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QUOJECTIONS AND THE PROBLEM OF TOPOLOGIES OF GROTHENDIECK

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Dedicated to the memory of Professor Gottfried Köthe

Abstract. *We show that the problem of topologies of Grothendieck has a negative answer in the case of quojections: we construct a separable reflexive quojection* E *such that* $E\widehat{\otimes}_\pi l^2$ *does not have property* (BB) *in the sense of [T1]. This result implies for example the existence of a strict* (LB)*-space without a local partition of unity in the sense of Hollstein.*

1. INTRODUCTION AND PRELIMINARIES

Most of the concrete Fréchet spaces appearing in the analytic applications of Functional analysis are nuclear Fréchet spaces or Banach spaces. Relevant classes containing the nuclear spaces are Schwartz spaces and Montel spaces whereas the class of quojections, which has recently received much attention (see for example the survey article [M-M]), constitutes a way to generalize the structure of Banach spaces to a wider setting. During the last few years the classical problem of topologies of Grothendieck (open question 2 in [G]) has been considered in most of the above mentioned classes of Fréchet spaces (see [T1,2,3]). In this note we treat the problem in the last open case, namely in the setting of quojections.

Recall that by definition, section 5 in [B-Du], a Fréchet space is a *quojection* if it is isomorphic to a projective limit of a sequence of surjective operators on Banach spaces. Section 2 contains the main result of this note: there exists a separable quojection E such that $E\widehat{\otimes}_\pi l_2$ does not have property (BB), i.e. the *bounded* sets of $E\widehat{\otimes}_\pi l_2$ are not contained in the bounded sets of the canonical form $\overline{\Gamma(B \otimes C)}$, where $B \subset E$ and $C \subset l_2$ are bounded and Γ means the absolutely convex hull. As in many of the results of [T1,2,3] the crucial steps in this construction consist of considerations concerning tensors in finite dimensional Banach spaces (see section 2).

According to [Bo-M] every quojection is a quotient of a countable product of copies of $l_1(I)$ for a suitable index set I. Consequently, our example shows that a quotient of a countable product of Banach spaces need not be an (FBa)-space in the sense of [T3].

We call a Fréchet space E *quasinormable,* if an arbitrary 0-neighbourhood U contains another 0-neighbourhood V such that for all $\lambda > 0$ we can find a bounded set $B \subset E$ with $V \subset B + \lambda U$. It is known that all quojections are quasinormable spaces. Using the preceding result we thus get negative answers to the Grothendieck problem also in the setting

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of quasinormable spaces. In particular, we see that it is impossible to generalize the proof of Proposition 2.2.2 of [T3] for quasinormable spaces.

In section 3 we use duality to show that for the quojection E constructed in section 2 the spaces $E'_b \otimes_\epsilon l_2$ and $L_b(l_2, E'_b)$ are not (gDF)-spaces. We also show that there exists a strict (LB)-space without a local partition of unity in the sense of [H], section 2.

Our notation and terminology are standard and we follow [K1,2] and [T1]. We recall that the tensor product of seminorms $\|\cdot\|_E$ and $\|\cdot\|_F$ is defined by

$$(\|\cdot\|_E \otimes \|\cdot\|_F)(z) = \inf \sum_{i=1}^{n} \| a_i \|_E \| b_i \|_F,$$

where E and F are vector spaces and the infimum is taken over all representations $z = \sum a_i \otimes b_i \in E \otimes F$. The complete projective and injective tensor products and the ϵ-product of Schwartz for the locally convex spaces E and F are denoted by $E \widehat{\otimes}_\pi F$, $E \widehat{\otimes}_\epsilon F$ and $E \epsilon F$, respectively. The space of linear continuous mappings from E to F endowed with the topology of uniform convergence on the bounded sets of E is denoted by $L_b(E,F)$. The reader unfamiliar with the basic techniques of the projective tensor norm in Banach spaces is asked to consult [T1], section 4.1.

