In particular, we have

$$\forall_{e^* \in \mathfrak{E}_{\sigma,v_0}^0 \cup E_0^*} \quad e^*(v) > 0.$$

In view of Theorem 3.3, v_0 is not weakly K_0-efficient, which is a contradiction. ∎

Corollary 3.6. *Let $\sigma \in \mathfrak{M}_0$. If v_0 in Z_σ is K_0-efficient, then there exists a functional x_0^* in*

$$\operatorname{cone}\left(\Sigma_{\sigma,v_0}^0\right) + B_0^*$$

such that $x_0(Y) = 0$. ∎

If Y has finite dimension, there is a representation of the functional x_0^*. Choose a basis

$$v_1, v_2, \ldots, v_N$$

of the linear space Y and define for each $x^* \in Y^*$ the vector

$$G\left(x^*\right) := \left(x^*\left(v_1\right), x^*\left(v_2\right), \ldots, x^*\left(v_N\right)\right).$$

With this notation we have:

Theorem 3.7. *Let σ in $\mathfrak{M}_0$ and assume $\dim Y = N$. Then an element v_0 in Z_σ is weakly K_0-efficient if and only if*

$$0 \in \text{cone}\ \left(\left\{G\left(e^*\right) \in \mathbb{R}^N | e^* \in \mathfrak{G}^0_{\sigma,v_0}\right\}\right) + \text{con}\ \left(\left\{G\left(e^*\right) \in \mathbb{R}^N | e^* \in E_0^*\right\}\right).$$

Proof. The mapping $G : Y^* \to \mathbb{R}^N$ is a continuous linear mapping. If C is a compact set in Y^*, then we have

$$\text{ext}\ (G(C)) \subset G(\text{ ext}(C)).$$

Since $G(C)$ is a compact set in a finite-dimensional linear space, the last inclusion implies

$$G(C) = \text{con}\ G(\text{ ext}(C))$$

and also

$$(*) \qquad \text{cone}\ G(C) = \text{cone}\ G(\text{ ext}(C)).$$

$(\Leftarrow)$. Assume

$$0 \in \text{cone}\ \left(G\left(\mathfrak{G}^0_{\sigma,v_0}\right)\right) + \text{con}\ \left(G\left(E_0^*\right)\right).$$

By (*), we have

$$0 \in \text{cone}\ \left(G\left(\Sigma^0_{\sigma,v_0}\right)\right) + \text{con}\ \left(G\left(B_0^*\right)\right).$$

Thus, the set

$$G^{-1}(0) \cap \left(\text{cone}\ \left(\Sigma^0_{\sigma,v_0}\right) + B_0^*\right)$$

contains a functional x_0^* such that $x_0^*(Y) = 0$. By Theorem 3.5, v_0 is weakly K_0-efficient.
$(\Rightarrow)$. Assume v_0 is weakly K_0-efficient. By Theorem 3.5, there exists a functional x_0^* such that

$$x_0^* \in \text{cone}\ \left(\Sigma^0_{\sigma,v_0}\right) + B_0^* \ \&\ x_0^*(Y) = 0.$$

By (*), we have

$$\text{cone}\ \left(G\left(\Sigma^0_{\sigma,v_0}\right)\right) = \text{cone}\ \left(G\left(\mathfrak{G}^0_{\sigma,v_0}\right)\right)$$

and

$$\text{con}\ \left(G\left(B_0^*\right)\right) = \text{con}\ \left(E_0^*\right)$$

which implies

$$0 \in \text{cone}\ \left(G\left(\mathfrak{G}^0_{\sigma,v_0}\right)\right) + \text{con}\ \left(E_0^*\right).$$

■

Theorems 3.5 and 3.7 imply the following corollaries:

Corollary 3.8. *Let $\sigma \in \mathfrak{M}_0$. If there exists a functional x_0^* such that*

$$x_0^* \in B_0^* \ \& \ x_0^*(Y) = 0$$

then each point Z_σ is weakly K_0-efficient.

Corollary 3.9. *Let $\sigma \in \mathfrak{M}_0$. If*

$$0 \in \text{ con } \left(\left\{G\left(e^*\right) \in \mathbb{R}^N | e^* \in E_0^*\right\}\right),$$

then each point in Z_σ is weakly K_0-efficient.

