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ABELIAN p-GROUPS OF ARBITRARY LENGTH AND THEIR ENDOMORPHISM RINGS

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Dedicated to the memory of Professor Gottfried Köthe

The second author enjoyed listening to many brilliant lectures on pure and applied mathematics by Professor G. Köthe at Frankfurt University. Gottfried Köthe refereed Ulm's Habilitationsschrift in 1936 and in 1988 he helped completing the paper [«Helmut Ulm: His work and its impact on recent mathematics», by R. Göbel, pp. 1-10 in Contemporary Mathematics vol. 87 (1989)]. This paper which deals with abelian p-groups of arbitrary infinite Ulm length is dedicated to the memory of Gottfried Köthe.

1. INTRODUCTION

In this paper we want to generalize results on endomorphism rings of separable abelian $p-$ groups (= abelian p-groups of length ω) and of abelian p-groups of length λ with λ cofinal to ω (see [2]) to abelian p-groups of arbitrary infinite length. The length $l(G) = \lambda$ of an abelian p-groups G is the first ordinal λ with Ulm subgroup $p^\lambda G$ to be 0; cf. [10, vol. 1, p. 154]. In order to deal with decomposition properties of abelian groups, it turned out to be very useful to prescribe rings as endomorphism rings.

The general question, which will be investigated, can be summarized as follows.

Can we find a p-groups G of length λ (for a given infinite ordinal λ) such that the endomorphism ring End G of G modulo the ideal of all small endomorphisms becomes isomorphic to a prescribed ring A?

The necessity of some non-trivial «natural» ideal reflects the presence of well-known decomposition theorems like Gauß' Fundamental Theorem, which has substantial influence on all abelian p-groups.

The idea for such realization theorems goes back to two classical papers of Corner's, concerning torsion-free abelian groups [3] and abelian p-groups (of length ω) [4] respectively. His idea has been exploited in a number of subsequent papers as [7], [9], [11], [18] and [19]. After having investigated endomorphism rings of abelian p-groups of length cofinal to ω in [2], see also Goldsmith [11] for the case $\lambda = \omega + 1$, we want to derive realization theorems for endomorphism rings of abelian p-groups of any infinite length.

If the length λ of the groups under investigation is a limit ordinal cofinal to ω, say $cf\lambda = \omega$, then the p^λ-topology becomes the central tool: The p^λ-topology on groups G of length λ is defined by taking the subgroups $p^\alpha G (\alpha < \lambda)$ as basis of neighborhoods for $0 \in G$.

This topology is a metrizable Hausdorff topology under the assumption that λ is cofinal to ω.

Metrizable topologies have decent properties. In particular, infinite direct sums will never be complete. Using now topological arguments, the transition from separable abelian p-groups becomes natural and takes place in the frame work of the p^λ-topology. With some care, e.g. replacing cyclic p-groups by generalized Prüfer groups and by now standard combinatorial methods, the desired results can be established in this case [2]. The fact, that the p^λ-topology on p-groups of length λ with $cf\lambda \neq \omega$ is no longer metrizable, makes the topological approach completely useless.

A striking example of such a group can be constructed as follows. Let B_α be the generalized Prüfer group of length α (see § 2) and let λ be an infinite cardinal of cofinality $\kappa > \omega$. There exists a strictly increasing sequence $\beta_\nu (\nu \in \kappa)$ with supremum λ. Then we consider the torsion subgroup B of

$$\prod_{\alpha<\lambda}^{<\kappa} B_\alpha$$

the cartesian product with elements of support $< \kappa$. The p^λ-completion $\widehat{B}$ is the inverse limit of the system of groups $B/p^{\beta_\nu}B$ with $(\nu \in \kappa)$. An easy exercise shows however that B is not even dense in $\widehat{B}$.

Dealing with p-groups of length not cofinal to ω, we need different methods to obtain a realization theorem. These methods are quite simple. We replace the underlying «dense submodule B» of the construction in [7] by $B' = H \oplus B$ with H of length λ. Then we proceed as before making sure that B' remains pure (even isotype) in the final extension G in such a way that $G/p^\omega G = G/p^\omega$ looks like the old separable case [7].

In order to state our main theorems, and to describe the «natural» ideal mentioned above, we recall Pierce's [16] well-known notion of a **small homomorphism**.

A homomorphism $\varphi : G \to H$ between two groups G, H is small if the following holds:

$$(*) \qquad \forall \kappa < \omega \; \exists n = n_\kappa < \omega \;\; \text{with} \;\; p^n G\left[p^\kappa\right] \varphi = 0.$$

The set of all small homomorphisms $\varphi : G \to H$ is denoted by Small (G, H). We also write Small (G,G) = Small G, , which is a two sided ideal of End G.

The following properties of Small (G, H) have been observed by Pierce [16].

Small$_\lambda(G, H)$ is a pure and closed subgroup of Hom (G, H) equipped with the p-adic topology. The quotient group Hom (G, H)/Small (G, H) is torsion-free and complete in the p-adic topology.

A converse of Pierce's observations will be our main result which is the following

Realization-Theorem. *Let A be a torsion-free ring with 1 which is complete and Hausdorff in the p-adic topology and let λ be a limit ordinal. Then there exists a p-group of length λ such that End $G = A \oplus$ Small G.*

Moreover, the cardinality of G can be any cardinal $\mu > \max(|\lambda|, |A|)$ with $\mu^{\aleph_0} = \mu$.

Using slightly modified arguments we obtain maximal rigid systems of groups as in the Theorem. A pair of groups is rigid, if homomorphisms between them are small. The Theorem extends results in [2], where we restricted λ to be cofinal to ω. In [2] we replaced **all** (basic) cyclic groups of G by generalized Prüfer groups. This allows to prescribe the endomorphism ring even on the layers $p^\alpha G$, however we have to pay for this. We had to replace Small G by a larger ideal $\text{Small}_\lambda G$. The Theorem gives rise to various pathological decompositions of groups of length λ depending on the choice of well-known rings A. As special cases we derive the existence of essentially indecomposable groups of length λ of arbitrarily large cardinality.

All application can be derived similar to [7] and references given there.

While direct sums of cyclic groups are the basis in constructing separable abelian p-groups with prescribed endomorphism rings, here we will need a generalized Prüfer group of length λ as well. It is made into an A-module, and the desired abelian p-groups will be extensions of direct sums of such groups. The combinatorical arguments which are needed to get rid of unwanted endomorphisms are of course similar to those used in the case of separable abelian p-groups and depend on Shelah's Black Box, see [7].