2. CONSTRUCTION OF THE QUOJECTION AND MAIN RESULTS

We now define a Fréchet space E which is a quojection such that $E \widehat{\otimes}_\pi l_2$ does not have property (BB). The construction resembles that of [T1], section 4, and we try to use analogous notation. However, to prove a form of Lemma 4.3, [T1], in the setting of quojections we need some completely new ideas.

We fix the number $p, 1 \leq p < \infty$. Our space E will depend on p; for $1 < p < \infty$ we get a reflexive space.

For all $n, k \in \mathbf{N}$, $k \geq 2$, let M_{nk} and $N_{nk,2}$ be the k-dimensional Hilbert spaces l_2^k. We choose for all n and k the space N_{nk} to be the finite dimensional Banach space E_k defined in the beginning of section 3.1 of [T2]. Recall that then M_{nk} is contained (isometrically) in N_{nk} and that the projection constant $\inf\{\| P \| \mid P$ is a projection from N_{nk} onto $M_{nk}\}$ is for each fixed n asymptotically equal to $\sqrt{k}$. We fix for all n and k a continuous projection P_{nk} from N_{nk} onto M_{nk} with $\| P_{nk} \| \leq 2\sqrt{k}$ (this is possible by [P], 28.2) and denote $N_{nk,1} := P_{nk}(N_{nk})$.

Remark. Let $(e_i)_{i=1}^k$ be the canonical basis of M_{nk}. The inequality (4.1) of [T2] says that the tensors

$$x_{nk} = \sum_{i=1}^{n} e_i \otimes e_i \in M_{nk} \otimes M_{nk}$$

satisfy for all n and k

$$(1) \qquad \left(\|\cdot\|_{M_{nk}} \otimes \|\cdot\|_{M_{nk}}\right)(x_{nk}) \geq C\sqrt{k}\left(\|\cdot\|_{N_{nk}} \otimes \|\cdot\|_{M_{nk}}\right)(x_{nk}),$$

where C is an absolute constant; we can assume here $0 < C < 1$. On the other hand, the norm of x_{nk} can be calculated in the space $M_{nk}\widehat{\otimes}_\pi M_{nk}$; by [K2], 42.6(1) we have

$$(2) \qquad \left(\|\cdot\|_{M_{nk}} \otimes \|\cdot\|_{M_{nk}}\right)\left(\sum_{i=1}^{k} e_i \otimes e_i\right) = k.$$

We still denote $G_{nk} = N_{nk} \oplus N_{nk,2}$ and define the subspace $\widehat{N}_{nk} \subset G_{nk}$ by

$$(3) \qquad \widehat{N}_{nk} := \operatorname{sp}\left(e_i + (Cn\sqrt{k})^{-1} f_i | i = 1, \dots, k\right),$$

where $(e_i)_{i=1}^k \subset M_{nk}$ and $(f_i)_{i=1}^k \subset N_{nk,2}$ are the canonical orthonormal bases and C is as in the Remark above.

Now for all n, k we define the following two norms in the space G_{nk}:

$$(4) \qquad \widetilde{g}_{nk}(x) = \left(\| x_{N_{nk}} \|_{N_{nk}}^2 + \| x_{N_{nk,2}} \|_{N_{nk,2}}^2\right)^{\frac{1}{2}},$$

and

$$(5) \qquad g_{nk}(x) = \inf_{y \in \widehat{N}_{nk}} \widetilde{g}_{nk}(x+y),$$

where $x_{N_{nk}}$ and $x_{N_{nk,2}}$ denote the $N_{nk}-$ and $N_{nk,2}$-components of $x \in G_{nk}$.