4. SOME EXAMPLES

Let S be a compact Hausdorff space. $S \neq \emptyset$, and let

$$I_k := \{x \in \mathbb{Z} \,|\, 0 < |x| \leq k\}.$$

Consider the compact Hausdorff space

$$T := I_k \times S$$

and assume that the $\mathbb{R}$-vector space $X := C[T]$ is partially ordered by the cone

$$K := \left\{x \in C[T] | \underset{t \in T}{\forall} x(t) \geq 0\right\}.$$

Let Y be a linear subspace of $C[S]$. Consider the parameter $\sigma := (A, \gamma, b)$, where
(i) $A : Y \to C[T]$ is a continuous linear mapping,
(ii) γ is an element of $K \setminus \{0\}$,
(iii) b is an element in $C[T]$.
Then we consider the minimization problem

MP (σ). Minimize the function $p : Y \times \mathbb{R} \to \mathbb{R}$ defined by $p(v, z) := z$ subject to the side-condition

$$A_v - b - \gamma z \in -K.$$

To characterize a solution (v_0, E_σ) of MP(σ) by using the Theorem 2.2 or 2.5, we have to determine the set of extreme points E^* of a basis of K^*. One can choose a basis B^* of K^* such that E^* consists of the positive evaluation functionals of $C[T]$, compare Holmes [5, 80-81]. Thus, we can identify the set $\mathfrak{G}_{\sigma, v_0}$ with

$$M_{\sigma, v_0} := \left\{(\eta, s) \in T | A_{v_0}(\eta, s) - \gamma(\eta, s) E_\sigma = b(\eta, s)\right\}.$$

Using Theorem 2.2 we obtain the following characterization of a solution of MP(σ):

Theorem 4.1. *Assume the parameter σ satisfies the Slater-condition. Then an element $(v_0, E) \in Y \times \mathbb{R}$ is a solution of $MP(\sigma)$ if and only if*

$$\forall_{v \in Y} \min_{(\eta,s) \in M_{\sigma,v_0}} A_v(\eta, s) \leq 0.$$

For finite-dimensional subspaces $Y \subset C[S]$ and an injective mapping A we obtain also a characterization theorem by using Theorem 2.5. Let

$$v_1, v_2, \ldots, v_N$$

be a basis of Y. Using the representation of the functionals of $\mathfrak{G}_{\sigma,v_0}$ the vectors $G(x^*)$ can be identified with the vectors

$$H(\eta, s) := \left(A_{v_1}(\eta, s), A_{v_2}(\eta, s), \ldots, A_{v_N}(\eta, s)\right).$$

Then, Theorem 2.5 implies

Theorem 4.2. *Assume the parameter σ satisfies the Slater-condition and that*

$$Y = \text{span } (v_1, v_2, \ldots, v_N).$$

Then, an element (v_0, E) in $Y \times \mathbb{R}$ is a solution of $\text{MP}(\sigma)$ if and only if

$$0 \in \text{con } \left(\left\{H(\eta, s) \in \mathbb{R}^N | (\eta, s) \in M_{\sigma,v_0}\right\}\right).$$

By suitable choices of the quantities $S, A, k,$ etc. we obtain from the Theorem 4.1 and 4.2 characterizations of best Chebyshev-approximations with various side-conditions.

Example 4.3. *Ordinary Chebyshev-approximation.*

Choose $k = 1$ and define a mapping

$$A : C[S] \to C[T]$$

by setting

$$\forall_{(\eta,s) \in T} x(\eta, s) := \eta y(s),$$

where $x := A_y$. Consider the parameter

$$\sigma = \left(A, \gamma, A_f\right)$$

where $\gamma(\eta, s) = 1$ for each $(\eta, s) \in T$ and f is a given function in $C[S]$. It is easy to see that the parameter σ satisfies the Slater-condition. Then, (v_0, E) is a solution of the minimization problem MP(σ) if and only if v_0 is a best Chebyshev approximation to the function f from Y with minimum distance E. The condition

$$(\eta, s) \in M_{\sigma, v_0}$$

is equivalent to

$$v_0(s) - f(s) = \eta E_\sigma, \text{ i.e. } \eta = sgn\big(v_0(s) - f(s)\big).$$

If we introduce the set

$$M_{f-v_0} := \{s \in S \mid |f(s) - v_0(s)| = E\}$$

and observe that Y contains with v also $-v$, we obtain the well-known Kolmogoroff criterion for best Chebyshev approximation:

An element $v_0 \in Y$ is a best Chebyshev approximation to the function f from Y if and only if

$$\underset{v \in Y}{\forall} \min_{s \in M_{f-v_0}} \big(f(s) - v_0(s)\big)\, v(s) \leq 0.$$

If $Y = \text{span}(v_1, v_2, \ldots, v_N)$ a similar consideration shows that the vector $H(\eta, s)$ can be represented as

$$H(\eta, s) = sgn\big(f(s) - v_0(s)\big) \begin{pmatrix} v_1(s) \\ v_2(s) \\ \vdots \\ v_N(s) \end{pmatrix}.$$

Thus, we conclude from Theorem 4.2, the well-known *zero in the convex hull* theorem:

An element v_0 of a finite-dimensional subspace $Y = span(v_1, v_2, \ldots, v_N)$ is a best Chebyshev approximation to the function f from Y if and only if

$$0 \in \text{con}\left(\left\{sgn\big(f(s) - v_0(s)\big) \begin{pmatrix} v_1(s) \\ v_2(s) \\ \vdots \\ v_N(s) \end{pmatrix} \in \mathbb{R}^N \,\middle|\, s \in M_{f-v_0}\right\}\right).$$

Example 4.4. *One-sided Chebyshev approximation.*

Choose $k = 1$ and define a mapping

$$A : C[S] \to C[T]$$

by setting

$$\underset{(\eta,s)\in T}{\forall} \; x(\eta, s) := \eta y(s)$$

where $x := A_y$. Consider the parameter

$$\sigma = \left(A, \gamma, A_f\right),$$

where

$$\gamma(\eta, s) = \frac{1-\eta}{2} \left(\text{resp. } \gamma(\eta, s) = \frac{1+\eta}{2}\right)$$

for each $(\eta, s) \in T$ and f is a given function in $C[S]$. The Slater condition is not always fulfilled for $\mathrm{MP}(\sigma)$, consider e.g. the case where the function $f, v_1, v_2, \dots, v_N$ have a common zero $s_0 \in S$. For the following we assume that σ satisfies the Slater condition. Then, $(v_0, E) \in Y \times \mathbb{R}$ is a solution of the minimization problem $\mathrm{MP}(\sigma)$ if and only if v_0 is a best one-sided Chebyshev approximation to the function f from the set Z_σ. Let us consider the case $\gamma(\eta, s) = \frac{1-\eta}{2}$; then the condition $(\eta, s) \in M_{\sigma, v_0}$ is equivalent to

$$v_0(s) - b(s) = \begin{cases} -E_\sigma & \text{if} \quad \eta = -1 \\ 0 & \text{if} \quad \eta = 1 \end{cases}.$$

Now define a function $\varepsilon : S \to \{-1, 0, 1\}$ by setting

$$\varepsilon(s) = \begin{cases} -1 & \text{if} \quad b(s) - v_0(s) = E_\sigma \\ 1 & \text{if} \quad b(s) - v_0(s) = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then Theorem 4.1 implies the following characterization:

Assume the parameter σ satisfies the Slater-condition. Then an element $v_0 \in Z_\sigma$ is a best one-sided Chebyshev approximation to the function f from Z_σ if and only if

$$\underset{v\in Y}{\forall} \; \min_{\varepsilon(s)\neq 0} \varepsilon(s) v(s) \leq 0.$$

If Y is finite-dimensional then one can prove also a *zero in the convex hull* theorem as in Example 4.3.

Example 4.5. *A general alternation theorem.*

Choose $k = 1$ and $S = [\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$. Define a mapping

$$A : C[\alpha, \beta] \to C[T]$$

by setting

$$\forall_{(\eta,s)\in T} \; x(\eta, s) = \eta y(s),$$

where $x := A_y$. Further assume that $Y \subset C[\alpha, \beta]$ satisfies the Haar condition, i.e.

$$\det \left(v_\nu \left(s_\mu\right)\right) \neq 0$$

for all points $\alpha \leq s_1 < s_2 < \ldots < s_N \leq \beta$. Consider the parameter

$$\sigma = \left(A, \gamma, A_f\right),$$

where f is a given function in $C[\alpha, \beta]$ and γ is a $K \setminus \{0\}$. Assume σ satisfies the Slater condition and the condition

$$\forall_{s\in S} \; \gamma(1, s) + \gamma(-1, s) > 0.$$

The last condition implies that

$$(\eta, s) \in M_{\sigma,v_0} \Rightarrow (-\eta, s) \notin M_{\sigma,v_0}.$$

If we introduce the set

$$M_{f-v_0} := \left\{s \in S | (\eta, s) \in M_{\sigma,v_0}\right\}$$

and the mapping $\eta : M_{f-v_0} \to \{-1, 1\}$ we can state the characterization Theorem 4.2 as follows:

An element $(v_0, E) \in Y \times \mathbb{R}$ *is a solution of MP*(σ) *if and only if*

$$(*) \qquad 0 \in con \left(\left\{\eta(s) \begin{pmatrix} v_1(s) \\ v_2(s) \\ \vdots \\ v_N(s) \end{pmatrix} \in \mathbb{R}^N | s \in M_{f-v_0}\right\}\right).$$

By Caratheodory's theorem and by the Haar condition the relation (*) implies that there exists $N + 1$ points

$$\alpha \leq s_0 < s_1 < \dots s_N \leq \beta$$

in M_{f-v_0} such that

$$\Sigma_{\nu=0}^{N} \rho_\nu \eta_\nu \begin{pmatrix} v_1(s_\nu) \\ v_2(s_\nu) \\ \vdots \\ v_N(s_\nu) \end{pmatrix} = 0 .$$

$\rho_0 + \rho_1 + \dots \rho_N = 1$, and $\rho_\nu > 0$, $\nu = 0, 1, \dots, N$. Making the same considerations as in Cheney [4, 74-75], we can conclude

$$\eta_\nu \eta_{\nu+1} < 0 ,$$

$\nu = 0, 1, \dots, N-1$. An easy calculation shows that

$$b(s_\nu) - v_0(s_\nu) = -\eta_\nu \gamma(\eta_\nu, s_\nu) E_\sigma,$$

$\nu = 0, 1, \dots, N$. Taking into account the inequalities $\eta_\nu \eta_{\nu+1} < 0$ if follows that

$$b(s_\nu) - v_0(s_\nu) = \varepsilon(-1)^{\nu+1} \gamma(\varepsilon(-1)^\nu, s_\nu) E_\sigma.$$

$\nu = 0, 1, \dots, N$ with a suitable chosen $\varepsilon \in \{-1, 1\}$. Thus, we have the following generalization of the classical alternation theorem.

Theorem 4.6. *Assume $Y \subset C[\alpha, \beta]$ is a Haar subspace and σ satisfies the Slater-condition and the condition*

$$\underset{s \in S}{\forall} \gamma(1, s) + \gamma(-1, s) > 0 .$$

Then, an element $(v_0, E) \in Z_\sigma \times \mathbb{R}$ is a best Chebyshev approximation to the function f if and only if there exists an $\varepsilon \in \{-1, 1\}$ and the set M_{f-v_0} contains $N + 1$ points

$$\alpha \leq s_0 < s_1 < \dots < s_N \leq b$$

such that

$$b(s_\nu) - v_0(s_\nu) = \varepsilon(-1)^{\nu+1} \gamma(\varepsilon(-1)^\nu, s_\nu) E.$$

By appropriate choices of γ we obtain well-known results, e.g. if we choose

$$\gamma(\eta, s) = 1$$

then we have

$$b\left(s_\nu\right) - v_0\left(s_\nu\right) = \varepsilon(-1)^{\nu+1} E,$$

which is the classical alternation theorem. By choosing

$$\gamma(\eta, s) = \frac{1-\eta}{2}$$

(one-sided best approximation) we obtain

$$b\left(s_\nu\right) - v_0\left(s_\nu\right) = \frac{1-\varepsilon(-1)^\nu}{2} E,$$

$\nu = 0, 1, \ldots, N$, i.e. there must be a sequence of points

$$\alpha \leq s_0 < s_1 < \ldots < s_N \leq \beta$$

in M_{f-v_0} such that the error function $f - v_0$ is alternately zero or attains its maximum value E.

Example 4.7. *Approximation of a function and its derivative.*

Choose $k = 2$ and $S = [\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$. Define a mapping

$$A : C^1[\alpha, \beta] \to C[T]$$

by setting

$$\forall_{(\eta,s)\in T} x(\eta, s) := \begin{cases} \eta y(s) & \text{if} \quad \eta \in \{-1, 1\} \\ \frac{1}{2}\eta y'(s) & \text{if} \quad \eta \in \{-2, 2\} \end{cases},$$

where $x = A_y$. Consider the parameter

$$\sigma = \left(A, \gamma, A_f\right),$$

where f is a given function in $C^1[\alpha, \beta]$ and γ is a function in K defined by $\gamma(\eta, s) = 1$.