2. PRELIMINARIES

First we will give the basic definitions and formulate the required combinatorical results. As indicated in § 1, we will have to generalize generalized Prüfer groups, passing from modules over the p-adic integers to modules over certain rings A.

We recall from [2] the following known result (cf. also [20], [17])

Corollary 2.1. *Let A be a ring with 1 such that A^+ is a torsion-free and p-reduced group. For all ordinals α there exists an A-module X_α such that*
(i) X_α is a reduced totally-projective p-group of length α,
(ii) $X_\alpha = \bigoplus_{\beta<\alpha} X_\beta$, if α limit ordinal,
(iii) $p^\alpha X_{\alpha+n} \cong A/p^n A$ for all $n < \omega$ and ordinals α,
(iv) $X_{\alpha+n}/p^\alpha X_{\alpha+n} \cong X_\alpha$ for all $n < \omega$ and ordinals α.
The isomorphisms in (iii) and (iv) are A-module isomorphisms.

We choose an A-module X_λ of length λ as in (2.1) and A-modules $X_n \cong A/p^n A$ of length p^n for all $n < \omega$. Let $\kappa := |X_\lambda| \geq |A| \geq \aleph_0$ and choose a cardinal λ' such that $\lambda'^\kappa = \lambda'$. In particular, by König's Lemma ([12], p. 45) follows $cf\lambda' > \kappa \geq \aleph_0$.

We consider the three $T' =^{\omega>} \lambda'$, choose elements $1_\sigma \in X_\sigma$ of order $p^{l(\sigma)}$ for each $\sigma \in T'$ and idenfity 1_σ with σ. Then we define our basic A-module

$$(*) \qquad B := X_{-1} \oplus \bigoplus_{\tau \in T'} X_\tau$$

where $X_\tau := X_n$ for all $n < \omega$ such that $n = l(\tau)$ and $X_{-1} := X_\lambda$. Hence $l(B) = \lambda$ and let $T = T' \cup \{-1\}$.

The final group G will be an extension of B which is constructed recursively declaring particular ω-tuples of elements in B as new elements of the extension of B. Hence the extensions can be described formally in a language L having constants for the elements in B and at most λ' function symbols with at most ω places to allow L to talk about the desired group extensions of B. The constructions then take place in an L-universe B^* of cardinality $\lambda'(= \lambda'^{\aleph_0})$ which can be obtained easily by induction on intermediate sets, say $B = B^0 \subset B^\alpha(\alpha \in \omega_1)$, $B^* = \cup B^\alpha$ taking closures of B^α under the operation of the function symbols on B^α; cf. Shelah [18, 19].

In order to formulate Shelah's Black Box in B^*, we will use a (preliminary) **support** of elements in B^*:
If $x \in B$, then $x = \sum_{\tau \in T} x_\tau$ with finitely many elements $0 \neq x_\tau \in X_\tau$ follows from (*). We let $[x]^* = \{\tau \in T : x_\tau \neq 0\}$ be the *-support of x. If the *-support of elements in B^α is already defined and $x \in B^{\alpha+1} \setminus B^\alpha$, then we can find an (as we will see even unique) ω-tuple $X = (x_i)_{i \in \omega} \in (B^\alpha)^\omega$ such that $x = f(X)$, where f is represented by a well determined function symbol in L. In this case we let $[x]^* = \bigcup_{i \in \omega} [x_i]^*$. The subset $[x]^*$ of T is the smallest set $[x]^*$ such that

$$x \in \prod_{\tau \in [x]^*} X_\tau.$$

The notion of a *-support can be naturally be extended to subsets of B^*.

In order to deal with singular cardinals λ' as well, we also have to introduce **norms** of elements in B^*. The *-norm $\| x \|^*$ of an element x will be the norm $\| [x^*] \|$ of the underlying support of x and it remains to define norms of subsets of T.
We choose a continuous strictly increasing function $\| \cdot \|$: $cf\lambda' + 1 \to \lambda' + 1$ such that $\| 0 \| = 0$ and $\| cf\lambda' \| = \lambda'$; moreover let $\| -1 \| = -1$. For any element $X \subset T$ let $\| X \| := \min\{\nu < cf\lambda' : X \subset^{\omega>} \| \nu \|\}$.

An A-module $P = \bigoplus_{\tau \in I} X_\tau \subset B$ with $I \subset T$, $|I| < \kappa$ is called a **canonical submodule** of B and a **trap** is a triple (f, P, φ), where P is a canonical submodule, $f :^{\omega>} \kappa \to T$ is a tree embedding, $[P]^*$ is a subtree of T, $cf \| P \| = \omega$, $\| v \|^* = \| P \|^*$, whenever $v \in$ $\in Br(Im\, f)$, $\varphi : B^* \to B^*$ is a partial map with $P \subset dom\varphi \subset [P]^*$ and $[P\varphi]^* \subset [P]^*$.

Here $Br(Im\, f)$ denotes the set of all branches in $Im\, f$.

Now we can state our main combinatorical tool (see [18, 19] and also (A.7) in [7] for a proof):

The Black Box. *For some ordinal* $\lambda^* < \lambda'^+$ *exists a transfinite sequence of traps* $(f_\alpha, P_\alpha, \varphi_\alpha)_{\alpha<\lambda^*}$ *such that for all* $\alpha, \beta < \lambda^*$,
(i) $\beta < \alpha \Rightarrow \| P_\beta \|^* \leq \| P_\alpha \|^*$,
(ii) $\beta \neq \alpha \Rightarrow Br(Im\, f_\alpha) \cap Br(Im\, f_\beta) = \emptyset$.
(iii) $\beta + \kappa^{\aleph_0} \leq \alpha \Rightarrow Br(Im\, f_\alpha) \cap Br([P_\beta]^*) = \emptyset$,
(iv) for all $X \subset B$ *with* $|X| \leq \kappa$, *any partial map* φ *such that* $B \subset dom\ \varphi$ *there exists* $\alpha \in \lambda^*$ *such that* $X \subset P_\alpha$, $\| X \|^* < \| P \|^*$, $\varphi\lceil P_\alpha = \varphi_\alpha$, *and* $[X\varphi]^* \subset [P_\alpha]^*$.

