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Subgroup embedding properties and the structure of finite groups

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Abstract. Our main aim in this paper is to present some results to help us better understand some different ways a subgroup can be embedded in a finite group and their impact on the group structure.

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1 Introduction

Our main aim in this paper is to present some results to help us better understand some different ways a subgroup can be embedded in a finite group and their impact on the group structure.

The following definition turns out to be central in our study.

Definition 1.1. A *subgroup embedding property* is a map f which associates with each group G (in some fixed universe) a subset $f(G)$ of $\mathcal{S}(G)$, the set of all subgroups of G, and satisfies

$$\alpha\big(f(G)\big) = f\big(\alpha(G)\big) \qquad (*)$$

for all group isomorphisms $\alpha\colon G \longrightarrow \alpha(G)$.

This definition is very general and represents the minimum requirement that a subgroup embedding property should be an invariant of each isomorphism class of groups.

Normal, subnormal or pronormal subgroups are typical examples of embedding properties of subgroups which are important in investigations of groups with a rich subgroup structure.

Most useful embedding properties of subgroups satisfy additional conditions which are useful in proofs using induction arguments in the universe of all finite groups. We collect them in the following definition.

Definition 1.2. Let f be a subgroup embedding property.

- We say that f is *quotient-invariant* if the equation (*) holds for all epimorphisms $\alpha\colon G \longrightarrow \alpha(G)$.
- We say that f is *subgroup-invariant* if, for all $S \leqslant G$, $f(S) = \{\, S \cap H \mid H \in f(G) \,\}$.
- We say that f is *persistent* if, for all G and all $S \leqslant G$, we have $H \in f(S)$ whenever $H \leqslant S$ and $H \in f(G)$.

In the sequel, we analyse three different subgroup embedding properties in the universe of all finite groups. Therefore the unspoken rule is that *all groups are finite.*

2 Supplements of normal subgroups

Our attention in this section is confined to study an embedding property of subgroups which has a strong influence in the study of the structure of soluble groups and it is defined as follows:

Let G be a group. We write:

$$f(G) = \{\, H \leqslant G \mid G = H\mathrm{F}(G) \,\}.$$

Here $\mathrm{F}(G)$ is the Fitting subgroup of G, that is, the subgroup generated by all nilpotent normal subgroups of G. Clearly f is a subgroup embedding property which is persistent but it is not quotient-invariant. However, it satisfies

$$\alpha\big(f(G)\big) \subseteq f\big(\alpha(G)\big)$$

for all group epimorphisms α.

Note that if G is soluble, then $\Phi(G)$, the Frattini subgroup of G, is a proper subgroup of $\mathrm{F}(G)$ ([7, A, 10.6]). Therefore there exists a maximal subgroup M of G such that $G = M\,\mathrm{F}(G)$, that is, $M \in f(G)$. However, if G is a Frattini extension of a non-abelian simple group, then $\Phi(G) = \mathrm{F}(G)$ and so $f(G) = \{G\}$.

More generally, for a normal nilpotent subgroup Q of a group G, we can define

$$f_Q(G) = \{\, H \leqslant G \mid G = HQ \,\}$$

It is clear that $f_Q(G) \subseteq f(G)$ and it satisfies:

(1) if $H \in f_Q(G)$ and $H \leqslant S$, then $H \in f_{S\cap Q}(S)$. More generally, if X is a subgroup of G, then $X \cap f_Q(G)$ is contained in $f_{X\cap Q}(X)$.

(2) if $N \trianglelefteq G$, then $f_Q(G)N/N \subseteq f_{QN/N}(G/N)$

In the following we give some significant properties of f.

Recall that a *formation* is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images and subdirect products. In particular, if $\mathfrak{F}$ is non-empty, every group G has a smallest normal subgroup with quotient in $\mathfrak{F}$ called the $\mathfrak{F}$*-residual* of G and denoted by $G^{\mathfrak{F}}$.

(1) (Bryant, Bryce, and Hartley [7, IV, 1.14]) *Every subgroup in $f(G)$ belongs to the formation generated by G.*

(2) ([7, IV, 1.17(b)]) *If $\mathfrak{F}$ is a formation, then $U^{\mathfrak{F}}$ is contained in $G^{\mathfrak{F}}$ for all $U \in f(G)$.*

   As a consequence, every formation composed of nilpotent groups is closed under taking subgroups, that is, it is a variety.

   Recall that a formation $\mathfrak{F}$ is *saturated* if it is closed under taking Frattini extensions.

(3) ([7, IV, 1.17(b)]) *If G is soluble and $\mathfrak{F}$ is a saturated formation such that $G \notin \mathfrak{F}$, there exists a maximal subgroup $M \in f(G)$ such that $G/M_G \notin \mathfrak{F}$.*

   This property allows Carter and Hawkes to define $\mathfrak{F}$-normalisers in every soluble group as an extension of Hall's system normalisers (see [7, V, Section 3]).

(4) *Every subgroup D in $f(G)$ has the cover and avoidance property in G.*

   Therefore the intersection of D with a chief series of G is a chief series of D and the automorphism groups induced on the corresponding chief factors are isomorphic.

As to whether some subgroups in $f(G)$ are G-conjugate has been an important theme in group theory. In fact, fundamental results on the theory of Schunck classes and projectors of soluble groups depend on the conjugacy of some elements of $f(G)$.

Given a class of groups $\mathfrak{X}$, a subgroup H of a group G is $\mathfrak{X}$*-maximal* in G if

(1) $H \in \mathfrak{X}$ and

(2) if $H \leqslant L \leqslant G$ and $L \in \mathfrak{X}$, then $H = L$.

A subgroup H of a group G is said to be an $\mathfrak{X}$*-projector* of G if HN/N is $\mathfrak{X}$-maximal in G/N for all normal subgroups N of G.

Denote by $\mathrm{Proj}_{\mathfrak{X}}$ the subgroup embedding property associating with each group G the set of all $\mathfrak{X}$-projectors of G.