We still denote

$$H := \left\{ x = (x_{nk})_{n,k \in \mathbb{N}, k \geq 2} \,|\, x_{nk} \in G_{nk}, \sum_{n,k} \widetilde{g}_{nk}^p(x_{nk}) < \infty \right\},$$

and define two seminorms on H:

$$\widetilde{h}(x) := \left(\sum_{n,k} \widetilde{g}_{nk}(x_{nk})^p\right)^{\frac{1}{p}}, \qquad h(x) := \left(\sum_{n,k} g_{nk}(x_{nk})^p\right)^{\frac{1}{p}}.$$

The corresponding unit balls are denoted by

$$\widetilde{U} := \left\{ x \in H | \widetilde{h}(x) \leq 1 \right\}, \qquad U := \{ x \in H | h(x) \leq 1 \}.$$

Note that $(H, \widetilde{h})$ is a Banach space.

We denote by V the closed unit ball of l_2 and for all $k \geq 2$ we denote by M_k some fixed k-dimensional subspaces of l_2.

Lemma 1. *For all* $n \in \mathbb{N}$, $n > (24/C)^2$, *and all* $s \in \mathbb{R}$

$$\Gamma(U \otimes V) \cap \Gamma(n\widetilde{U} \otimes V) \not\subset \Gamma(\sqrt{n}U \cap s\widetilde{U}) \otimes V.$$

Proof. It is not a restriction to assume that $s \geq 1$. By the remark above the tensors

$$z_{nk} := \frac{Cn}{\sqrt{k}} \sum_{i=1}^{k} e_i \otimes e_i \in M_{nk} \otimes M_k \subset H \otimes l_2$$

satisfy

$$\left(\|\cdot\|_{M_{nk}} \otimes \|\cdot\|_{M_k}\right)(z_{nk}) \geq C\sqrt{k}\left(\|\cdot\|_{N_{nk}} \otimes \|\cdot\|_{M_k}\right)(z_{nk}). \tag{6}$$

(It does not cause any confusion to denote also some orthonormal basis of M_k by $(e_i)_{i=1}^k$.) By (2) we then have the normalization

$$\left(\|\cdot\|_{M_{nk}} \otimes \|\cdot\|_{M_k}\right)(z_{nk}) = Cn\sqrt{k}. \tag{7}$$

Then it follows immediately from (6) and (7) that $z_{nk} \in \Gamma(n\widetilde{U} \otimes V)$ for all n and k. Since each z_{nk} has the representation

$$z_{nk} = \sum_{i=1}^{k} \frac{Cn}{\sqrt{k}} e_i \otimes e_i,$$

and on the other hand since for all i

$$g_{nk}(e_i) = \inf_{y \in \widehat{N}_{nk}} \widetilde{g}_{nk}(e_i + y) \leq \widetilde{g}_{nk}\left(e_i - \left(e_i + (Cn\sqrt{k})^{-1} f_i\right)\right) =$$

$$= (Cn\sqrt{k})^{-1} \widetilde{g}_{nk}(f_i) = (Cn\sqrt{k})^{-1} \| f_i \|_{N_{nk,2}} = (Cn\sqrt{k})^{-1},$$

we get the estimate

$$\left(g_{nk} \otimes \|\cdot\|_{M_k}\right)(z_{nk}) \leq \sum_{i=1}^{k} \frac{Cn}{\sqrt{k}} g_{nk}(e_i) \| e_i \|_{M_k} \leq$$

$$\leq \sum_{i=1}^{k} \frac{Cn}{\sqrt{k}} (Cn\sqrt{k})^{-1} = 1;$$

hence, $z_{nk} \in \Gamma(U \otimes V)$ for all n and k.

We show that for all $n > (24/C)^2$ and s there exists a k such that

$$z_{nk} \notin \Gamma\left(\sqrt{n}U \cap s\widetilde{U}\right) \otimes V.$$

Let k be such that

$$16C^{-2}s^2 \leq k \leq 64C^{-2}s^2; \tag{8}$$

this choice is possible since $0 < C < 1$ and $s \geq 1$. Suppose that z_{nk} has a representation as a finite sum

$$z_{nk} = \sum_i t_i a_i \otimes b_i;$$

where $\sum_i |t_i| \leq 1$, $a_i \in \sqrt{n}U \cap s\widetilde{U}$ and $b_i \in V$. By definition we may assume that $a_i \in G_{nk}$ and $b_i \in M_k$.

