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Graded Algebras, Polynomial Identities and Generic Constructions

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Abstract. In these lecture we present some results which intertwine topics as graded algebras, polynomial identities and algebras of generic elements. Some of these connections are classical and well known to different communities (e.g. crossed products, Galois cohomology, algebra of generic matrices, general group gradings on finite dimensional algebra). Some other connections among these topics are relatively new where these are realized via the theory of group graded polynomial identities. In particular, using (G-graded) asymptotic PI theory, we outline the proof of a conjecture of Bahturin and Regev on regular gradings on associative algebras.
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1 Introduction

The purpose of these lectures is to combine three topics, namely gradings on associative algebras, polynomial identities and Brauer groups (division algebras).

There are well known connections among these topics. For instance, Galois cohomology is one of the main tools in the study of finite dimensional k-central simple algebras and Brauer groups. One way to realize this connection is via "crossed product structures" one can put on (certain) k-central simple algebras. Another way is via "Galois descent".

Question: Can one introduce a crossed product G-grading on every k-central simple algebra and in particular on every k-central division algebra?

Amitsur gave a negative answer to that question in 1972 by constructing "noncrossed product" division algebras (see [8]). His remarkable idea was to use a "generic construction" and show that the "generic division algebra" is in general not a crossed product. The generic division algebra can be constructed by means of polynomial identities and this can serve as a bridge between Brauer groups and polynomial identities. In a similar way one can use G-graded polynomial identities to construct "generic crossed products". In these lectures I will firstly recall these notions (e.g. G-gradings, crossed products, Brauer groups, polynomial identities) and secondly I'll present relatively more recent results on G-graded polynomial identities which provide new "bridges" among the topics mentioned above. In particular, at the end, I will present a positive solution of a conjecture of Bahturin and Regev on group gradings on associative algebras. We start our journey with G-gradings on associative algebras.

2 Group gradings and Brauer groups

Let A be an associative algebra over a field F and G any group. We say that A is G-graded if there exists a vector space decomposition

$$A \cong \oplus_{g \in G} A_g$$

such that for any $g, h \in G$ we have $A_g A_h \subseteq A_{gh}$. We refer to A_g, $g \in G$, as the homogeneous component of degree g.

We say that the G-grading on A is *strong* if $A_g A_h = A_{gh}$ for every $g, h \in G$. We say that the algebra A is a (ring theoretic) *G-crossed product* over A_e (the identity component) if and only if the homogeneous component A_g contains an invertible element for every $g \in G$. Note that if A is a (ring theoretic) G-crossed product, then it is necessarily strongly graded. The converse is false as the following example shows.

Example 2.1. Let A be the algebra of 3×3-matrices over a field F and G the group with two elements (denoted by e, σ). Consider the G-grading on A given by

$$A_e = span_F\{e_{11}, e_{12}, e_{21}, e_{22}, e_{33}\}$$
$$A_\sigma = span_F\{e_{13}, e_{23}, e_{31}, e_{32}\}.$$

It is easy to check that the grading is strong whereas the σ component has no invertible elements.

The example above is a very special case of a general type of G-grading on the algebra of $n \times n$-matrices over a field F.

Definition 2.2. (Elementary grading) Let A be the algebra of $n \times n$-matrices over a field F and let G be any group. Fix an n-tuple $\alpha = (g_1, \ldots, g_n) \in G^n$. For every $g \in G$ we determine the g-homogeneous component of A to be

$$A_g = span_F\{e_{i,j} : g = g_i^{-1} g_j\}.$$

One checks easily that this indeed determines a G-grading on A.

Remark 2.3. 1) Note that $\alpha = (e, e, \sigma) \in \{e, \sigma\}^3$ yields the grading considered in Example 2.1.

We observe (in the definition above) that since the algebra A is simple, the G-graded algebra A is G-simple (i.e. no nontrivial G-graded two sided ideals).