If $\mathfrak{X} = \mathfrak{S}_p$, with p a prime, $\mathrm{Proj}_{\mathfrak{X}}(G) = \mathrm{Syl}_p(G)$. More generally, if $\mathfrak{X} = \mathfrak{E}_\pi$, with π a set of primes, $\mathrm{Proj}_{\mathfrak{X}}(G) = \mathrm{Hall}_\pi(G)$ for all π-separable groups G. A classical result of Carter shows that the nilpotent self-normalising subgroups of a soluble group are exactly the projectors for the class $\mathfrak{N}$ of all nilpotent groups ([7, III, 4.6]).

A class of groups $\mathfrak{H}$ is a *Schunck class* if $\mathfrak{H}$ is closed under taking epimorphic images and a group G belongs to $\mathfrak{H}$ if and only if every primitive epimorphic image of G belongs to $\mathfrak{H}$.

The following theorem was proved by Gaschütz and Schunck in the soluble case, and it is a consequence of Förster's results in the general case ([7, III, Section 3]).

Theorem 2.1. *Let $\mathfrak{X}$ be a class of groups. Then* $\mathrm{Proj}_{\mathfrak{X}}(G) \neq \emptyset$ *for all groups G if and only if $\mathfrak{X}$ is a Schunck class. Moreover, if G is soluble,* $\mathrm{Proj}_{\mathfrak{X}}(G)$ *is a conjugacy class of subgroups of G. In particular,* $\mathrm{Proj}_{\mathfrak{X}}$ *is a persistent* Q*-invariant subgroup embedding property in the soluble universe.*

The conjugacy of projectors associated to Schunck classes in the soluble universe depends heavily on the following lemma due to Gaschütz (see [7, III, 3.14]).

Lemma 2.2 (Gaschütz, [8])**.** *Let $\mathfrak{H}$ be a Schunck class and let Q be a nilpotent normal subgroup of G. If H is an $\mathfrak{H}$-maximal subgroup in $f_Q(G)$, then $H \in \mathrm{Proj}_{\mathfrak{H}}(G)$.*

More recently, Parker and Rowley [14] proved the following result:

Theorem 2.3. *Let G be a soluble group and Q a nilpotent normal subgroup of G such that no G-chief factor of G/Q is G-isomorphic to a G-chief factor of Q. If U, $V \in f_Q(G)$ and $U \cap Q = V \cap Q$, then U and V are G-conjugate.*

The authors claimed that this result arose during investigations into 2-minimal subgroups of classical groups. In fact, they describe a typical situation in which the above theorem applies. Let X be the wreath product

$$3 \wr \underbrace{2 \wr \cdots \wr 2}_{a} \wr \mathrm{Sym}(4) \wr \underbrace{2 \wr \cdots \wr 2}_{b}$$

of order $2^{2^{a+b-2}-1} 3^{2^{a+b+2}+2^b}$. Let Q be the base group of this wreath product. Hence Q has order $3^{2^{a+b+2}}$. The X-chief factors in Q have orders 3, $3, \ldots,$ $3^{2^{b-1}}$, $3^{2^b+2^{b+1}}$, $3^{2^{b+2}}, \ldots,$ $3^{2^{a+b+1}}$, whereas the X-chief factors in X/Q are all 2-groups except for a single X-chief factor of order 3^{2^b}. Thus X satisfies the hypothesis of Theorem 2.3. Hence any two subgroups complementing Q are X-conjugate.

Theorem 2.3 can be interpreted in terms of Schunck classes and projectors and it can be deduced directly from Gaschütz's lemma and conjugacy of projectors of soluble groups.

Proof of Theorem 2.3. Consider the Schunck class $\mathfrak{H}$ of all soluble groups whose primitive epimorphic images belong to the class of all primitive epimorphic images of G/Q. If we argue by minimal counterexample, then $G = \langle U, V^g \rangle$, for all $g \in G$, and $U \cap Q = V \cap Q = 1$. Gaschütz's lemma implies that U and V are contained in $\mathfrak{H}$-projectors U^* and V^* of G respectively. Hence $G \in \mathfrak{H}$, a contradiction proving the result. QED

In the following we shall show that it is possible to go much further in the conjugacy problem for elements in $f(G)$.

If we turn the situation on its head and look for stuctural conditions on a normal subgroup of a group having a conjugacy class of supplements composed of maximal subgroups we have:

Theorem 2.4 (Ballester-Bolinches, Ezquerro, [4])**.** *Suppose that G is a group and Q is a normal subgroup of G such that any two maximal subgroups of G supplementing to Q in G are G-conjugate. Then Q is a soluble group of nilpotent length at most* 2*.*

The bound of the previous theorem is best possible as the following example shows:

Example 2.5. Consider the group $X = \mathrm{SL}(2,3)$ acting on a 2-dimensional vector space V over the Galois field GF(3). Construct the semidirect product $G = [V]X$. If $Z = \mathrm{Z}(X)$, the centre of X, then $Q = ZV$ is a supersoluble non-nilpotent normal subgroup of G. The set of maximal subgroups supplementing Q in G is the conjugacy class of all core-free maximal subgroups of G complementing V.

We present now some results which can be viewed as partial converses of the above theorem in the case when Q is a normal nilpotent subgroup of G. They are, therefore, results providing sufficient conditions to ensure conjugacy of subgroups in $f_Q(G)$.

The following example shows that imposing some conditions on the intersections such as local conjugacy seems quite reasonable.

Example 2.6. Let $G = \langle a, b, x \colon a^3 = b^3 = x^2 = 1 = [a,b], a^x = a^{-1}, b^x = b^{-1} \rangle$. If $Q = \langle a, b \rangle \cong C_3 \times C_3$, then the subgroups $U = \langle a, x \rangle$ and $V = \langle b, x \rangle$ are two supplements to Q in G which are not G-conjugate. In this case $U \cap Q$ and $V \cap Q$ are two different normal subgroups of G.

Definition 2.7. Two subgroups A and B of a group G are *locally G-conjugate* if every Sylow subgroup of A is G-conjugate to a Sylow subgroup of B.

Our next theorem describes a minimal configuration encountered in the study of conjugacy of supplements of normal nilpotent subgroups of soluble groups, from which sufficient conditions and counterexamples emerge.