Let us denote $R_{nk} = P_{nk}\widetilde{P}_{nk}$, where $\widetilde{P}_{nk}$ is the canonical projection from G_{nk} onto N_{nk}. Since $\sum_i t_i R_{nk} a_i \otimes b_i$ is also a representation of z_{nk}, the normalization (7) implies the existence of i_0 such that

$$\| R_{nk} a_{i_0} \|_{M_{nk}} \geq Cn\sqrt{k} > 4ns. \tag{9}$$

Now we have also

$$\| \left(id_{N_{nk}} - P_{nk}\right) \widetilde{P}_{nk} a_{i_0} \|_{N_{nk}} > 2ns; \tag{10}$$

otherwise

$$\begin{aligned} &\widetilde{g}_{nk}\left(a_{i_0}\right) \geq \| \widetilde{P}_{nk} a_{i_0} \|_{N_{nk}} \geq \\ &\geq \| R_{nk} a_{i_0} \|_{M_{nk}} - \| \left(id_{N_{nk}} - P_{nk}\right) \widetilde{P}_{nk} a_{i_0} \|_{N_{nk}} \geq 2ns, \end{aligned} \tag{11}$$

which would contradict the assumption $a_{i_0} \in s\widetilde{U}$. But now (10) and (8) and the choice of n

in Lemma 1 imply

$$g_{nk}\left(a_{i_0}\right) = \inf_{x\in\widehat{N}_{nk}} \left\{\widetilde{g}_{nk}\left(a_{i_0}+x\right)\right\} =$$

$$= \inf_{x\in\widehat{N}_{nk}} \left\{\left(\| \left(id_{G_{nk}} - \widetilde{P}_{nk}\right)\left(a_{i_0}+x\right)\|^2_{N_{nk,2}} + \| \widetilde{P}_{nk}\left(a_{i_0}+x\right)\|^2_{N_{nk}}\right)^{\frac{1}{2}}\right\} \geq$$

$$(12) \qquad \geq \inf_{x\in\widehat{N}_{nk}} \left\{\| \widetilde{P}_{nk}\left(a_{i_0}+x\right)\|_{N_{nk}}\right\} \geq$$

$$\geq \| id_{N_{nk}} - P_{nk}\|^{-1} \inf_{x\in\widehat{N}_{nk}} \left\{\| \left(id_{N_{nk}} - P_{nk}\right)\widetilde{P}_{nk}\left(a_{i_0}+x\right)\|_{N_{nk}}\right\} =$$

$$= \| id_{N_{nk}} - P_{nk}\|^{-1} \| \left(id_{N_{nk}} - P_{nk}\right)\widetilde{P}_{nk} a_{i_0}\|_{N_{nk}} \geq 2ns/(3\sqrt{k}) \geq \frac{Cn}{12} > 2\sqrt{n}.$$

This contradicts the assumption $a_{i_0} \in \sqrt{n}U$. Hence, $z_{nk} \notin \Gamma(\sqrt{n}U \cap s\widetilde{U}) \otimes V$.

Using the space H and the norms h and $\widetilde{h}$ we now construct the Fréchet space E as in section 4.4 of [T1]; the only change is to replace $\eta_m(x) = \sum \widehat{h}_m(x_n)$ of [T1] by

$$\eta_m(x) = \left(\sum \widehat{h}_m\left(x_n\right)^p\right)^{\frac{1}{p}}.$$

To show that E is a quojection we shall use the fact that E is a Moscatelli-type Fréchet space in the terminology of [Bo-Di]. We first establish this statement in more detail.