Next we present a "completely" different type of G-gradings on semisimple algebras which turn into G-simple algebras. Let G be a finite group and F a field of characteristic zero or p and p does not divide the order of the group. One knows (by Maschke's theorem) that the group algebra FG is semisimple. Furthermore, since every nonzero homogeneous element is invertible, the group algebra FG is G-simple. More generally, we may twist the product in FG by means of a 2-cocycle on G with coefficients in F^* (recall that a function $f : G \times G \to F^*$ is a 2-cocycle if for every $\sigma, \tau, \nu \in G$ we have that $f(\sigma\tau, \nu)f(\sigma, \tau) = f(\sigma, \tau\nu)f(\tau, \nu)$) and obtain the twisted group algebra F^fG. It is well known that F^fG is a semisimple (associative) algebra. Furthermore, we have that every nonzero homogeneous element is invertible and hence F^fG is G-simple. We can extend this construction a bit more by taking a finite subgroup H of any group G and considering the twisted group algebra F^fH as a G-graded algebra where the g homogeneous component is 0 if $g \in G \smallsetminus H$ (here: f is a 2-cocycle on H). Clearly, we obtain an H-simple algebra as above but we note that the algebra F^fH is also G-simple. In case the field F is algebraically closed of characteristic zero, we have that these two examples are the building blocks of any finite dimensional G-simple algebra. This is a theorem of Bahturin, Sehgal and Zaicev.

Theorem 2.4 ([10]). *Let A be a finite dimensional G-graded simple algebra. Then there exists a subgroup H of G, a 2-cocycle $\alpha : H \times H \to F^*$ where the action of H on F is trivial, an integer r and an r-tuple $\mathbf{g} = (g_1, g_2, \ldots, g_r) \in G^r$ such that A is G-graded isomorphic to $\Lambda = F^\alpha H \otimes M_r(F)$ where $\Lambda_g = span_F\{\pi_h \otimes e_{i,j} \mid g = g_i^{-1}hg_j\}$. Here $\pi_h \in F^\alpha H$ is a representative of $h \in H$ and $e_{i,j} \in M_r(F)$ is the (i, j) elementary matrix.*
In particular the idempotents $1 \otimes e_{i,i}$ as well as the identity element of A are homogeneous of degree $e \in G$.

Remark 2.5. Clearly, the G-graded algebra A is determined up to a G-graded isomorphism by the *presentation* $P_G = (\mathbf{g} = (g_1, g_2, \ldots, g_r), H, \alpha)$.

An interesting question that arises here is the isomorphism problem, namely what can we say about presentations $P_{G,1} = (\mathbf{g}_1, H_1, \alpha_1)$ and $P_{G,2} = (\mathbf{g}_2, H_2, \alpha_2)$ of two G-graded algebras A_1 and A_2 if we know they are G-graded isomorphic? Definitely it is *not* true that the tuples $\mathbf{g}_1$ and $\mathbf{g}_2$ defining the corresponding elementary grading must be the same. They also need not be equal up to permutation. Similarly, the finite subgroups H_1 and H_2 which determine the fine grading need not be equal as subgroups of G but they must be conjugate in G. It turns out that there are 3 basic moves on any presentation P_G of a G-graded algebra A which yield presentations of algebras which are G-graded isomorphic to A. The theorem says that any two algebras with given presentations are G-graded isomorphic if and only if one can get from one presentation to the other applying a finite finite number of moves of that kind. In case G is abelian this result was proved by Koshlukov and Zaicev (see [22]). Later, in a joint work with Darrell Haile (see [5]), we proved it for any group G (i.e. not necessarily abelian).

An important elementary G-grading is the so called "crossed product grading" (note: this is a very special case of the "ring theoretic crossed product" we mentioned above). Let G be a finite group of order n and let A be the algebra of $n \times n$-matrices over an algebraically closed field F. We consider the elementary G-grading on A where the n-tuple $\mathbf{g} = (g_1, \ldots, g_n)$ consists of all elements of G. It is easy to see that for any $g \in G$, the g-homogeneous components is of dimension n and is obtained by the product of all diagonal matrices with a suitable permutation matrix (namely, the permutation matrix whose entry (i, j) is 1 whenever $g = g_i^{-1}g_j$ and zero otherwise). We note that the e-component is commutative and for any $g \in G$ we have $a_g b_{g^{-1}} c_g = c_g b_{g^{-1}} a_g$, where $a_g, c_g \in A_g$ and $b_{g^{-1}} \in A_{g^{-1}}$.