Theorem 2.8 (Ballester-Bolinches, Ezquerro, [4]). *Let $\mathfrak{X}$ be a Q-closed class of groups, and*

$$\mathfrak{F} = (G\colon G/M \in \mathfrak{X} \text{ for some nilpotent normal subgroup } M \text{ of } G).$$

Let G be a soluble group of minimal order in $\mathfrak{F}$ among the groups satisfying the following property:

> (†) *there exists a nilpotent normal subgroup Q of G and non-G-conjugate elements U and V in $f_Q(G)$ such that $U \cap Q$ is locally G-conjugate to $V \cap Q$.*

Then G is a p-group for some prime p.

The above theorem allows us to obtain a number of results on conjugacy of supplements of nilpotent normal subgroups of soluble groups, all of them proved in [4]. They allows us to confirm that local conjugacy is a good subgroup embedding property to study the conjugacy problem for subgroups in $f(G)$. The first one is an extension of Theorem 2.3.

Corollary 2.9. *Let G be a soluble group and Q a nilpotent normal subgroup of G such that no G-chief factor of G/Q is G-isomorphic to a G-chief factor of Q.*

If U, $V \in f_Q(G)$ such that $U \cap Q$ and $V \cap Q$ are locally G-conjugate, then U and V are G-conjugate.

An advantage of Parker and Rowley's procedure in Theorem 2.3 is that the condition $U \cap Q = V \cap Q$ holds in subgroups containing both U and V. This is not longer true in the case of local conjugacy.

We shall show now by an example that no statement of similar kind is possible if we remove the hypothesis on the chief factors.

Example 2.10. Let Q be a group isomorphic to the quaternion group of order 8. Consider a subgroup T of $\mathrm{Aut}(Q)$ isomorphic to S_3. Write $T = \langle b, c\colon b^3 = c^2 = 1, b^c = b^{-1}\rangle$. Set $B = \langle b\rangle$ and $C = \langle c\rangle$. Construct the semidirect product

$G = [Q]T$. Write $Z = \mathrm{Z}(Q) = \langle z \rangle$. Note that G/QB is a complemented central 2-chief factor of G over Q and $Z/1$ is also a central G-chief factor of G below Q. Consider the subgroups $U = \langle b, c \rangle$ and $V = \langle b, zc \rangle$. Then U and V are two non-conjugate supplements to Q in G such that $U \cap Q = 1 = V \cap Q$.

Corollary 2.11. *Suppose that G is a soluble group and Q is a nilpotent normal subgroup of G. If $U, V \in f_Q(G)$ such that U and V are locally G-conjugate, then U and V are G-conjugate.*

Let $\mathbb{F}$ be a saturated formation. If G is a group and $G \notin \mathbb{F}$, then the $\mathbb{F}$-residual $G^{\mathbb{F}}$ of G is a non-trivial normal subgroup of G which is supplemented in G by every $\mathbb{F}$-projector of G.

Corollary 2.12. *Let $\mathbb{F}$ be a saturated formation and let G be a soluble group whose $\mathbb{F}$-residual $G^{\mathbb{F}}$ is nilpotent. Then any two supplements U and V of $G^{\mathbb{F}}$ in G are G-conjugate provided $U \cap G^{\mathbb{F}}$ and $V \cap G^{\mathbb{F}}$ are locally G-conjugate.*

The case when $G^{\mathbb{F}}$ is abelian is particularly interesting. In this case, $G^{\mathbb{F}}$ is complemented in G and its complements form a conjugacy class of subgroups of G ([7, IV, 5.18]). In this case, Corollary 2.12 is equivalent to the fact that the complements of $G^{\mathbb{F}}$ are conjugate in G.

Corollary 2.13. *Let $\mathbb{F}$ be a saturated formation and let G be a soluble group whose $\mathbb{F}$-residual $G^{\mathbb{F}}$ is abelian. The following conditions are equivalent:*

(1) Any two supplements U and V of $G^{\mathbb{F}}$ in G are G-conjugate provided $U \cap G^{\mathbb{F}}$ and $V \cap G^{\mathbb{F}}$ are locally G-conjugate.

(2) Any two complements U and V of $G^{\mathbb{F}}$ in G are G-conjugate.

3 Subgroups of hypercentral type

The focus of this section relates to the influence of minimal subgroups, i. e. subgroups of prime order, on the structure of a group. It is known that the embedding of minimal subgroups of a group often gives a good insight into the group structure: a theorem of Itô about the p-nilpotence of a group in which the subgroups of order p or order 4 if $p = 2$ are central is a good example.

A typical situation one can find in this context is the following: Let f be a persistent subgroup embedding property. Suppose we would like to prove a result of the following type:

> *A group G belongs to a class $\mathfrak{X}$ provided that the minimal subgroups of G belong to $f(G)$.*

Through the approach of a minimal counterexample, $G \notin \mathfrak{X}$ and every proper subgroup of G belongs to $\mathfrak{X}$, that is, G is an $\mathfrak{X}$-critical group. Hence if we wanted to prove a result of the above type, we would need to have a good structural knowledge of the groups in such minimal classes. Therefore it is convenient to settle the following definition.

Definition 3.1. Let $\mathfrak{X}$ be a class of groups. A group G is said to be $\mathfrak{X}$-*critical* (or *critical for* $\mathfrak{X}$) if $G \notin \mathfrak{X}$, but all proper subgroups of G belong to $\mathfrak{X}$.

It is clear that a detailed knowledge of the $\mathfrak{X}$-critical groups is likely to give some insight into just what makes a group to belong to $\mathfrak{X}$.

One of the most popular critical groups are the ones associated to the class of all nilpotent groups. These groups were investigated by Schmidt in 1924 and so they are usually called *Schmidt groups*. By a result of Itô, every critical group for the class of all p-nilpotent groups, p a prime, is a Schmidt group. The structure of the $\mathfrak{N}$-critical groups is very restricted as the following theorem shows.