Then, the element $(v_0, E) \in Y \times \mathbb{R}$ is a solution of $\mathrm{MP}(\sigma)$ if and only if v_0 is a best approximation to the function f from Y with respect to the norm

$$\| f \|_{\infty}^{1} := \max \left(\| f \|_{\infty}, \| f' \|_{\infty} \right)$$

and E is the minimum distance. Since the parameter σ satisfies the Slater-condition, we obtain from Theorem 4.1 the following characterization:

An element $v_0 \in Y$ is a best approximation to the function f from Y with minimum distance E_σ if and only if

$$\forall_{v \in Y} \min \left[\min_{s \in M_{f-v_0}} \left(f(s) - v_0(s) \right) v(s), \right.$$

$$\left. \min_{s \in M_{f'-v_0'}} \left(f'(s) - v_0'(s) \right) v(s) \right] \leq 0 .$$

Here M_{f-v_0} resp. $M_{f'-v_0'}$ denote the sets

$$M_{f-v_0} := \{ s \in [\alpha, \beta] \mid | f(s) - v_0(s) | = E_\sigma \}$$

resp.

$$M_{f'-v_0'} := \{ s \in [\alpha, \beta] \mid | f'(s) - v_0'(s) | = E_\sigma \} .$$

Analogously as in the previous cases one can obtain a *zero in the convex hull* theorem.

Example 4.8. *Restricted range approximation.*

Choose $k = 2$, $S = [\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$, and define a mapping

$$A : C[\alpha, \beta] \to C[T]$$

by setting

$$\forall_{(\eta, s) \in T} \; x(\eta, s) := \begin{cases} \eta y(s) & \text{if} \quad \eta \in \{-1, 1\} \\ \frac{1}{2} \eta y(s) & \text{if} \quad \eta \in \{-2, 2\} \end{cases} ,$$

where $x = A_y$. Further let be given functions $u, l,$ in $C[\alpha, \beta]$ such that

$$\forall_{s \in [\alpha, \beta]} \; l(s) < u(s) .$$

For a given function f in $C[\alpha,\beta]$ define a function b in $C[T]$ by setting

$$b(\eta,s) := \begin{cases} \eta f(s) & \text{if} \quad \eta \in \{-1,1\} \\ u(s)-f(s) & \text{if} \quad \eta = -2 \\ -l(s)+f(s) & \text{if} \quad \eta = 2 \end{cases} .$$

Consider the parameter $\sigma = (A,\gamma,b)$, where γ is a function in $K \setminus \{0\}$ defined by

$$b(\eta,s) := \begin{cases} 1 & \text{if} \quad \eta \in \{-1,1\} \\ 0 & \text{if} \quad \eta \in \{-2,2\} \end{cases} .$$

Then, the element $(v_0, E) \in Y \times \mathbb{R}$ is a solution of MP(σ) if and only if v_0 is a *restricted range approximation* to the function f from Y with minimum distance E, i.e. the error function $f - v_0$ satisfies the side-conditions

$$\forall_{s\in S}\; l(s) \leq f(s) - v_0(s) \leq u(s).$$

To state characterization theorems for the restricted range approximation we introduce the set

$$\begin{aligned} M_{f-v_0} := \{s \in S \mid\; & |f(s)-v_0(s)| = E_\sigma \\ & \text{or } |f(s)-v_0(s)| = l(s) \\ & \text{or } |f(s)-v_0(s)| = u(s)\} \end{aligned}$$

and the mapping $\varepsilon : S \to \{-1,0,1\}$ defined by

$$\varepsilon(s) := \begin{cases} -sgn\big(f(s)-v_0(s)\big) & \text{if} \quad \big(sgn\big(f(s)-v_0(s)\big), s\big) \in M_{\sigma,v_0} \\ 1 & \text{if} \quad f(s)-v_0(s) = u(s) \\ -1 & \text{if} \quad f(s)-v_0(s) = l(s) \\ 0 & \text{otherwise} \end{cases} .$$