Finally we extend B (inside B^*) in order to get rid of «essentially all» endomorphisms except scalar multiplication by elements in A. We do this «locally first» and consider the following

Definition 2.2. *Let* $D := \bigoplus_{\kappa<\omega} x^\kappa A$ *be the direct sum of* ω *copies of* A. *Furthermore, let* x_κ *be an element of order at most* $p^{\kappa+1}$ *of* B *for each* $\kappa < \omega$, *let*

$$X := < px^0, px^{\kappa+1} + x_\kappa - x^\kappa,\ \kappa < \omega >$$

be an A*-submodule of* $D \oplus H$ *where* H *is an* A*-invariant* p*-group containing* B. *Then call* $H' := H \oplus \bigoplus_{\kappa<\omega} x^\kappa A)/X$ *a* **rk-1 extension** *of* H *by* $x_\kappa (\kappa < \omega)$.

Write $H' = \ll H, x_\kappa, \kappa < \omega \gg$ for the A-module generated by H and x^κ subject to the relations X and call (x^κ) the **chain defined by** x_κ with $\kappa < \omega$. Such rk-1 extensions will arise from branches v of T. If $\sigma_\kappa = v\lceil\kappa$ and $x_\kappa = \sigma_\kappa$, $v^\kappa = \sum_{n\geq\kappa} p^{n-\kappa}\sigma_n$, then $B' = \ll B, x_\kappa, \kappa < \omega \gg$ is an example. We will also say that (x^κ) is **a chain defined by** v.

The following properties of rk-1 extensions are easily verified.

Lemma 2.3. *Let* H' *be a rk-1 extension of* H *by* $x_\kappa (\kappa < \omega)$ *and let* φ *be an endomorphism of* H *with* $p^n\varphi = 0$. *Then the following holds for all* $\kappa < \omega$.
(i) H'/H *is divisible and* $H \cap x^\kappa A = 0$,
(ii) $x^\kappa = x_\kappa + px^{\kappa+1} = \sum_{m=\kappa}^n p^{m-\kappa} x_m + p^{n+1-\kappa} x^{n+1}$ *and* $o(x^\kappa) = p^{\kappa+1}$, $n \geq \kappa$,
(ii) If* $g \in H'$, *then* $g = b + p^n a x^m$, *for an* $a \in A$, $b \in H$ *and* $m < \omega$.
(iii) There exists an extension $\varphi' : H' \to H$ *of* φ *with* $p^n\varphi' = 0$ *and*
(*) $g\varphi' := b\varphi$ *for all* $g \in H'$ *with* $g = b + p^n a x^m$ *for an* $a \in A$, $m < \omega$ *and* $b \in H$.

Proof. (i) and (ii) follow by construction of H' and (ii*) by iterated application of (ii).
(iii) We first define a homomorphism $\phi : Y := H \oplus \bigoplus_{\kappa<\omega} x^\kappa A \to H$. If $p^n\varphi = 0$, then we set $\phi\lceil H = \varphi$ and $x^\kappa a\phi := \sum_{m=0}^n p^m x_{\kappa+m} a\varphi$. The homomorphism ϕ is well defined, because $\bigoplus_{\kappa<\omega} x^\kappa A$ is a free A-module and H is an A-module. We show that X (as above) is contained in ker ϕ and consider the canonical A-generators of X.

$px^0 a\phi = p\sum_{m=0}^n p^m x_m a\varphi = \sum_{m=0}^n p^{m+1} x_m a\varphi = 0$, since $o(x_m)$ divides p^{m+1} and $(px^{\kappa+1} - x^\kappa + x_\kappa)a\phi = p\sum_{m=0}^n p^m x_{\kappa+1+m} a\varphi - \sum_{m=0}^n p^m x_{\kappa+m} a\varphi + x_\kappa a\varphi = p^{n+1} x_{\kappa+1+n} a\varphi + -x_\kappa a\varphi + x_\kappa a\varphi = 0$, since φ is p^n-bounded. Hence $X \subset \ker \phi$.

The induced homomorphism $\Phi : H' \to H$ of ϕ extends φ because $\phi\lceil H = \varphi$. By (ii*) we can write $g = b + p^n a x^\kappa$ for some $a \in A$, $b \in H$ and $\kappa < \omega$. Hence $g\Phi = b\varphi + p^n a x^\kappa = = b\varphi$ by $p^n\varphi = 0$.

Lemma 2.3 has an immediate consequence.

Definition/Lemma 2.4. *Let H' be a rk-1 extensin of B. The projection $\pi_\tau : B \to \tau A$ with $\tau \in T$ and $l(\tau) = n$ has a unique extension*

$$\widehat{\pi}_\tau : H' = \ll B, x_\kappa, \kappa < \omega \gg \to \tau A \left(b + p^n a x^\kappa \to b\widehat{\pi}_\tau\right).$$

We call $\widehat{\pi}_\tau = \pi_\tau$.

Now we are ready to replace the preliminary *-support by a refined support. This will follow by induction (§ 3) based on the

Definition 2.5. *Let H be a group contained in B^* and containing $[H]^*$. Suppose π_τ : $H \to \tau A$ is a given projection extending $\pi_\tau : \bigoplus_{\sigma\in[H]^*} \sigma A \to \tau A$.*
If $H' = \ll H, x_\kappa, \kappa < \omega \gg$ is a rk-1 extension of H and $h \in H'$, then let

$$[h] := \{\tau \in T, h\pi_\tau \neq 0\} \cup \{-1\} \quad \textit{be the support of} \quad h.$$

Similarly let $\| h \| := \min\{\nu < cf\lambda' : [h] \subset^{\omega>} \rho(\nu)\}$ be the norm of h, which extends naturally to subsets of H'.

The two notions of supports are related, which follows by (2.6) and an induction in § 3. The basic step (2.6) is very easy.

Lemma 2.6. *(a) For any subset Y of B we have $[Y] = [Y]^*$.*
(b) For any element x of a rk-1 extension H of B we have $[x] \subseteq [x]^*$.