Theorem 3.2 (Schmidt, [15]). *(1) If every proper subgroup of a group G is nilpotent, then G is soluble.*

(2) Assume that every proper subgroup of G is nilpotent, but G is not nilpotent. Then G satisfies:

a.
- $|G| = p^a q^b$ *for prime numbers* $p \neq q$,
- *the Sylow p-subgroup is normal in G,*
- *the Sylow q-subgroups are cyclic, and*
- *for every Sylow q-subgroup Q of G,* $\Phi(Q) \leqslant \mathrm{Z}(G)$.

b. The nilpotency class of the Sylow p-subgroup P of G is at most two. Moreover, $\Phi(P) \leqslant \mathrm{Z}(G)$.

c.
- *For $p > 2$, P has exponent p;*
- *for $p = 2$, the exponent of P is at most* 4.

In the sequel, we shall discuss a couple of recent results in which the above method of proof could be applied. To this end, we need the following notation.

Let P be a p-group. If k is a natural number we denote

$$\Omega_k(P) = \langle x \in P \colon x^{p^k} = 1 \rangle, \quad \text{and} \quad \Omega(P) = \begin{cases} \Omega_1(P) & \text{if } p \text{ is odd}, \\ \Omega_2(P) & \text{if } p = 2. \end{cases}$$

We consider the subgroup embedding property f defined by:

$$f(G) = \{ H \leqslant G \mid H \leqslant \mathrm{Z}_\infty(G) \}.$$

Here $\mathrm{Z}_\infty(G)$ is the hypercentre of G, that is, the last term of the ascending central series of the group G.

It follows that f is a persistent subgroup embedding property.

Let p be a prime and let $\mathfrak{X}$ be the saturated formation of all p-nilpotent groups. Assume that G is a group such that $\Omega(P) \subseteq f(G)$ for some $P \in \mathrm{Syl}_p(G)$. Then $\mathrm{O}^p(G) \leqslant \mathrm{Cent}_G\bigl(\Omega(P)\bigr)$ and therefore G cannot be an $\mathfrak{X}$-critical group by Theorem 3.2. Therefore G should belong to $\mathfrak{X}$. This proves:

Theorem 3.3 (González-Sánchez, Weigel, [10, Theorem A])**.** *Let p be an odd prime and let G be a p-central group of height $k \geqslant 1$. Then G is p-nilpotent.*

Here a group G is said to be *p^i-central of height k* if $\Omega_i(P) \leqslant \mathrm{Z}_k(G)$, where $P \in \mathrm{Syl}_p(G)$ and $\mathrm{Z}_k(G)$ is the kth term of the ascending central series of G.

The above theorem does not hold for $p = 2$.

Example 3.4. Let G be the semidirect group of the quaternion group of order 8 with a cyclic group of order 3 permuting the subgroups of order 4 of the quaternion group. Then the unique subgroup of G of order 2 is central in G and G is not 2-nilpotent.

Let $\mathfrak{D}$ be a class of p-groups, p a prime. We say that a subgroup H of a group G *controls fusion of $\mathfrak{D}$-groups in G* if

(1) any $\mathfrak{D}$-subgroup of G is conjugate to a subgroup of H, and

(2) for any $\mathfrak{D}$-subgroup A of G and for any $g \in G$ such that A, $A^g \leqslant H$, there exists $x \in H$ such that for all $a \in A$, $a^g = a^x$.

If p is a prime, let $\mathfrak{D}_p$ denote:

(1) the class of cyclic groups of order p, if p is odd, and

(2) the class of cyclic groups of order 2 or 4, if $p = 2$.

Let

$$f_p(G) = \{\, H \leqslant G \mid H \text{ controls fusion of } \mathfrak{D}_p\text{-subgroups} \,\}.$$

If G is p-nilpotent, it is clear that $\mathrm{Syl}_p(G)$ is contained in $f_p(G)$. Conversely, assume, arguing by contradiction, that G is not p-nilpotent and $\mathrm{Syl}_p(G) \subseteq f_p(G)$. Then G contains a subgroup C which is critical for the class of all p-nilpotent groups. By Theorem 3.2, $C = AB$, where A is a normal p-subgroup of C and $\exp A = p$ if p is odd, or $\exp A \leqslant 4$ if $p = 2$, and $B = \langle g \rangle$ is a cyclic Sylow q-subgroup of C, where $q \neq p$. The minimality of C implies that $A = [A, g]$. Moreover the hypothesis on G implies that there exists a Sylow p-subgroup P of G such that $A \leqslant P$ and there exists $x \in P$ such that $a^x = a^g$ for every $a \in A$. This means that $A \leqslant \mathrm{Z}(P)$ and the condition on G implies that $A = 1$. This contradiction proves:

Theorem 3.5 (González-Sánchez, [9, Main Theorem]). *Let p be a prime. A group G is p-nilpotent if and only if every Sylow p-subgroup of G belongs to $f_p(G)$.*

Example 3.4 shows that it is necessary to consider the subgroups of order 4 in the above theorem.

These results belong to a consolidated research project in which subgroups of hypercentral type are used as descriptors for characterising some structural properties of the groups. The theory of formations turns out to be a useful tool and provides a suitable language to analyse these phenomena.

Let me introduce some definition and results before stating our next theorem. They can be found in [7, IV, Section 4].

A formation $\mathbb{F}$ is said to be a *Baer-local formation* if there exists a function F which assigns to every simple group J a class of groups $F(J) \subseteq \mathbb{F}$ provided that $F(J)$ is a formation whenever the simple group J is abelian, such that $\mathbb{F}$ is equal to the class of all groups G such that for every G-chief factor H/K, $G/\operatorname{Cent}_G(H/K) \in F(J)$ if the composition factors of H/K are isomorphic to J. In addition, F can be chosen satisfying $\mathfrak{S}_p F(J) = F(J)$ if J is isomorphic to a cyclic group of order p.

In a group G, a G-chief factor H/K whose composition factors are isomorphic to a simple group J is said to be $\mathbb{F}$-*central in* G if $G/\operatorname{Cent}_G(H/K) \in F(J)$. Note that G belongs to $\mathbb{F}$ if and only if every chief factor of G is $\mathbb{F}$-central in G. More generally, a normal subgroup N of G is said to be $\mathbb{F}$-*hypercentral in* G if every G-chief factor below N is $\mathbb{F}$-central in G. The product of all normal $\mathbb{F}$-hypercentral subgroups of G is also $\mathbb{F}$-hypercentral in G. This subgroup is called the $\mathbb{F}$-*hypercentre* of G and it is denoted by $\mathrm{Z}_{\mathbb{F}}(G)$. For the class $\mathfrak{N}$ of nilpotent groups, we have $\mathrm{Z}_{\mathfrak{N}}(G) = \mathrm{Z}_\infty(G)$.