It is easy to see that the mapping ε is well-defined. Since σ satisfies the Haar condition we conclude from Theorem 4.1:

An element v_0 in Z_σ is a restricted range approximation to the function f if and only if

$$\forall_{v\in Y} \min_{\varepsilon(s)\neq 0} \varepsilon(s)v(s) \leq 0.$$

If Y is finite-dimensional then, by Theorem 4.2, we have:

An element v_0 in Z_σ is a restricted range approximation to the function f if and only if

$$(*) \qquad 0 \in \operatorname{con}\left(\left\{\varepsilon(s)\begin{pmatrix} v_1(s) \\ v_2(s) \\ \vdots \\ v_N(s) \end{pmatrix} \in \mathbb{R}^N \middle| \varepsilon(s) \neq 0\right\}\right).$$

If Y satisfies also the Haar condition then the relation (*) implies that there exist $N + 1$ points

$$\alpha \leq s_0 < s_1 < \ldots < s_N \leq \beta$$

in M_{f-v_0} such that

$$\Sigma_{\nu=0}^N \rho_\nu \varepsilon\left(s_\nu\right)\begin{pmatrix} v_1\left(s_\nu\right) \\ v_2\left(s_\nu\right) \\ \vdots \\ v_N\left(s_\nu\right) \end{pmatrix} = 0,$$

$\rho_0 + \rho_1 + \ldots + \rho_N = 1$, and $\rho_\nu > 0$, $\nu = 0, 1, \ldots, N$. As in Example 4.5, we can conclude

$$\left(s_\nu\right) \varepsilon\left(s_{\nu+1}\right) < 0.$$

$\nu = 0, 1, \ldots, N-1$, i.e. the error function $f - v_0$ alternates between the values $\min[E_\sigma, u(s_\nu)]$ and $\max[-E_\sigma, l(s_{\nu+1})]$. More precisely, this is stated in the following

Theorem 4.9. *Assume $Y \subset C[\alpha, \beta]$ is a Haar subspace. Then, an element $(v_0, E_\sigma) \in$ $\in Z_\sigma \times \mathbb{R}$ is a restricted range approximation to the function f if and only if there exists an $\varepsilon \in \{-1, 1\}$ and the set M_{f-v_0} contains $N + 1$ points*

$$\alpha \leq s_0 < s_1 < \ldots < s_N \leq \beta$$

such that

$$f\left(s_\nu\right) - v_0\left(s_\nu\right) = \left[\frac{1 - \varepsilon(-1)^\nu}{2} \min\left(u\left(s_\nu\right), E_\sigma\right) + \right.$$
$$\left. + \frac{1 - \varepsilon(-1)^{\nu+1}}{2} \max\left(l\left(s_\nu\right), E_\sigma\right)\right].$$

Example 4.10. *Scalarization of vector optimization problems.*

Scalarization of a vector optimization problem means to replace this problem by a family of scalar optimization problems such that a solution of such a scalar problem is weakly efficient and such that each weakly efficient point can be obtained in this way. In the following we extend the scalarization developed in [2,3] to the more general situation considered in this paper. For the scalarization we use the equivalent formulation of a weakly efficient point of Theorem 3.1.

To scalarize a given vector optimization problem choose an element γ in K_0 such that

$$\underset{e^* \in E_0^*}{\forall} e^*(\gamma) > 0;$$

this is always possible since we assume $\operatorname{int}(K_0) \neq \emptyset$. Further choose a functional $e_0^* \in E_0^*$, we can assume $e_0^*(\gamma) = 1$. Then, define the set of parameter

$$\Lambda := \{\lambda \in Y | e_0^*(\lambda) = 0\},$$

which implies $Y = \Lambda + \mathbb{R}\gamma$. Consider for each $\lambda \in \Lambda$ the following scalar minimization problem:

MPS (λ). Minimize the function $p : Y \times \mathbb{R} \to \mathbb{R}$ defined by $p(v, z) := z$ subject to the side-condition

$$v - \gamma z - \lambda \in -K_0$$

and

$$A_v - b \in -K.$$

The second side-condition describes the feasible set. This scalarization has the wanted properties, as the following theorems show. We denote by P_λ the set of all minimal points MPS(λ). Each element in P_λ is a pair (v_0, z_0) in $Z_\sigma \times \mathbb{R}$.