This grading takes us to Brauer group theory and specifically to the interpretation of the Brauer group $Br(k)$, k a field, as $H^2(G_k, k^*_{sep})$ where G_k is the absolute Galois group of the field k and k^*_{sep} is the separable closure of k. Hence, before we continue with gradings, let us make a short trip into Brauer groups theory and the theory of division algebras.

We fix a field k which is usually not algebraically closed and we consider finite dimensional central simple algebras over k. By Wedderburn's theorem, such an algebra is isomorphic to the algebra of $r \times r$ matrices over a finite dimensional division algebra D, whose center is equal to k. Moreover, the integer r is uniquely determined and the division algebra D is uniquely determined up to a k-algebras isomorphism.

The Brauer group of k (denoted by $Br(k)$) consists of equivalence classes of finite dimensional k-central simple algebras, where two algebras are equivalent if and only if they have underlying division algebras which are k-algebra isomorphic. So in fact, the elements $Br(k)$ are in one to one correspondence with the k-central division algebras. It looks a bit strange to define the Brauer group in that way. Why not just consider the division algebras themselves? An important reason is that we can introduce a natural multiplication on the set of classes, namely the tensor product of classes representatives over k. It is well known that the tensor product over k of two k-central division algebras is a

k-central simple algebra (and hence matrices over a k-central division algebra) but in general is not a division algebra. One shows (easily) that the product of classes, via the tensor product over k of their representatives, is well defined and it induces an abelian group structure on $Br(k)$. Indeed, the identity element is the class represented by k and the inverse of $[A] \in Br(k)$ is $[A^{op}]$ where A^{op} is the opposite algebra of A (one shows that $A \otimes A^{op} \cong M_d(k)$ where $d = dim_k(A)$). We refer the reader to [19] for a "ring theoretic" introduction to the theory of simple algebras and Brauer groups.

An important characterization of k-central simple algebras is given by "k-algebras that become a matrix algebra after extending scalars to an algebraically closed field". We say that k-central simple algebras are *twisted k-forms* of matrix algebras. This point of view will be very useful for us when we consider the third topic of these lectures, polynomial identities. We refer the reader to [29] for an introduction to the theory of Brauer groups using Galois descent.

We present now an important way to construct k-central simple algebras. Let L/k be a finite Galois extension with Galois group G. Consider the corresponding skew group algebra L_tG. It is isomorphic to the group algebra LG as a left L-vector space and hence its elements are expressed by $\sum_{\sigma\in G} a_\sigma u_\sigma$, where $a_\sigma \in L$ and $\{u_\sigma\}_{\sigma\in G}$ is a basis of L_tG over L. The multiplication in L_tG is defined as to satisfy the relation $au_\sigma bu_\tau = a\sigma(b)u_{\sigma\tau}$, where $a, b \in L$ and $\sigma(b)$ is the action of σ on b as determined by the Galois action of G on L. It is easy to show that any element of L_tG determines an endomorphism in $End_k(L)$ by $\sum_{\sigma\in G} a_\sigma u_\sigma(a) = \sum_{\sigma\in G} a_\sigma \sigma(a)$ and this correspondence induces an isomorphism of L_tG with $End_k(L) \cong M_n(k)$ where n is the degree of the extension L/k. We note in particular that L_tG represents the identity element in $Br(k)$. Now we wish to twist the product in L_tG by means of a 2-cocycle $f : G \times G \to L^*$ (note: unlike the definition above, the action of G on L is not trivial). Recall that a function $f : G \times G \to L^*$ is a 2-cocycle if for every $\sigma, \tau, \nu \in G$ we have $f(\sigma\tau, \nu)f(\sigma, \tau) = f(\sigma, \tau\nu)f(\tau, \nu)^\sigma$. Then by means of f we "change" the multiplication in L_tG as to satisfy the rule

$$au_\sigma bu_\tau = a\sigma(b)f(\sigma, \tau)u_{\sigma\tau}.$$

As above, $a, b \in L$ and $\sigma(b)$ is the action of σ on b. It is easy to show that the algebra we obtain is a k-central simple algebra of dimension n^2 over k. We denote it by L_t^fG and refer to it as a *crossed product* of G over L (see [19]).