Any saturated formation is a Baer-local formation. The class of all *generalised nilpotent groups* is a non-saturated Baer-local formation.

Theorem 3.6 (Ballester-Bolinches, Ezquerro, Skiba [5]). *Let $\mathbb{F}$ be a Baer-lo-cal formation. Given a group G and a normal subgroup E of G, let $\mathrm{Z}_{\mathbb{F}}(G)$ contain a p-subgroup A of E which is maximal being abelian and of exponent dividing p^k, where k is some natural number, $k \neq 1$ if $p = 2$ and the Sylow 2-subgroups of E are non-abelian. Then*

$$E/\mathrm{O}_{p'}(E) \leq \mathrm{Z}_{\mathbb{F}}\big(G/\mathrm{O}_{p'}(E)\big).$$

Suppose the result is false and let the group G provide a counterexample of least order. Among the normal subgroups of G for which the theorem fails we choose E of minimal order. Then E is a normal subgroup of G, containing a p-subgroup A which is maximal being abelian and of an exponent dividing p^k, where k is some natural number, $k \neq 1$ if $p = 2$ and the Sylow 2-subgroups of E are non-abelian such that $A \leqslant \mathrm{Z}_{\mathbb{F}}(G)$ but $E/\mathrm{O}_{p'}(E)$ is not contained in $\mathrm{Z}_{\mathbb{F}}(G/\mathrm{O}_{p'}(E))$.

Let $W = A^G$ be the normal closure of A in G and set $C = C_E(W)$. Let E_p be a Sylow p-subgroup of E containing A. Then $C_p = E_p \cap C$ is a Sylow p-subgroup of C. The contradiction follows after the following steps.

(1) $O_{p'}(E) = 1$.

(2) $E/C \leq \mathrm{Z}_{\mathbb{F}}(G/C)$.

(3) $C_p \neq 1$ and $\Omega_k(C_p) \leq A$.

(4) $C = E$. In particular, $\Omega_k(E_p) \leq A \leq \mathrm{Z}(E)$.

(5) $E = E_p$.

As an immediate deduction we have:

Corollary 3.7. *Let $\mathbb{F}$ be a Baer-local formation. Given a group G and a normal subgroup E of G such that $G/E \in \mathbb{F}$, let $\mathrm{Z}_{\mathbb{F}}(G)$ contain a p-subgroup A of E which is maximal being abelian and of exponent dividing p^k, where k is some natural number, $k \neq 1$ if $p = 2$ and the Sylow 2-subgroups of E are non-abelian. Then $G/\mathrm{O}_{p'}(E) \in \mathbb{F}$.*

Corollary 3.8. *Let $\mathbb{F}$ be a Baer-local formation. Consider a group G and a normal subgroup E of G. For each prime divisor p of $|E|$ assume that $\mathrm{Z}_{\mathbb{F}}(G)$ contains a p-subgroup A of E which is maximal being abelian of exponent dividing p^k, where k is some positive integer, $k \neq 1$ if $p = 2$ and the Sylow 2-subgroups of E are non-abelian. Then $E \leq \mathrm{Z}_{\mathbb{F}}(G)$.*

Since $\mathrm{Z}_{\mathbb{F}}(G)$ centralises the $\mathbb{F}$-residual of G ([7, IV, 6.10]), and the generalised Fitting subgroup contains its centraliser ([12, X, 13.12]), we have:

Corollary 3.9. *Let $\mathbb{F}$ be a Baer-local formation and $E = \mathrm{F}^*(G)$ the generalised Fitting subgroup of G. For each prime divisor p of $|E|$ let $\mathrm{Z}_{\mathbb{F}}(G)$ contain a p-subgroup A of E which is maximal being abelian and of exponent dividing p^n, where n is some natural number, $n \neq 1$ if the Sylow 2-subgroups of E are non-abelian. Then $G \in \mathbb{F}$.*

Some earlier results can be also deduced from Theorem 3.6.

Corollary 3.10 (Ballester-Bolinches, Pedraza-Aguilera, [6])**.** *Let K be a normal subgroup of a group G with G/K contained in the saturated formation $\mathbb{F}$. If every element of order p or 4 (if $p = 2$) lies in $\mathrm{Z}_{\mathbb{F}}(G)$, then $G/\mathrm{O}_{p'}(K)$ belongs to $\mathbb{F}$.*

Corollary 3.11 (Ballester-Bolinches, Pedraza-Aguilera, [6]). *Let* $\mathbb{F}$ *be a subgroup-closed saturated formation. Suppose that* G *is a group with a normal subgroup* N *such that* $G/N \in \mathbb{F}$*. If every minimal subgroup of* N *is contained in* $Z_{\mathbb{F}}(G)$ *and* N *has abelian Sylow 2-subgroups, then* G *is an* $\mathbb{F}$*-group.*

Corollary 3.12 (Yokoyama, [17]). *Let* $\mathbb{F}$ *be a saturated formation containing the class of all nilpotent groups. Let* N *be a normal subgroup of a soluble group* G *such that* $G/N \in \mathbb{F}$*. If every subgroup of* N *of prime order is contained in* $Z_{\mathbb{F}}(G)$ *and the Sylow 2-subgroups of* N *are quaternion-free, then* $G \in \mathbb{F}$*.*

Corollary 3.13 (Laue, [13]). *Let* $\mathbb{F}$ *be a local formation and* G *a soluble group. For each prime divisor* p *of* $|F(G)|$ *let* $Z_{\mathbb{F}}(G)$ *contain a* p*-subgroup* A *of* E *which is maximal being abelian and of exponent dividing* p^n*, where* n *is some natural number,* $n \neq 1$ *if* $p = 2$*. Then* $G \in \mathbb{F}$*.*

4 p-length and p-nilpotency

The focus of this section relates to some questions concerning p-length and p-nilpotency of p-soluble groups. The results we are going to present are motivated by the papers [10] and [16].