If we let $x_\kappa = 0$ $(\kappa < \omega)$ in (2.3), then H' is visibly not reduced. On the other hand we have:

Lemma 2.7. *Let H be a reduced extension of B contained in B^* and let v be a branch of T such that $v \cap [h]^*$ is finite for all $h \in H$ with $\| h \| = \| v \|$. If $x_\kappa \in B$ is of order at most p^κ and $H' = \ll H, x_\kappa, \kappa < \omega \gg$ is a rk-1 extension of H as above with $[x^\kappa]^* \cap v$ infinite for all $\kappa < \omega$, then H' is reduced.*

Proof. If H' is not reduced, then there are elements $z_n \in H' \setminus H$ of order p^{n+1} such that $pz_{n+1} = z_n$ for all $n < \omega$ and $pz_0 = 0$. These elements can be expressed as $z_n = h_n + a_n x^{n*}$

with $h_n \in H$, $a_n \in A \setminus pA$ and $n* < \omega$. Clearly $pz_{n+1} = ph_{n+1} + pa_{n+1}x^{(n+1)*} = h_n + a_n x^{n*} = z_n$ and $ph_{n+1} - h_n \equiv a_n x^{n*} - pa_{n+1}x^{(n+1)*} \in H \cap x^{(n+1)*}A \equiv 0$ modulo X, with X as in (2.2). It follows that $a_n x^{n*} - pa_{n+1}x^{(n+1)*} \in X$, hence $(n+1)* = n^* + 1$, $a_n \equiv a_{n+1} \mod p^{n*+1}A$ and (a_n) forms a coverging p-adic sequence in A. There exists a limit $a \in A$ by completeness of A and $a \equiv a_n \mod p^n A$. Moreover $a \notin pA$ from $a_n \notin pA$. Using $a \in A \setminus pA$ and $ph_{n+1} - h_n = a(x^{n*} - px^{n*+1}) = ax_{n*} \in B$ we derive $h_n = -a\sum_{\kappa=n*}^{n*+m} p^{\kappa-n*}x_\kappa + p^{m+1-n}h_{m+1}$ and $[h_n]^*$ is almost the same as v, which contradicts our assumption on v.

Lemma 2.8. *Let H be as in Definition 2.5 and let be $H' := \ll H, x_\kappa, \kappa < \omega \gg$ with (x^κ) a chain defined by x_κ and $v \in Br(T)$ such that $v \cap [h]$ finite for all $h \in H$ and*
(i) $x_\kappa = v_\kappa + b_\kappa, b_\kappa \in B[p^{\kappa+1}]$
(ii) $\| v \| > \sup\{\| b_\kappa \|, \kappa < \omega\}$.
Then H is isotype in H', $p^\omega H' = p^\omega H$ and $h_{H'}(p^m x^\kappa) = p^m$ for all $m < \kappa + 1$.

Proof. We first claim that H is **pure in** H' and show $p^n H' \cap H \subset p^n H$ for all $n < \omega$. If $h' = p^n y$ in $p^n H' \cap H$, we want $h' \in p^n H$. Since H' is a rk-1 extension of H, there are $h \in H$, $a \in A$ and $\kappa < \omega$ such that $y = h + ax^\kappa$. Hence $h' = p^n(h + ax^\kappa) \in p^n H' \cap H$ and $h' - p^n h = p^n a x^\kappa \in H \cap x^\kappa A = 0$ by (2.3).

Thus $p^n h = h' \in p^n H$ as desired.

In order to show $p^\omega H = p^\omega H'$, it remains to show that any $h' \in p^\omega H'$ is in $p^\omega H$. Since H' is a rk-1 extension of $H, h' = h + ax^m \in p^\omega H'$ for some $h \in H, m < \omega$ and $a \in A$. If $ax^m = 0$, then $h' = h \in p^\omega H' \cap H = p^\omega H$ by purity. Now suppose $ax^m \neq 0$ for contradiction. We calculate with (2.3) $(h + x^m a) = h - \sum_{\kappa=m}^{n}(p^{\kappa-m}v_\kappa + p^{\kappa-m}b_\kappa)a + p^{n+1-\kappa}x^{n+1}a$.

By assumption on v we find n, such that $\sigma \in v\setminus[h]$, $l(\sigma) = n$ and $\| \sigma \| > \| b_\kappa \|$ for all $\kappa < \omega$. It then follows $\sigma \notin [b_\kappa]$ and we derive $(h + x^m a)\pi_\sigma = p^{n-m-1}\sigma a$ and $a = p^r a'$ for some $r \leq m$, $a' \in a \setminus pA$ from $x^m a \neq 0$. Thus $p^{n-m-1}\sigma a = \sigma p^{n-m-1}p^r a' \neq 0$. Since σA is a direct summand of H', also $h_{H'}(h + x^m a) = h_{H'}(p^{\kappa-m-1}\sigma a) = h_{H'}(p^{\kappa-m-1-r}\sigma) = p^{\kappa-m-1-r} < p^\omega$.

This contradiction proves our claim.

The same calculations show $p^m x^\kappa = p^{m+1}x^{\kappa+1} + p^m x_\kappa$ for all $m \leq \kappa + 1$. Using purity we derive

$$h_{H'}(p^m x^\kappa) = h_{H'}\left(p^{m+1}x^{\kappa+1} + p^m x_\kappa\right) \leq \min\left\{p^{m+1}, h_H\left(p^m x_\kappa\right)\right\} \leq p^m.$$

Since heights in the sum are different, we derive equality $h_{H'}(p^m x^\kappa) = p^m$ for all $m < \kappa + 1$ as desired.

Now it is immediate from purity and the last height equations that H is an isotype subgroup of H'.

3. CONSTRUCTION OF ABELIAN p-GROUPS OF LENGTH λ AND PROOF OF THE THEOREM

a. The construction. We will proceed similar to the construction given in [7]. If $(f_\alpha, P_\alpha, \varphi_\alpha)_{\alpha<\lambda^*}$ is a sequence of traps given by the Black Box (§ 2), then we will construct inductively a group G in B^* as the union of a continuous chain $G_\alpha (\alpha < \lambda^*)$. At the same time we define the notion of supports of elements in G and determine a subset $S \subset \lambda^*$ of «strong ordinals». By continuity we only have to deal with non-limit ordinals $< \lambda^*$. Let $G_0 = B$ (from § 2) and suppose G_β and $S \cap \beta$ have been determined for all $\beta \leq \alpha$ and some $\alpha < \lambda^*$. In order to define $G_{\alpha+1}$ and $S \cap \alpha + 1$, we consider $(f_\alpha, P_\alpha, \varphi_\alpha)$ from the Black Box.