Remark 2.6. We emphasize once again that the terminology "crossed product" means different type of algebras for different researchers. The general definition (for ring theorists and hence "ring theoretic crossed product") means a G-graded algebra where every homogeneous component contains an invertible element (see

[23]). The more restrictive terminology (used by researchers in Brauer groups and division algebras) means a k-central simple algebra of the form $L_t^f G$.

Any 2-cocycle determines a cohomology class in $H^2(G, L^*)$ where 2-cocycles f and g are cohomologically equivalent (cohomologous) if there exists a 1-parameter family $\{\lambda_\sigma\}_{\sigma \in G} \subset L^*$ such that for any σ and τ in G we have

$$f(\sigma, \tau) = \lambda_\sigma \sigma(\lambda_\tau) \lambda_{\sigma\tau}^{-1} g(\sigma, \tau).$$

It is easy to show that up to a G-graded isomorphism the algebra $L_t^f G$ depends only on the cohomology class $\alpha = [f] \in H^2(G, L^*)$ and not on the representative f of α. We therefore write $L_t^\alpha G$ where $\alpha \in H^2(G, L^*)$. The crossed product algebras play a key role in Brauer group theory since *any* Brauer class can be represented by a crossed product algebra. In fact, a k-crossed product algebra $B = L_t^\alpha G$ becomes trivial when extending scalars to L, that is $B_L = B \otimes_k L = M_n(L)$ where $n = ord(G)$, and one shows that the map

$$H^2(G, L^*) \to Br(k),$$

determined by sending a cohomology class α to the class represented by the crossed product algebra $L_t^\alpha G$ induces an isomorphism between $H^2(G, L^*)$ and $Br(L/k) = \{[A] \in Br(k) : A \otimes_k L = M_r(L), \text{ some } r\}$.

The fact that any Brauer class may be represented by a crossed product algebra says that for any k-central finite dimensional division algebra there exists an integer n such that the algebra of $n \times n$-matrices over D is k-isomorphic to a crossed product algebra. But what about the division algebra itself? It was an open question for many years whether any k-central division algebra is a crossed product. This is known to be true for local or global fields (e.g. finite extensions of $\mathbb{Q}$). In 1972, Amitsur showed that this is false in general by using generic constructions (see [8]). What is the relevance of all this to us? Recall that we started our discussion with the description of gradings on $M_n(F)$ and in particular we considered elementary gradings (crossed product gradings) where the tuple $(g_1, \ldots, g_n)$ consists precisely of all elements of the group G (with multiplicity 1). It turns out and not difficult to prove, that if we take a crossed product algebra $L_t^\alpha G$ and extend scalars to F (the algebraic closure of k), we obtain $M_n(F)$ with the crossed product grading just mentioned. So, the crossed products are G-graded twisted k-forms of the matrix algebra with the elementary grading. This will play a role in the sequel.

Before turning to our 3rd topic, namely polynomial identities, let me present some results which concern with "fine gradings". It is very well known that a group algebra FG is semsimple (F of characteristic zero) and it is never simple (unless G is of order 1). What about the twisted group algebra $F^f G$ where f is a 2-cocycle? Can it be a simple algebra? For instance, consider the quaternion algebra over the complex field F. It is isomorphic to $M_2(F)$. On the other hand it is isomorphic to a twisted group algebra with the Klein 4-group, where the 2-cocycle is determined such that the generators of the group anticommute. More generally, let k be a field which contains a primitive nth root of unity ζ. For any $a, b \in k^*$ and integer $n \geq 2$, we consider the "symbol algebra" $(a,b)_{n,k}$ of degree n over the field k. It is given by $(a,b)_{n,k} = \langle x, y : x^n = a, y^n = b, yx = \zeta_n xy \rangle$ (i.e., the k-algebra generated by x, y subject to the relations $x^n = a, y^n = b, yx = \zeta_n xy$). It is not difficult to show that any symbol algebra is k-central simple. A fundamental result of Merkurjev and Suslin says that if k-contains enough roots of unity then all elements of $Br(k)$ are represented by tensor products of symbol algebras. More precisely, if k contains a primitive n-th root of unity, then any element in $Br(k)$ whose order divides n is represented by the tensor product of symbol algebras (over k). Note that the symbol algebra $(a,b)_{n,k}$ is isomorphic to $k^f Z_n \times Z_n$ with a suitable 2-cocycle f. Moreover, it is easy to see that tensor product of symbol algebras is isomorphic to a twisted group algebra of the form $k^f U \times U$, where U is a (finite) abelian group. So we see that twisting a group algebra FG with a 2-cocycle where the group G is abelian may give a matrix algebra and more generally provides important examples of k-central simple algebras.