In the sequel, p will denote a prime number.

The p-nilpotency of a group G is a property which can be read off from the structure of the Sylow p-subgroups of G and the way in which they are embedded in G. For instance, if a Sylow p-subgroup P of a finite group is abelian, then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent. This is a classical result of Burnside ([11, IV, 2.6]), which was extended to modular Sylow p-subgroups, i. e. groups with modular subgroup lattice, by Esteban-Romero and the author of this paper (see [1, 2, 2.2.5]). This result is extremely useful in establishing relationships between local characterisations of T-, PT- and PST-groups.

Following [16], we say that a class of groups $\mathfrak{X}$ *determines p-nilpotency locally* if a group G with a Sylow p-subgroup P in $\mathfrak{X}$ is p-nilpotent whenever $N_G(P)$ is p-nilpotent.

Of course, not every class of p-groups determines p-nilpotency locally. It is enough to consider a class containing a p-group P which can be embedded in a non-p-nilpotent group G as a Carter subgroup. On the positive side, the above-mentioned results show that the class of all abelian p-groups and the class of all modular p-groups are both examples of subgroup-closed classes of p-groups determining p-nilpotency locally. Regular p-groups ([11, III, Section 10]) is also a subgroup-closed class of p-groups which determines p-nilpotency locally by virtue of a result of Hall and Wielandt ([11, IV, 8.1]). Every finite p-group of nilpotency class less or equal to $p-1$ and every finite p-group of exponent p are regular. Therefore the class of all p-groups of nilpotency class at most $p-1$ and the class of all p-groups of exponent p are subgroup-closed classes determining p-nilpotency locally.

Weigel [16] proved that if p is odd, there exists a subgroup-closed class of p-groups which determines p-nilpotency locally and contains every subgroup-closed class of finite p-groups with this property. It is defined as follows.

Definition 4.1. Let $E = \langle g_1, g_2, \dots, g_p \rangle$ be an elementary abelian group of order p^p. Let $C = \langle x \rangle$ be a cyclic group of order p^m acting on E, where m is a natural number, in such a way $g_i^x = g_{i+1}$ for $1 \leqslant i \leqslant p-1$ and $g_p^x = g_1$. Let $Y_p(m) = [E]C$ be the corresponding semidirect product.

Note that $Y_p(1)$ is just the regular wreath product $C_p \wr C_p$.

We say that a p-group P is *slim* if P contains no subgroup isomorphic to $Y_p(m)$ for all $m \geq 1$.

By [16, Main Theorem], the class of all slim p-groups, p odd, determines p-nilpotency locally and it contains every subgroup-closed class finite of p-groups which determines p-nilpotency locally ([16, 4.3]).

For the proof of his Main Theorem, Weigel considers the semidirect product $S = [V]S_0$, where V is a faithful and irreducible S_0-module over $\mathrm{GF}(p)$, the finite field of p-elements and $S_0 = [Q]C_p$ is the semidirect product of C_p with a faithful and irreducible C_p-module Q over $\mathrm{GF}(q)$ for a prime $q \neq p$ (here C_p is the cyclic group of order p). These groups are called *pqp-sandwich groups* in [16]. Weigel also deals with groups X with a normal p-subgroup N contained in the Frattini subgroup $\Phi(X)$ of X such that $X/N \cong S$ and $N \cap \mathrm{O}^p(X) \leqslant \mathrm{Z}(\mathrm{O}^p(X))$. The corresponding natural map $\tau\colon X \longrightarrow S$ is called there a *p-Schur-Frattini extension*. The main result of [16] follows from the interesting fact that such a group X always possesses a subgroup isomorphic to $Y_p(m)$ provided that p is odd ([16, Section 3.4 and Proposition 3.5]). This fact is useful in some arguments by minimal counterexample.

Unfortunately, we have found some delicate points in the proof of the above statement. For instance, the image of the form defined in Equation (3.14) is not contained in general in $\mathrm{GF}(p^e)$ because we cannot assure in general that this image is fixed by the corresponding Frobenius-type automorphism. Moreover, in the construction of the subgroup isomorphic to $Y_p(m)$ in Case 1.B in the proof of [16, Proposition 3.5], it is not sufficient to ensure that the chosen element x is not fixed under the automorphism $x \mapsto x^{p^f}$, because x could be taken as an element of the maximal submodule of the regular module and hence x might generate a non-regular submodule.

We have been unable to overcome those difficulties just following Weigel's proof and so we have tried to solve them by presenting an alternative proof of Proposition 3.5 of [16]. This is the main result of the paper [2].

Suppose that k is a natural number and let $\mathfrak{X}_k$ be the class of all p-groups P such that $\Omega(P) \leqslant \mathrm{Z}_k(P)$. It is clear that $\mathfrak{X}_k$ is a subgroup-closed class of p-groups.

If p is odd, then every $P \in \mathfrak{X}_{p-1}$ has to be slim. Therefore we have:

Corollary 4.2 (González Sánchez, Weigel, [10, Theorem D]). *Let p be an odd prime. Then $\mathfrak{X}_{p-1}$ determines p-nilpotency locally.*

Our aim in the sequel is to describe a completely different approach based on the classical theory of Hall and Higman and focusing the attention on the p-length and moving from here to p-nilpotence. The next result is a structural theorem about p-soluble groups of minimal order among the groups belonging to a subgroup-closed class of groups and whose p-length is greater than 1. Such groups are critical for the subgroup-closed saturated Fitting formation $\mathfrak{L}_p$ of all p-soluble groups of p-length at most 1.

Theorem 4.3 (Ballester-Bolinches, Esteban-Romero, Ezquerro, [3]). *Let $\mathcal{P}$ be a subgroup-closed class of p-groups and let $\mathfrak{Y}(\mathcal{P})$ denote the class of all p-soluble groups whose Sylow p-subgroups are in $\mathcal{P}$. Suppose that $\mathfrak{Y}(\mathcal{P}) \not\subseteq \mathfrak{L}_p$, and let G be a p-soluble group of minimal order in $\mathfrak{Y}(\mathcal{P}) \setminus \mathfrak{L}_p$. If P is a Sylow p-subgroup of G, then $\Phi(G)$, the Frattini subgroup of G, is contained in P and one of the following holds.*

(1) If p is not a Fermat prime or the Hall p'-subgroups of G are abelian, then the nilpotency class of $P/\Phi(G)$ is greater or equal than p.