Theorem 4.11. *Let v_0 be a weakly efficient point in Z_σ. Then there exist a parameter $\lambda_0 \in$ $\in \Lambda$ and a real number z_0 such that (v_0, z_0) is a minimal point of MPS(λ_0).*

Proof. The element v_0 has a representation $v_0 = \mu_0 + \alpha\gamma$ with $\mu_0 \in \Lambda$ and $\alpha \in \mathbb{R}$. Then, define

$$\lambda_0 := v_0 - \alpha\gamma \;\&\; z_0 := \alpha.$$

If (v_0, z_0) is not a minimal point of MPS(λ_0) then there exists a point v in Z_σ and real number $z < z_0$ such that

$$v - \gamma z - \lambda_0 \in -K_0 \;\&\; A_v - b \in -K.$$

Then, it follows that

$$\forall_{e^* \in E_0^*} e^*(v) < e^*\left(v_0\right),$$

which contradicts the weak efficiency of v_0. ∎

Since each efficient point is also weakly efficient we have the

Corollary 4.12. *Let v_0 be an efficient point in Z_σ. Then there exist a parameter $\lambda_0 \in \Lambda$ and a real number z_0 such that (v_0, z_0) is a minimal point of $MPS(\lambda_0)$.*

Theorem 4.13. *Let λ in Λ be given. Then, each point $v \in Y$ with $(v, z_0) \in P_\lambda$ is weakly efficient.*

Proof. Clearly, each $v \in Y$ with $(v, z_0) \in P_\lambda$ is contained in Z_σ. Assume now, there is a point $(v_0, z_0) \in P_\lambda$ such that v_0 is not weakly efficient. Then there exists a point $\overline{v}$ in Z_σ such that

$$\forall_{e^* \in E_0^*} e^*(\overline{v}) < e^*\left(v_0\right),$$

which implies that there exists a $\delta > 0$ such that

$$\forall_{e^* \in E_0^*} e^*\left(\overline{v} - v_0\right) \leq -\delta e^*(\gamma).$$

Then it follows that

$$\forall_{e^* \in E_0^*} e^*(\overline{v}) - e^*(\lambda) - \left(z_0 - \frac{\delta}{2}\right) e^*(\gamma) =$$

$$= e^*\left(\overline{v} - v_0\right) + e^*\left(v_0\right) - e^*(\lambda) - \left(z_0 - \frac{\delta}{2}\right) e^*(\gamma) \leq$$

$$\leq -\delta e^*(\gamma) + \frac{\delta}{2} e^*(\gamma) = -\frac{\delta}{2} e^*(\gamma) \leq 0,$$

i.e. $(\overline{v}, z_0 - \delta/2)$ is a better solution, which is a contradiction. ∎

The set P_λ has a representation $Q_\lambda \times \{z_0\}$ such that $Q_\lambda \subset Z_\sigma$ and $z_0 \in \mathbb{R}$.

Theorem 4.14. *If the set P_λ contains an element (v_0, z_0) such that v_0 is efficient with respect to the set Q_λ, then v_0 is also efficient with respect to the set Z_σ.*

Proof. Choose a point v in Z_σ such that

$$\forall_{e^* \in E_0^*} e^*(v) \leq e^*\left(v_0\right).$$

Then we have also

$$e^*(v) - e^*(\lambda) - e^*(\gamma) z_0 \leq e^*\left(v_0\right) - e^*(\lambda) - e^*(\gamma) z_0 ,$$

i.e. (v, z_0) is in P_λ and so v in Q_λ. Since v_0 is efficient with respect to Q_λ, it follows that

$$\underset{e^* \in E_0^*}{\forall} e^*(v) = e^*\left(v_0\right),$$

which proves the claim. ∎

If we choose $Y = \mathbb{R}^N$ then we have the semi infinite-vector optimization problem with respect to the cone K_0. As theorem 4.13 and 4.14 show the scalarization can be carried out with a linear parameter set of dimension at most $N-1$. In the case of $e \leq N-1$ objective functions the dimension of the parameter space can be reduced to $e-1$ as the results of Brosowski, Conci [2] show. Thus, we have the

Theorem 4.15. *Consider the semi-infinite linear vector optimization problem in $\mathbb{R}^N$ with respect to a cone K_0 or with respect to e linear objective functions. Then, the $B\&C$ scalarization can be carried out with a linear parameter space of the dimension at most* $\min(e-1, N-1)$.

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Received December 28, 1990
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