Question: Suppose G is nonabelian. Can we twist a group algebra FG into a matrix algebra? It turns out that the answer is positive.

Definition 2.7. A finite group G is of *central type* (nonclassically) if it admits a *nondegenerate* 2-cocycle f with values in $\mathbb{C}^*$ such that $\mathbb{C}^f G \cong M_n(\mathbb{C})$ for some n.

Clearly, a group of central type must be of square order. Howlett and Isaacs proved in 1982, using the classification of finite simple groups, that such a group must be solvable (see [20]).

Let me present an example of a group of central type of order 36. Consider the semidirect product $G = Sym(3) \ltimes C_6$ where $Sym(3)$ is the symmetric group of order 6 (generated by an element σ of order 3 and by an involution τ) and C_6 is the cyclic group of order 6 (generated by x). We let $Sym(3)$ act on C_6 via the involution τ (that is via the image $Sym(3)/ < \sigma >$) where $\tau(x) = x^{-1}$. One shows with this set up that there is a bijective 1-cocycle from $Sym(3)$ onto C_6 and using that function one can construct a nondegenerate 2-cocycle on G. It turns out that this is a special case of a rather general construction introduced by Etingof and Gelaki (see [14]). Let H be a group of order n and suppose it acts on an abelian group A of order n making A and hence A^*, the dual of A, into an H-module. Suppose there exists a bijective 1-cocycle $\pi : H \to A^*$.

Then the semidirect product $H \ltimes A$ is a group of central type. Note that by the result of Isaacs and Howlett mentioned above, any group H admiting a bijective 1-cocycle onto an abelian group must be solvable. Let us remark here that the last statement can be proved without the classification of finite simple groups.

The construction of Etingof and Gelaki was extended by Ben David and Ginosar. Using a bijective 1-cocycles from H onto A^* as above, one can construct central type groups which are nonsplit extensions of H and A. Based on that theory, jointly with Angel del Rio, we found an example of a group of central type of order 64 which cannot be expressed as a semidirect product of two groups of order 8 (see [12]).

3 Polynomial identities

We now present the 3rd topic in this series of lectures, namely polynomial identities of associative algebras and G-graded polynomial identities of G-graded associative algebras. We'll work throughout over a field F of characteristic zero. "Most of the time" the field F will represent an algebraically closed field but if we want to connect PI theory with Brauer theory we drop that assumption. In particular we will be interested in finding, roughly speaking, "small field" over which our algebras are defined. This will take us to a task which we call minimal fields of definition of a given algebra. It is obvious that a matrix algebra is defined over the rationals, but what about the G-simple algebras over the complex field? What is the minimal field of definition? In general we don't know the answer to that question. However, we have an answer in case the group G is abelian, or in case the grading is elementary or fine. In order to prove this we need "graded polynomial identities".