(2) If p is a Fermat prime, then the nilpotency class of $P/\Phi(G)$ is greater or equal than $p-1$.

Our group G satisfies the following structural conditions:

(1) $\mathrm{O}_{p'}(G) = 1$. Therefore if F is the Fitting subgroup of G, then $F = \mathrm{O}_p(G)$ and $\mathrm{Cent}_G(F) \leqslant F$.

(2) $G/\Phi(G)$ is primitive and so $F/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$ is a chief factor of $G/\Phi(G)$.

(3) If P is a Sylow p-subgroup of G, then $N_G(P)$ is the unique maximal subgroup of G containing P.

(4) G is a $\{p, q\}$-group for some prime $q \neq p$. Then there exist a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G such that $G = PQ$.

(5) Write $A = \mathrm{O}_{p,q}(G)$. If $N/\mathrm{O}_p(G) = \Phi(A/\mathrm{O}_p(G))$, then A/N is the unique minimal normal subgroup of G/N and $N_G(P) = PN$. Moreover $\mathrm{O}^p(G) \leqslant A$.

(6) Let M be a maximal subgroup of G complementing $F/\Phi(G)$. Write $B = P \cap M$ and let Q be a Sylow q-subgroup of G contained in M. We have:

a. B is a Sylow p-subgroup of M and $M = QB$.

b. $B/\Phi(G)$ is a cyclic p-group. Hence P/F is cyclic.

c. $M = \mathrm{N}_G(Q)$ and $\mathrm{Z}(M/\Phi(G))$ is cyclic.

d. $[\mathrm{O}^p(G), \Phi(G)] = 1$.

e. $B \leqslant \mathrm{Cent}_G(\Phi(Q))$.

f. $\mathrm{Z}_\infty(G) = \Phi(G)$.

We focus now our attention on the quotient group $\overline{G} = G/\Phi(G)$. For any subgroup X of G we will write $\overline{X}$ to denote the image of X in $\overline{G}$: $\overline{X} = X\Phi(G)/\Phi(G)$.

(7) $\overline{Q}$ is either elementary abelian or an extraspecial q-group.

$\overline{F}$ can be regarded as an irreducible and faithful $\overline{M}$-module over K, the finite field of p-elements. Let $\overline{F}_{\overline{B}}$ denote the subgroup $\overline{F}$ regarded as $\overline{B}$-module over K by restriction.

(8) If $\overline{Q}$ is abelian, then $\overline{F}_{\overline{B}}$ is a direct sum of copies of the regular $K\overline{B}$-module. Assume that $\overline{Q}$ is extraspecial.

(9) If p is not a Fermat prime, then regular $K\overline{B}$-module is a direct summand of $\overline{F}_{\overline{B}}$.

(10) If p is a Fermat prime then two possibilities arise:

- either the regular $K\overline{B}$-module is a direct summand of $\overline{F}_{\overline{B}}$,
- or $\overline{F}_{\overline{B}}$ is a direct sum of copies of the Jacobson radical, $J(K\overline{B})$, of the regular $K\overline{B}$-module.

Write $W = C_p \wr C_p$. Note that $\mathrm{Z}(W)$ is of order p, W' is elementary abelian of order p^{p-1} and the nilpotency class of W is p. Hence the nilpotency class of $W/\mathrm{Z}(W)$ is $p-1$.

a. Suppose that p is not a Fermat prime or $\overline{Q}$ is abelian. Then a direct summand of $\overline{F}_{\overline{B}}$ is isomorphic to the regular $K\overline{B}$-module. In this case $P/\Phi(G)$ contains a subgroup isomorphic to W. Then the nilpotency class of $P/\Phi(G)$ is greater or equal than p.

b. Suppose that p is a Fermat prime. Then it could occur that $\overline{F}_{\overline{B}}$ is a direct sum of indecomposable $K\overline{B}$-modules isomorphic to $J(K\overline{B})$. In this case $P/\Phi(G)$ contains a subgroup isomorphic to $W/\operatorname{Z}(W)$ and so the nilpotency class of $P/\Phi(G)$ is greater or equal than $p-1$.

Example 4.4. The group of automorphisms of $Q \cong C_{11}$ has a subgroup isomorphic to $H = C_5$. Let $S = [Q]H$ be the corresponding semidirect product. Let V be an irreducible and faithful module for S over the field of 5 elements. The dimension of V as a GF(5)-vector space is 5. Let $G = [V]S$ be the corresponding semidirect product.

The Sylow 5-subgroup of G is isomorphic to $[V]H$, which is isomorphic to the wreath product $C_5 \wr C_5$. The nilpotency class of P is exactly 5. Moreover, the maximal subgroups of G are isomorphic to S, to $[V]S$ or to $[V]Q$, all of them of 5-length one. Since $\Phi(G) = 1$, the bound of Theorem 4.3 cannot be improved in general.

Example 4.5. Let Q be a central product of a quaternion group of order 8 and a dihedral group of order 8 with $|Q| = 32$. Let g_1 be an automorphism of Q of order 5 and let $R = [Q]\langle g_1\rangle$. The group R can be regarded as a group of automorphisms of an extraspecial group E of order 5^5 and exponent 5. The semidirect product $G = [E]R$ has order $|G| = 2^5 \cdot 5^6 = 500{,}000$. Then G is soluble of 5-length 2, but every maximal subgroup of G is of 5-length 1. The nilpotency class of $P/\Phi(G)$ is $4 = 5-1$. This shows that the bound of Theorem 4.3 cannot be improved for the Fermat prime $p = 5$.