Suppose we can choose a branch $v_\alpha \in Br(Im\ f_\alpha)$, a chain $(g_\alpha^\kappa)_{\kappa<\omega}$ defined on the branch v_α by $g_{\alpha,\kappa} \in B[p^{\kappa+1}]$ such that the following conditions hold for

$$G_{\alpha+1} = \ll G_\alpha, g_{\alpha,\kappa}, \kappa < \omega \gg .$$

$(\alpha_1)\, g_\alpha^\kappa = v_\alpha^\kappa + b_\kappa$ for some $b_\kappa \in B[p^{\kappa+1}]$ and $\sup_{\kappa<\omega} \| b_\kappa \| < \| v_\alpha \|$

(α_α) If the partial homomorphism φ extends φ_α, then we can find $n < \omega$ such that $g_\alpha^n \varphi \notin G_{\alpha+1}$.

(β_α) If $\beta \in S \cap \alpha$ (was strong) and a partial homomorphism φ extends φ_β such that $P_\alpha^* \cap G_{\alpha+1} \subset dom\varphi$, then we can find $n < \omega$ such that $g_\alpha^n \varphi \notin G_{\alpha+1}$.

In this case we say that α is strong, put α into S and choose the extension $G_{\alpha+1}$ as above.

If $\alpha \in S$ is not possible, then we choose $G_{\alpha+1}$ as above without the requirement (α_α). In this case we call α a weak ordinal.

It follows by now standard arguments that the weak case - in particular requirement (β_α) - is always possible. This is normally referred to as the statement that there are no useless ordinals, see [7] and (3.5). Using (2.4) to (2.6), the notion of support extends inductively to G. This finishes the construction of G and G is fixed for the rest of this paper.

b. Proof of the Theorem. Finally we want to show that the constructed p-group G satisfies the condition of the Theorem. We begin with some of its algebraic properties summarized in the Theorem and (3.6) and finish with End $G = A \oplus Small\, G$.

As in [7], we can easily show that elements in G have a very special support.

Recognition Lemma 3.1. *If $g \in G \setminus B$, then there is a unique $\alpha \in \lambda^*$ such that $g \in G_{\alpha+1} \setminus G_\alpha$. Moreoveer, there exists a strictly decreasing sequence of ordinals $\alpha = \alpha_0 > \ldots > \alpha_r \in \lambda^*$ such that $\| P_{\alpha_i} \| = \| P_\alpha \|$ for $i \leq r$ and there is $\nu < \| P_\alpha \|$ with ${}_\nu[g] = F \cup \bigcup_{i\leq r} {}_\nu[v_{\alpha_i}]$ (a disjoint union), where F is a finite set of elements of T each of norm greater than $\| P_\alpha \|$. Furthermore for each $\beta < \lambda$ with $\| P_\beta \| = \| P_\alpha \|$ there exist an $a \in A$ and $\kappa < \omega$ such that $g\pi_\sigma = \sigma(p^{l(\sigma)-\kappa}a)$ for almost all $\sigma \in v_\beta$.*

In particular, elements in B and on branches can be recognized.

Lemma 3.2. *There exists an ordinal $\nu <\| v_\alpha \|$ such that ${}_\nu[g_\alpha^\kappa] \subset v_\alpha$ and for all $a \in A$.*

$$a \in p^{\kappa+1} A \Longleftrightarrow g_\alpha^\kappa a = 0 \Longleftrightarrow {}_\nu[g_\alpha^\kappa a] \text{ is finite},$$

In particular $Ann_A(g_\alpha^\kappa) = p^\kappa A$ and $b \in B$ if and only if $[b]$ is finite.

Definition 3.3. *For any $\eta < \lambda$ the constant branch $w(\eta)$ is the branch represented by the constant function $\omega \to \{\eta\}$, hence $w(\eta) =^{\omega>} \{\eta\}$.*

Every branch $v_\alpha(\alpha < \lambda^*)$ has norm $\| v_\alpha \|$ a limit ordinal and $\| b \|$ is a sucessor ordinal. We derive a

Corollary 3.4. *Let $g \in G$. Then $[g]$ contains no infinite subset of a constant branch w, such that $\| w \|=\| g \|$.*

The next Lemma is similar to [7] but it is the crucial part for proving that there are no useless ordinals. The final argument is in [7].

Lemma 3.5. *Let $\alpha < \lambda^*$, $\nu <\| P_\alpha \|$ and for each branch $v \in Br(Im\ f_\alpha)$ let $(g_v^\kappa)_{\kappa<\omega}$ be a chain defined by $g_{v,\kappa} \in B[p^{\kappa+1}](\kappa < \omega)$ such that for all $\kappa < \omega$ ${}_\nu[(g_v^\kappa - v_\alpha^\kappa)] = \emptyset$. Then there exists a branch $v \in Br(Im\ f_\alpha)$ such that $g_\beta^o\varphi \notin G_{\alpha+1}(v)$ for all strong ordinals $\beta < \alpha$, where $G_{\alpha+1}(v) =\ll G_\alpha, g_{v,\kappa}, \kappa < \omega\text{g}$.*