Let us denote by $\widehat{N}$ the closure of $\oplus_{n,k}\widehat{N}_{nk}$ in $(H,\widetilde{h})$. It is easy to see that $\widehat{N}$ is equal to kern(h) in H. We claim that for all $x \in H$

$$(13) \qquad h(x) = \inf_{y\in\widehat{N}} \widetilde{h}(x+y).$$

Indeed, for all $x \in H$

$$h(x) = \left(\sum_{n,k} g_{nk}\left(x_{nk}\right)^p\right)^{\frac{1}{p}} = \left(\sum_{n,k}\left(\inf_{z\in\widehat{N}_{nk}} \widetilde{g}_{nk}\left(x_{nk}+z\right)\right)^p\right)^{\frac{1}{p}} \leq$$

$$\leq \inf_{y\in\widehat{N}} \left(\sum_{n,k} \widetilde{g}_{nk}\left(x_{nk}+y_{nk}\right)^p\right)^{\frac{1}{p}} = \inf_{y\in\widehat{N}} \widetilde{h}(x+y),$$

where x_{nk} is the G_{nk}-coordinate of x etc. On the other hand, if $x \in H$ and $\varepsilon > 0$ is arbitrary and $J \subset \mathbb{N} \times (\mathbb{N} - \{1\})$ is a finite subset such that

$$\sum_{(n,k) \notin J} \widetilde{g}_{nk} \left(x_{nk}\right)^p < \varepsilon^p,$$

then

$$\inf_{y \in \widehat{N}} \widetilde{h}(x+y) = \inf_{y \in \widehat{N}} \left(\sum_{n \in \mathbb{N}, k \in \mathbb{N}-\{1\}} g_{nk} \left(x_{nk} + y_{nk}\right)^p \right)^{\frac{1}{p}} \leq$$

$$\leq \inf_{y \in \widehat{N}} \left(\sum_{(n,k) \in J} g_{nk} \left(x_{nk} + y_{nk}\right)^p \right)^{\frac{1}{p}} + \left(\sum_{(n,k) \notin J} g_{nk} \left(x_{nk}\right)^p \right)^{\frac{1}{p}} =$$

$$= \left(\sum_{(n,k) \in J} \left(\inf_{z \in \widehat{N}_{nk}} g_{nk} \left(x_{nk} + z\right) \right)^p \right)^{\frac{1}{p}} + \left(\sum_{(n,k) \notin J} g_{nk} \left(x_{nk}\right)^p \right)^{\frac{1}{p}} \leq h(x) + \varepsilon,$$

since J is finite. This implies (13).

It follows now from (13) and Proposition 1.4 of [Bo-Di] that E is canonically isomorphic to the Fréchet space of Moscatelli-type with respect to l_p, $(H_k/\widehat{N}, \overline{h})$, $(H_k, \widetilde{h})$ and q_k, where we use the terminology of Definition 1.3 of [Bo-Di] and H_k is equal to $H, \overline{h}$ is the quotient norm of $\widetilde{h}$ with respect to $\widehat{N}$ and q_k is the continuous quotient mapping $(H_k, \widetilde{h}) \to$ $\to (H_k/\widehat{N}, \overline{h})$.

Lemma 2. *The space E is a quojection.*

Proof. In view of the definitions this follows now immediately from Proposition 2.10, (5) and (6), of [Bo-Di].

Theorem 3. *There exists a reflexive quojection E such that $E \widehat{\otimes}_\pi l_2$ does not have property (BB).*

Proof. Using the proof of Theorem 4.5 of [T1] and Lemma 1 we see that $E \widehat{\otimes}_\pi l_2$, where E is as above, does not have property (BB). Moreover, for $1 < p < \infty$ the space E is reflexive.

Recall that in the terminology of [T3] this means that quojections are not in general (FBa)-spaces.

The following consequence completes the considerations of section 2.1 of [T3].