Let us start our presentation with ordinary polynomial identities. Let A be an algebra over a field F. We let $F\langle X\rangle$ be the free algebra over F with a countable set of variables X. Elements of the free algebra will be called polynomials in noncommuting variables and we say that a nonzero polynomial f is an identity of the algebra A if the polynomial vanishes upon any evaluation on A. We also say that the algebra A satisfies the polynomial identity f (or simply say that A satisfies the polynomial f). So for example $[x, y] = xy - yx$ is a polynomial identity of any commutative algebra and clearly this characterizes commutative algebras. A more interesting example is the Wagner identity of $M_2(F)$. Consider the polynomial $[x, y]$. Clearly, any evaluation of this polynomial on 2×2-matrices is a trace zero matrix and hence the eigenvalues are both zero or of opposite sign. If both eigenvalues are zero then the square of the matrix is zero and hence in that case $[x, y]^2$ represents the zero matrix. If the eigenvalues are of opposite sign, then the matrix is diagonalizable, and so our matrix is similar to a diagonal matrix of the form $(d, -d)$. Hence, in that case, the square of the matrix is similar to (d^2, d^2) which is a scalar matrix and nonzero. It follows that our matrix is scalar. This will be important for us later on. The polynomial $[x, y]^2$ is an example of a central (nonidentity) polynomial. The existence of such polynomials for $n \times n$-matrices is a highly nontrivial problem which was solved independently by Formanek and Razmyslov. Later I will present a central polynomial that was constructed by Regev. But where is the polynomial identity? Wagner's identity is $[[x, y]^2, z]$. We note that this is an homogeneous nonmultilinear polynomial of degree 5. It is not difficult to show (left to the reader) that the algebra of 2×2-matrices has an identity of smaller rank, namely $s_4 = \sum_{\sigma \in Sym(4)} sgn(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$. More generally, the algebra of $n \times n$-matrices over a field satisfies the polynomial identity $s_{2n} = \sum_{\sigma \in Sym(2n)} sgn(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(2n)}$. This is the famous Amitsur-Levitzki theorem. Furthermore, and this is easy to show, the algebra $M_n(F)$ does not satisfy any nonzero identity of degree $< 2n$. Let me show for instance that $M_n(F)$ does not satisfy s_{2n-1}. Consider the product of the $2n - 1$ elementary $n \times n$-matrices $e_{1,1} e_{1,2} e_{2,2} \cdots e_{(n-1),n} e_{n,n}$. We note that this product is equal to $e_{1,n}$ and hence nonzero. On the other hand, any nontrivial permutation vanishes as some of the indices do not match. We therefore see, by evaluating the variables of s_{2n-1} as above, that we get $e_{1,n}$ and hence nonzero.

We say that an algebra is PI if it satisfies at least one nontrivial identity. As we saw above, commutative algebras are PI. Finite dimensional algebras are also PI since any algebra of dimension n satisfies any polynomial which alternates on $n + 1$ variables. “In particular” any finite dimensional algebra over F satisfies the Capelli polynomial

$$c_{n+1} = \sum_{\sigma \in Sym(n+1)} sgn(\sigma) y_0 x_{\sigma(1)} y_1 x_{\sigma(2)} y_2 \cdots x_{\sigma(n+1)} y_{n+1}.$$

Does every PI algebra satisfies a Capelli polynomial c_n for some n? The answer is negative. A counter example which is key for the whole theory of polynomial identities is the Grassmann algebra E of countable rank. By definition

$$E = F\langle x_1, x_2, \ldots, x_n, \ldots \rangle / < x_i x_j + x_j x_i : i, j \leq 1 > .$$

The algebra E is spanned by multilinear monomials where the monomials of even length and odd length determine a $\mathbb{Z}_2$-grading in a natural way. It satisfies the identity $[[x, y], z]$ and this identity generates all identities in a sense I'll soon describe. It is not difficult to prove that E does not satisfy any Capelli polynomial.

Given an algebra A which is PI, we consider the set of all polynomial identities it satisfies. We denote this set by $Id(A)$. It is easily seen that $Id(A)$ is a 2-sided ideal of $F\langle X \rangle$. Furthermore, $Id(A)$ is closed under substitutions (or equivalently, it is closed under endomorphisms of $F\langle X\rangle$). We call an ideal with that property, a T-ideal. Note that any T-ideal is the T-ideal of identities of an algebra. Indeed, given a T-ideal I we may consider the corresponding relatively free algebra $F\langle X\rangle/I$. It is easy to see that $Id(F\langle X\rangle/I) = I$.