Proof. Suppose the conclusion is false. Then for each branch $v \in Br(Im\ f_\alpha)$ there exists an ordinal $\beta = \beta(v) < \alpha$, $m = m(\beta)$, a homomorphism $\widehat{\varphi}_\beta$ and $g_\beta^m\widehat{\varphi}_\beta \in G_{\alpha+1}(v)$. By definition of $G_{\alpha+1}$ there are elements $a = a_v \in A$, $\kappa = \kappa(v)$ with $b_\beta - g_v^\kappa a \in G_\alpha$ and it follows that $g_\alpha^\kappa a \neq 0$. For large enough ν we get ${}_\nu[g_v^\kappa a] = {}_\nu[v^\kappa a]$ which is empty or an infinite subset of v by (3.1). If ${}_\nu[v^\kappa a]$ is empty, then $a \in p^{\kappa+1}A$ and $g_v^\kappa a = 0$ by (3.2) which contradicts $g_v^\kappa a \neq 0$. Therefore ${}_\nu[g_v^\kappa a]$ is an infinite subset of v. The branches $v_\gamma(\gamma < \alpha)$ have norm at most $\| P_\alpha \|=\| v \|$ and thus are different from v. We derive $(b_\beta - g_\alpha^\kappa a)\pi_\sigma = 0$ for almost all $\sigma \in v$ and $b_\beta\pi_\sigma = g_\alpha^\kappa\pi_\sigma \neq 0$ for infinitely many $\sigma \in v$ from (3.1). Thus an infinite subset of v lies in $[b_\beta] \subset [P_\beta]$ and since $[P_\beta]$ is a subtree of T, the branch v is contained in $[P_\beta]$. Hence $v \in Br(Im\ f_\alpha) \cap [P_\beta])$. This implies that $\beta < \alpha < \beta + \kappa^{\aleph_0}$ by the Black Box. Thus we have proved that for each $v \in Br(Im\ f_\alpha)$ there exists an ordinal $\beta(v)$ and an $a_v \in A$ such that $\beta(v) < \alpha < \beta(v) + \kappa^{\aleph_0}$ and $b_{\beta(v)} - g_v^{\kappa(v)} a_v \in G_\alpha$. If β_0 is the least ordinal such that $\beta_0 < \alpha < \beta_0 + \kappa^{\aleph_0}$, then $\beta_0 \leq \beta(v) < \alpha < \beta_0 + \kappa^{\aleph_0}$ and $\beta(v)$ assumes less than $\kappa^{\aleph_0} = |Br(Im\ f_\alpha)|$ values. There are two different branches $v, w \in Br(Im\ f_\alpha)$ such that $\beta(v) = \beta(w)$. Subtracting the corresponding equations, we get $g_w^{\kappa(w)} - g_v^{\kappa(v)} \in G_\alpha$. Arguing as before we conclude that an infinite subset of v is contained in ${}_\nu[g_w^{\kappa(w)} a_w] \subset w$. But this is impossible because v and w are different.

The constructed abelian group G has further properties collected in a

Lemma 3.6. *G is a reduced abelian p-group containing G_α as an isotype subgroup ($\alpha < < \lambda^*$). Moreover $p^\omega G_\alpha = p^\omega B$ for all $\alpha < \lambda^*$.*

Proof. First we will show that G_α is isotype in $G_{\alpha+1}$ and assume that

$$G_\alpha \neq G_{\alpha+1} = \ll G_\alpha, g_{\alpha,\kappa}, \kappa < \omega \mathrm{g},$$

and $g_{\alpha,\kappa} = v_{\alpha,\kappa} + b_\kappa \in B[p^{\kappa+1}]$ for some $b_\kappa \in B$ and some branch $v_\alpha \in Br(Im\ f_\alpha)$ such that $\| v_\alpha \| > \sup_{\kappa<\omega} \| b_\kappa \|$. We apply Lemma 2.8 with $H' = G_{\alpha+1}$, $H = G_\alpha$, $v = v_\alpha$ and $x_\kappa = g_{\alpha,\kappa}$. The assumtions in (2.8 are either obvious or follow from (3.1). Thus G_α is isotype in $G_{\alpha+1}$ and G_α is also isotype in G by induction.

We will now show $p^\omega G_\alpha = p^\omega B$ by induction on α. The statement is trivial for $\alpha = = 0$, and we assume $p^\omega G_\alpha = p^\omega B$. We have shown above that G_α and $G_{\alpha+1}$ satisfy the assumptions (2.8), hence $p^\omega G_{\alpha+1} = p^\omega G_\alpha$ which is $p^\omega B$ by induction hypothesis. If α is a limit-ordinal, let $x \in p^\omega G_\alpha \subseteq \bigcup_{\rho<\alpha} G_\rho$. Then $x \in p^\omega G_\alpha \cap G_\rho$ for some $\rho < \alpha$. Since G_ρ is isotype in G_α, also $x \in p^\omega G_\rho$. It follows $p^\omega G_\alpha \subset p^\omega B$. The reverse inclusion is trivial.

If D is a divisible subgroup of G, then $D \subset p^\omega G = p^\omega B$ from above. B is reduced, hence $D = 0$ and G is reduced as well.

Lemma 3.7. *Let $G' = \ll G, x_\kappa, \kappa < \omega \mathrm{g}$, $\varphi \in End\ G$ and $\varphi_i : G' \to Im\ \varphi_i \subset X$ two extensions $\varphi_i \supset \varphi$, for $i = 1,2$ and some group X. If X is reduced, then $\varphi_1 = \varphi_2$.*

Proof. Let (x^κ) be the chain defined by x_κ. Then we have $px^{\kappa+1}\varphi_i = x^\kappa\varphi_i - x_\kappa\varphi$, for $i = 1,2$. Setting $\mathrm{d}^\kappa := x^\kappa(\varphi_1 - \varphi_2)$, we get

$$p\mathrm{d}^{\kappa+1} = px^{\kappa+1}(\varphi_1 - \varphi_2) = x^\kappa(\varphi_1 - \varphi_2) - x_\kappa(\varphi - \varphi) = x^\kappa(\varphi_1 - \varphi_2) = \mathrm{d}^\kappa \in X,$$

for all $\kappa < \omega$. But $< \mathrm{d}^\kappa, \kappa < \omega > \cong Z(p^\infty)$, if $\mathrm{d}^\kappa \neq 0$ for infinitely many κ. Since X is reduced, this forces $\mathrm{d}^\kappa = 0$ for almost all $\kappa < \omega$, hence $\mathrm{d}^\kappa = 0$ for all $\kappa, x^\kappa\varphi_1 = x^\kappa\varphi_2$, for all $\kappa < \omega$ and $\varphi_1 = \varphi_2$ follows.

Next we observe that small endomorphisms of G can be recognized on B.

Lemma 3.8. *If $\varphi \in End\ G$ then $\varphi\lceil B$ is small if and only if φ is small.*

Proof. We have to show that $\varphi \in$ End G is small if the restriction $\varphi\lceil B$ is small. For any $\kappa < \omega$ there exists $\kappa^* < \omega$ such that $p^{\kappa^*}B[p^\kappa]\varphi = 0$. We may assume $\kappa^* < (\kappa + 1)^*$ and want to show $p^{\kappa^*}G[p^\kappa]\varphi = 0$.

Using the Recognition Lemma 3.1, any $g \in p^{\kappa^*}G[p^\kappa]$ can be expressed as $g = b+ + \sum_{i=0}^m a_i x_i^{\kappa_i}$ with $x_i \in G$ coming from a branch v_{α_i} in the construction of G_{α_i+1} and $b \in B$.