Corollary 4. *Quotients of (FBa)-spaces need not be (FBa)-spaces.*

Proof. By [Bo-M] every quojection is a quotient of the countable product of Banach spaces $l_1(I)$ for some index set I. Countable products of Banach spaces are (FBa)-spaces by [T1], Proposition 3.5 so that the result follows from Theorem 3.

Corollary 5. *Quasinormable Fréchet spaces need not be (FBa)-spaces in the sense of [T3]. In particular, there exist quasinormable Fréchet spaces E and F such that $E\hat{\otimes}_\pi l_2$ and $F\hat{\otimes}_\pi F$ do not have property (BB).*

Proof. Quojections are quasinormable by [D-Z], Remark $b(\alpha)$ on p. 552, so we use Theorem 3. For the last statement we take $F = E \times l_2$ and use section 3.4 of [T1].

Remark. It follows easily from Proposition 3.5 of [T1] that our quojection is not trivial, i.e., it is not a product of Banach spaces.

3. OTHER CONSEQUENCES OF THE EXAMPLE

Using duality we derive some more consequences of the example constructed in the previous section.

For the definition of (gDF)-spaces we refer to [J], Section 12.4.

Proposition 6. *Let E be a quojection which is the surjective limit of the sequence of Banach spaces $(E_n)_{n\in\mathbb{N}}$. If we put $F_n := (E_n)'_b$, $F := E'_b$ and if G is a Banach space, then*

(i) $L_b(G,F)$ and the strict inductive limit $indL_b(G,F_n)$ coincide algebraically, they have the same bounded subsets and they induce the same topology on the bounded subsets,

(ii) the space $E\hat{\otimes}_\pi G$ is quasinormable and

$$\left(E\hat{\otimes}_\pi G\right)'_b = indL_b\left(G,F_n\right)$$

holds topologically, and

(iii) the following conditions are equivalent:

(1) $L_b(G,F)$ is a (gDF)-space

(2) $L_b(G,F)$ is a (DF)-space

(3) $L_b(G,F)$ is barreled

(4) $L_b(G,F)$ = is (ultra)bornological

(5) $L_b(G,F) = indL_b(G,F_n)$ holds topologically

(6) $G\hat{\otimes}_\pi E$ has property (BB) of [T1].

Proof. The statement *(i)* follows from [Di], section 4, and *(ii)* is due to Grothendieck. Concerning the proof of *(iii)*, (5) and (6) are equivalent by *(ii)*, whereas *(i)* shows that (1) implies (5). Finally, the implications (5) $\to$ (4) $\to$ (3) $\to$ (2) $\to$ (1) are well-known.

Propositon 7. *Let E be a quojection which is the surjective limit of the sequence of Banach spaces $(E_n)_{n\in\mathbb{N}}$. If we put $F_n := (E_n)'_b$, $F := E'_b$ and G is a Banach space, then*

(i) $G\epsilon F$ and the strict inductive limit ind $(G\epsilon F_n)$ coincide algebraically, they have the same bounded subsets and they induce the same topologies on the bounded subsets, and

(ii) the following conditions are equivalent:

(1) $G\epsilon F$ is a (gDF)-space

(2) $G\epsilon F$ is a (DF)-space

(3) $G\epsilon F$ is barreled

(4) $G\epsilon F$ is (ultra)bornological

(5) $G\epsilon F = ind(G\epsilon F_n)$ holds topologically.

If G has the approximation property (in the sense of [K2], chapter 43), then the ϵ-product can be replaced here by the complete injective tensor product.

(iii) If G'_b has the approximation property and G is reflexive, then (1) to (5) of (ii) imply

(6) the space $G'_b\widehat{\otimes}_\pi F'_b = G'_b\widehat{\otimes}_\pi E''_b$ has the property (BB).