Let $\mathfrak{Y}(\mathfrak{X}_k)$ denote the class of all p-soluble groups whose Sylow p-subgroups are in $\mathfrak{X}_k$, k a natural number. Assume that $\mathfrak{Y}(\mathfrak{X}_k)$ is not contained in $\mathfrak{L}_p$. If G is a group of minimal order in $\mathfrak{Y}(\mathfrak{X}_k) \setminus \mathfrak{L}_p$ then G is a group described in Theorem 4.3. We use the same notation.

Consider the normal subgroup A. Suppose that every element of order p of A is in $\Phi(G)$. Then $\Omega(F) \leqslant \operatorname{Z}_\infty(G) \cap A \leqslant \operatorname{Z}_\infty(A)$. Then A is p-nilpotent. This implies that $Q \leqslant \operatorname{Cent}_G(F) \leqslant F$, and this is not true. Therefore there exists an element of order p, or order 2 or 4 if $p = 2$, say g, in $F \setminus \Phi(G)$.

Since $F/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$, then the normal closure of $\langle g\Phi(G)\rangle$ in $G/\Phi(G)$ is $F/\Phi(G)$. Hence $\langle g\rangle^G\Phi(G) = F$. In fact, since $g \in F$, then $\langle g\rangle^G \leqslant F$ and then $\langle g\rangle^G \leqslant \Omega(P)$. Hence $F = \langle g\rangle^G\Phi(G) \leqslant \Omega(P)\Phi(G)$.

Since $\Omega(P) \leqslant \operatorname{Z}_k(P)$, then

$$F/\Phi(G) \leqslant \Omega(P)\Phi(G)/\Phi(G) \leqslant \operatorname{Z}_k(P)\Phi(G)/\Phi(G) \leqslant \operatorname{Z}_k(P/\Phi(G)).$$

Since $\overline{P}/\overline{F} \cong \overline{B}$ is a cyclic group, we have that the nilpotency class of $P/\Phi(G)$ is lesser or equal to k.

Consequently, the class $\mathfrak{Y}(\mathfrak{X}_k)$ is contained in $\mathfrak{L}_p$ for all $k < p-1$. If $k = p-1$ and p is not a Fermat prime, $\mathfrak{Y}(\mathfrak{X}_{p-1})$ is contained in $\mathfrak{L}_p$ either. Moreover, every group G in $\mathfrak{Y}(\mathfrak{X}_{p-1})$ whose Hall p'-subgroups of G are abelian is of p-length at most 1.

Therefore we have:

Corollary 4.6. *Let p be a prime.*

(1) If p is odd, then $\mathfrak{X}_{p-2}$ determines p-length locally.

(2) If p is not a Fermat prime, then $\mathfrak{X}_{p-1}$ determines p-length locally.

(3) If p is odd, then $\mathfrak{X}_{p-1}$ determines p-length locally in groups with abelian Hall p'-subgroups.

Corollary 4.7. *Suppose that p is a prime. Let G be a group and P a Sylow p-subgroup of G. Assume that $\mathrm{N}_G(P)$ is p-nilpotent.*

(1) If $\Omega(P) \leqslant \mathrm{Z}_{p-1}(P)$, then G is p-nilpotent.

(2) If $p = 2$, and either $\Omega(P) \leqslant \mathrm{Z}(P)$, or $\Omega_1(P) \leqslant \mathrm{Z}(P)$ and P is quaternion-free, then G is 2-nilpotent.

These results improve Theorem E and Theorem D of [10] respectively.

References

[1] A. Ballester-Bolinches, R. Esteban-Romero, and M. Asaad. *Products of finite groups*, volume 53 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter, Berlin, 2010.

[2] A. Ballester-Bolinches, R. Esteban-Romero, and L. M. Ezquerro. On a p-Schur-Frattini extension of a finite group. Preprint.

[3] A. Ballester-Bolinches, R. Esteban-Romero, and L. M. Ezquerro. On the p-length of some finite p-soluble groups. *Israel J. Math.*, in press.

[4] A. Ballester-Bolinches and L. M. Ezquerro. On conjugacy of supplements of normal subgroups of finite groups. *Bull. Aust. Math. Soc.*, in press, `http://dx.doi.org/10.1017/S0004972713000506`.

[5] A. Ballester-Bolinches, L. M. Ezquerro, and A. N. Skiba. On subgroups of hypercentral type of finite groups. *Israel J. Math.*, in press.

[6] A. Ballester-Bolinches and M. C. Pedraza-Aguilera. On minimal subgroups of finite groups. *Acta Math. Hungar.*, 73(4):335–342, 1996.

[7] K. Doerk and T. Hawkes. *Finite Soluble Groups*, volume 4 of *De Gruyter Expositions in Mathematics*. Walter de Gruyter, Berlin, New York, 1992.

[8] W. Gaschütz. Selected topics in the theory of soluble groups. Canberra, 1969. Lectures given at the 9th Summer Research Institute of the Australian Math. Soc. Notes by J. Looker.

[9] J. González-Sánchez. A p-nilpotency criterion. *Arch. Math. (Basel)*, 94:201–205, 2010.

[10] J. González-Sánchez and T. S. Weigel. Finite p-central groups of height k. *Israel J. Math.*, 181:125–143, 2011.

[11] B. Huppert. *Endliche Gruppen I*, volume 134 of *Grund. Math. Wiss.* Springer Verlag, Berlin, Heidelberg, New York, 1967.

[12] B. Huppert and N. Blackburn. *Finite groups III*, volume 243 of *Grund. Math. Wiss.* Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[13] R. Laue. Dualization of saturation for locally defined formations. *J. Algebra*, 52:347–353, 1978.

[14] C. Parker and P. Rowley. A note on conjugacy of supplements in finite soluble groups. *Bull. London Math. Soc.*, 42:417–419, 2010.

[15] O. J. Schmidt. Über Gruppen, deren sämtliche Teiler spezielle Gruppen sind. *Mat. Sbornik*, 31:366–372, 1924.

[16] T. S. Weigel. Finite p-groups which determine p-nilpotency locally. *Hokkaido Math. J.*, 41:11–29, 2012.

[17] A. Yokoyama. Finite solvable groups whose $\mathfrak{F}$-hypercenter contains all minimal subgroups. *Arch. Math. (Basel)*, 26(1):123–130, 1975.