We can write $g = b' + p^{\kappa*}\sum_{i=0}^{m} a_i x_i^{\kappa_i+\kappa*}$ for some $b' \in B$ and $b' \in p^{\kappa*}B[p^\kappa]$, $p^{\kappa*}a_i x_i^{\kappa_i+\kappa*} \in$ $\in p^{\kappa*}G[p^\kappa]$. Hence $b'\varphi = 0$ and it is sufficient to show $ax^m\varphi = 0$ for $ax^m \in p^{\kappa*}G[p^\kappa]$. This will follow from

(*) If $ax^m \in p^{\kappa*}G[p^\kappa]$, we find $z = rax^n \in p^{(\kappa+1)*}G[p^{\kappa+1}]$ such that $(pz)\varphi = (ax^m)\varphi$.

Since (x^κ) is a chain, we can write $ax^m = u + v$ with $u = \sum_{n=0}^{(\kappa+1)*} p^n ax_{n+m}$ and $v =$ $= p^{(\kappa+1)*+1}ax^{(\kappa+1)*+1+m}$.

This yields $u \in p^{\kappa*}B[p^\kappa]$ and if $z = p^{(\kappa+1)*}ax^{(\kappa+1)*+1+m}$, then $ax^m\varphi = v\varphi = pz\varphi$ and (*) holds.

If $z_0 = ax^m \in p^{\kappa*}G[p^\kappa]$, then we find $z_1 \in p^{(\kappa+1)*}G[p^{\kappa+1}]$ such that $(pz_1)\varphi = z_0\varphi$ by (*). Inductively we obtain elements z_i such that $z_i \in p^{(i+\kappa)*}G[p^{i+\kappa}]$ and $p(z_{i+1})\varphi = z_i\varphi \in G$. If $z_i\varphi \neq 0$, then $< z_i\varphi, i < \omega > \cong Z(p^\infty)$ is a subgroup of G which is impossible. We derive $z_o\varphi = ax^m\varphi = 0$ which completes the proof.

Some easy calculations show

Lemma 3.9. *If φ is a small endomorphism of G and G' is a rk-1 extension of G, then φ extends to a homomorphism of G' into G.*

The proof of the converse is more complicated, but it follows by standard arguments for separable abelian p-groups [7].

Lemma 3.10. *If $\varphi \in End\ G \setminus Small\ G$, then there are $x_\kappa \in B[p^{\kappa+1}] (\kappa < \omega)$ and $\kappa^* < \omega$ such that the following holds:*
$x^{\kappa*}\widehat{\varphi} \notin G$ for all extensions $\widehat{\varphi} \supset \varphi$ with $G' = \ll G, x_\kappa, \kappa < \omega \gg \subset dom\ \widehat{\varphi}$.

Proof. Since φ is not small, by Lemma 3.8 there is a minimal d $< \omega$ such that $p^\kappa B[p^d]\varphi \neq 0$ for all $\kappa < \omega$. We find $e_\kappa \in p^\kappa B[p^d]$ such that $0 \neq e_\kappa\varphi =: h_\kappa \in p^\kappa B[p]$. We may assume that all e_κ have the same order p^d and choose $x_\kappa \in B$ such that $p^{\kappa+1-d}x_\kappa = e_\kappa$, hence $o(x_\kappa) = p^{\kappa+1}$ and $p^{\kappa+1-d}x_\kappa\varphi = e_\kappa\varphi \neq 0$.

Passing to a subsequence of ω, we may assume

(*) $h_G(y_\kappa)$ is strictly increasing for $y_\kappa = p^{\kappa-t}x_\kappa\varphi \in G$ and some $t \geq$ d.

We will distinguish two cases:

Case 1: $y_\kappa \notin H_{-1}$, for infinitely many $\kappa < \omega$ and

Case 2: $y_\kappa \in H_{-1}$, for almost all $\kappa < \omega$.

In case 1 we may assume that $y_\kappa \notin H_{-1}$, for all $\kappa < \omega$. Again, passing to subsequences, we can choose elements $\sigma_\kappa \in [y_\kappa] \setminus \{-1\}$ such that $\sup_{\kappa<\omega} \| \sigma_\kappa \| = \sup_{\kappa<\omega} \| y_\kappa \|$. Similarly, the sequences $(\| y_\kappa \|)_{\kappa<\omega}$ and $(\| \sigma_\kappa \|)_{\kappa<\omega}$ are non-decreasing and

(1) $y_{\kappa+1} \in p^{l(\sigma_\kappa)}G$,

(2) if infinitely many of the $\sigma_\kappa' s$ lie in one single branch, then all do so.

Note, that by (1) each element of $[y_{\kappa+1}]$ has a height larger than $l(\sigma_\kappa)$. Therefore $y_{\kappa+1}\pi_\rho = 0$ for every element $\rho \in T$ with $l(\rho) < l(\sigma_\kappa)$. In particular $l(\sigma_{n+1}) > l(\sigma_n)$ and $\sigma_{n+1} \neq \sigma_n$ for all $n < \omega$. If $\epsilon_\kappa \in \{0,1\}$ and $q > l(\sigma_n) \leq t$, then

(3) $z_n := (\sum_{\kappa=1}^{q} \epsilon_\kappa y_\kappa)\pi_{\sigma_n} = \sum_{\kappa=1}^{l(\sigma_n)} \epsilon_\kappa y_\kappa)\pi_{\sigma_n}$.

Thus z_n only depends on ϵ_κ for $\kappa \leq l(\sigma_n)$. Furthermore $z_n \in G[p^m]$, where $m = t - \mathrm{d} + 1$ and

(4) $z_n = \sigma_n c_n$, with $c_n \in p^{l(\sigma_n)-m}A$.

Suppose $\epsilon_1, \ldots, \epsilon_{n-1}$ are constructed. As z_n and c_n only depend on the ϵ_κ for $\kappa \leq l(\sigma_n)$, we can find an ϵ_n such that $z_n\pi_{\sigma_n}$ is different from any prescribed element of $\sigma_n A$. We choose ϵ_n such that

($\dagger$) $z_n = p^{l(\sigma_n)-m}c_n \neq \sigma_n p^{l(\sigma_n)-m}c_{n-1}$, if n is odd and $z_n \neq 0$, if n is even.