(iv) If G has the approximation property, then (1)-(5) in (ii) are also equivalent to

(1') $G\otimes_\epsilon F$ is a (gDF)-space

(2') $G\otimes_\epsilon F$ is a (DF)-space

(3') $G\otimes_\epsilon F$ is quasibarreled

(4') $G\otimes_\epsilon F$ is bornological

(5') $G\otimes_\epsilon F = ind(G\otimes_\epsilon F_n)$ holds topologically.

Proof. The statement *(i)* is a consequence of [Bi-M], 5.10, and *(ii)* follows directly from *(i)*. So we prove now *(iii)*.

According to [D-F], Proposition 2

$$\left(G\widehat{\otimes}_\epsilon F\right)'_b = G'_b\widehat{\otimes}_\pi F'_b$$

holds algebraically. If $G\otimes_\epsilon F$ is quasibarreled (DF)-space, then this identity also holds topologically. Consequently, if C is any bounded subset of $G'_b\widehat{\otimes}_\pi F'_b$, it is strongly bounded in $(G\widehat{\otimes}_\epsilon F)'$, hence it is $(G\widehat{\otimes}_\epsilon F)$-equicontinuous. We can apply again [D-F], Proposition 2, to obtain neighbourhoods U and V in G and F such that

$$C \subset \overline{\Gamma\,(U^\circ \otimes V^\circ)}.$$

(Here U° denotes the polar of U and the closure is taken in $E'_b\widehat{\otimes}_\pi F'_b$.) Hence, $G'_b\widehat{\otimes}_\pi F'_b$, has property (BB).

Concerning *(iv)* it is enough to observe that (5) implies (5') by density, since G has the approximation property, that the implications (5') → (4') → (3') → (2') → (1') are trivial, and that (1') → (1) follows again by a density argument.

Corollary 8. *(i) There exists a quojection E such that neither $E'_b \otimes_\epsilon l_2$ nor $L_b(l_2, E'_b)$ is a (gDF)-space.*

(ii) There is a strict (LB)-space $F = ind F_n$, where $(F_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces, such that $l_2 \otimes_\epsilon F$ and $ind(l_2 \otimes_\epsilon F_n)$ do not coincide topologically, hence F is a strict (LB)-space without a local partition of unity in the sense of [H], section 2.

Proof. In the former section we constructed a quojection E such that $E\widehat{\otimes}_\pi l_2$ does not have property (BB). Proposition 6 implies that $L_b(l_2, E'_b)$ is not (gDF). Taking E reflexive Proposition 7 implies that $E'_b\widehat{\otimes}_\epsilon l_2$ is not a (gDF)-space and $E'_b = ind(E_n)'_b$ is a strict (LB)-space such that $l_2 \otimes_\epsilon E'_b$ and ind $(l_2 \otimes_\epsilon (E_n)'_b)$ do not coincide topologically. For the statement concerning the local partition of unity, see [H], Proposition 3.2.

Remarks. *1.* A result related to *(i)* was proved in [Bo-G] using a different method.

2. The space F of Corollary 8 is the first example of a strict (LB)-space without a local partition of unity.

3. The class of Fréchet spaces satisfying the *density condition of Heinrich* was studied in [Bi-Bo1]. A Fréchet space has the density condition if and only if the bounded subsets of the strong dual are metrizable. Every Fréchet-Montel space and every quasinormale Fréchet space have the density condition. In [Bi-Bo2] it is shown that if E and F are Fréchet spaces with the density condition and $E\widehat{\otimes}_\pi F$ has property (BB), then $E\widehat{\otimes}_\pi F$ has also the density condition. Our examples above show that there are quasinormable Fréchet spaces E and F (hence such that $E\widehat{\otimes}_\pi F$ is quasinormable) such that $E\widehat{\otimes}_\pi F$ does not have property (BB). As a consequence, the converse of the result mentioned above does not hold.

Note added in proofs. After the present paper was submitted, a simplified ccouterexample and nice complementary results were obtained by J.C. Diaz and G. Metafune in «The problem of topologies of Grothendieck for quojections», Results Math. **21**, 1992, 299-312.

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