Define $G' := \ll G, s_\kappa : \kappa < \omega \gg$, where $s_\kappa := \epsilon_\kappa x_\kappa$ and let (s^κ) be the chain defined by s_κ. Suppose there is an extension $\widehat{\varphi} \supset \varphi$ with $G' \subset dom\ \widehat{\varphi}$ such that $s^t\widehat{\varphi} \in G$, then $(s^\kappa\widehat{\varphi})\pi_\rho = \sum_{\kappa=1}^{l(\sigma)} p^{\kappa-t}\epsilon_\kappa x_\kappa \varphi\pi_\rho = \sum_{\kappa=1}^{l(\sigma)} \epsilon_\kappa y_\kappa \varphi\pi_\rho = z_n\pi_\rho$ for some $\sigma = \sigma_n$. By the choice of $\epsilon_\kappa(\kappa < \omega)$ we have $\sigma_n \in [s^t\widehat{\varphi}]$ for even n. Moreover $\sup_{n<\omega} \| \sigma_n \| = \| s^t\widehat{\varphi} \|$. The Recognition Lemma 3.1 and (2) force that there are an $a \in A, \kappa < \omega$ such that $(s^t\widehat{\varphi})\pi_{\sigma_n} = \sigma_n p^{l(\sigma_n)-\kappa}a$ for all large enough $n < \omega$. Because of $o(s^t\widehat{\varphi}) \leq p^m$ we can assume that $\kappa = m$ and (4) implies $a - c_n \in p^{m+1}A$ for all large enough n. Finally we conclude

$(s^t\widehat{\varphi})\pi_{\sigma_n} = \sigma_n p^{l(\sigma_n)-m}a_n = \sigma_n p^{l(\sigma_n)-m}a = \sigma_n p^{l(\sigma_n)-m}a_{n-1}$

which contrdicts ($\dagger$) for all large odd n.

In case 2 we may assume $y_\kappa \in H_{-1}$ for all $\kappa < \omega$ and (*) and an easy analysis of generalized Prüfer groups (2.1) show that some subsequence $y_\kappa(\kappa \in I \subset \omega)$ generates a direct sum $\bigoplus_{\kappa\in I} y_\kappa A$ in H_{-1}. We replace I and ω and choose a rk-1 extension $G' = \ll G, x_\kappa, \kappa < \omega \gg$ of G accordingly.

If $\widehat{\varphi}$ is an extension of φ such that $x^t\widehat{\varphi} \in G$, then we obtain

(5) $x^t\widehat{\varphi} = \sum_{\kappa=t}^{n} p^{\kappa-t}x_\kappa\varphi + p^{n+1}x^{n+1}\widehat{\varphi} = \sum_{\kappa=t}^{n} y_\kappa\varphi + p^{n+1}x^{n+1}\widehat{\varphi}$.

This implies $[x^t\widehat{\varphi}] \subset [y_\kappa\widehat{\varphi}] \subset \{-1\}$, $x^t\widehat{\varphi} \in H_{-1}$ and only a finite number of y_κ contribute to $x^t\widehat{\varphi}$. On the other hand, an easy argument shows $h(p^{\kappa+1-t}x^{\kappa+1}\widehat{\varphi}) > h(p^{n-t}x_n\varphi)$ $\forall\, n \leq \kappa$, and an infinite number of y_κ must contribute to $x^t\widehat{\varphi}$, a contradiction.

The final proof follows [7], see [1] for details. Here are the main steps.

Lemma 3.11. *Let w be a constant branch and (w^κ) the chain defined by w. Then the following holds:*

$$w^\kappa a \in G \Longleftrightarrow a \in p^{\kappa+1}A$$

Corollary 3.12. *$A\cap$ Small $G = 0$*

Lemma 3.13. *Let $\varphi \in End\ G$, w a constant branch, $G' = \ll G, w_\kappa, \kappa < \omega \gg$ and (w^κ) the chain defined by w. If $\widehat{\varphi} \supset \varphi : G' \to G'$, then there is an $a \in A$ such that*

$$w^\kappa(\widehat{\varphi} - a) \in G \text{ for all } \kappa < \omega.$$

Lemma 3.14. *If $\varphi \in End\ G \setminus A\oplus$ Small G, then there are $x_\kappa \in B[p^{\kappa+1}](\kappa < \omega)$ and a chain (x^κ) defined by x_κ such that*
(*) $\forall\widehat{\varphi} \supset \varphi \exists n < \omega$ with dom $\widehat{\varphi} \supset G'$, where $G' = \ll G, x_\kappa \kappa < \omega \gg$.

Finally, as in [7] we can see that G satisfies End $G = A\oplus$ *Small* G.

Proof. Assume there is a $\varphi \in End\ G \setminus A\oplus$ *Small* G. By (3.14) there is a chain (x^κ) defined by $x_\kappa \in B[p^{\kappa+1}]$ such that $G' := \ll G, x_\kappa \kappa < \omega \gg$ is reduced and $\forall\widehat{\varphi} \supset \varphi$ such that dom $\widehat{\varphi} \supset G'$ there is $n = n(\widehat{\varphi})$ with $x^n\widehat{\varphi} \notin G'$. The Black Box yields an $\alpha < \lambda^*$ and a trap $(f_\alpha, P_\alpha, \varphi_\alpha)$ such that $x_\kappa, x_\kappa\varphi \in P_\alpha \forall\kappa < \omega$, $\sup\{\| x^\kappa \|, \| x^\kappa\varphi \|, \kappa < \omega\} < \| P_\alpha \|$ and $\varphi_\alpha = \varphi\lceil P_\alpha$.

According to the construction of G and Lemma 3.5 there is a chain (g_α^κ) such that α is either weak or strong. We want to show that α is strong.

If $v \in Br(Im\ f_\alpha)$, then the following holds (see [7] and use the lemmata above , or see [1]) $\exists\epsilon \in \{0, 1\}\forall\widehat{\varphi} \supset \varphi\exists n = n(\widehat{\varphi})$ such that $(v^n + \epsilon x^n)\widehat{\varphi} \notin \ll G_\alpha, v_\kappa + \epsilon x_\kappa, \kappa < \omega \gg$.

This shows that α is strong, $g_\alpha^\kappa\varphi \notin G$ and $\varphi \notin End\ G$, a final contradiction, which completes the proof our Theorem. ∎

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