Now, we wish to find "simpler" generators of $Id(A)$. Using a Vandermonde argument one can show easily that if f is a polynomial identity of A, then if we decompose f into a sum of polynomials $f_1+f_2+\ldots+f_r$ where each f_i is the sum of *all* monomial of f with exactly the same variables, then f_i is an identity for every i. Next, using a well known multilinearization process, all identities are consequences of polynomials identities which are multilinear. Since this is important for us let me illustrate how this is done. Take, for instance, the polynomial x^2. Suppose it is an identity of an algebra A. Then $(x_1+x_2)^2 = x_1^2+x_1x_2+x_2x_1+x_2^2$ is also an identity of A. The terms x_1^2 and x_2^2 are identities for the same reason and so the multilinear polynomial $x_1x_2+x_2x_1$ is an identity of A. We can deduce that the multilinear polynomial $x_1x_2 + x_2x_1$ is a consequence of (i.e. belongs to the T-ideal generated by) the polynomial x^2. But what about the converse? Indeed, we would like to see whether our original polynomial x^2 is a consequence of $x_1x_2+x_2x_1$. Putting $x_1 = x_2 = x$ we obtain $2x^2$ and not x^2. In order to get x^2 we need to divide by 2 and so we need our field to be of characteristics not 2. This is a major problem and indeed, in positive characteristics we don't have generation of the T-ideal of identities by multilinear polynomials. There are good reasons to deal with multilinear polynomials and with multilinear identities. One of them which is important for us, is that identities "do not change" upon extension of scalars. This implies that the T-ideal of identities of a k-central simple algebra of degree n^2 is the same as the ideal of identities of $n \times n$-matrices.

One of the main questions in PI theory is the so called Specht problem, namely whether the T-ideal of identities is finitely generated as a T-ideal? (see [30]). This was established in the positive by Kemer in the mid 80's. Kemer proved a fantastic result which says that if S is an algebra over F (F is algebraically closed of characteristic zero) which satisfies a Capelli polynomial then there exists a finite dimensional algebra A over F which is PI-equivalent to S. This is the representability theorem for affine algebras. With some efforts, the representability theorem implies a positive solution of Specht problem, namely the finite generation of any T-ideal (see [21]).

What happens if S does not satisfy a Capelli polynomial (e.g. the infinite dimensional Grassmann algebra)? Then it cannot be PI-equivalent to a finite dimensional algebra (since the latter does satisfy a Capelli polyniomial). It turns out the the infinite Grassmann algebra is basically the only example with that property in the sense that there exists a $\mathbb{Z}_2$-graded finite dimensional algebra $A = A_0 \oplus A_1$ such that its Grassmann envelope $E(A) = E_0 \otimes A_0 \oplus E_1 \otimes A_1$

is PI equivalent to S. This is the representability theorem for arbitrary (i.e. not necessarily affine) associative algebras over an algebraically closed field of characteristic zero. Using this result one can give a positive solution to the Specht problem.

The fact that the T-ideal of identities is finitely generated says nothing about finding explicit polynomials which generate $Id(A)$. Such generators are known for an extremely short list of algebras. For instance, it is a nontrivial fact that the two identities mentioned above for $M_2(F)$, namely the Wagner identity and the standard identity s_4 of degree 4 are generators of the T-ideal of identities (this was shown by Ramyslov and Drensky). It is not difficult to show that the identity $[[x, y], z]$ generates the T-ideal of identities of the infinite dimensional Grassmann algebra. But already for the algebra of 3×3-matrices no such generators are known. Furthermore, knowing a generating set says not much about the problem of describing explicitly the elements in $Id(A)$. Therefore, it seems more effective to calculate invariants of $Id(A)$ and in particular invariants related to the size of $Id(A)$.

The "codimension sequence" of an algebra A.

We saw that the T-ideal of identities of an algebra A is generated as a T-ideal by multilinear identities. Therefore, while considering polynomial identities of degree n, we can restrict ourselves to the intersection of a T-ideal I with the $n!$-dimensional space

$$P_n = span_F\{x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} : \sigma \in Sym_n\}.$$

Clearly, $\dim(I \cap P_n) > 0$ for some n if and only if the algebra A is PI and it turns out, roughly speaking, that in that case, the "magnitude" of $\dim(I \cap P_n)$ is "close" to $n!$. It is therefore more informative to measure the *nonidentities*, meaning $\dim_F(P_n/(P_n \cap I))$. We denote this dimension by c_n. Regev (in his pioneering work) showed in 1972 that if I is nontrivial, then c_n is exponentially bounded, that is

$$\overline{lim}_{n\to\infty} \sqrt[n]{c_n} < \infty.$$

Amitsur conjectured that the limit exists and is a (nonnegative) integer. This was proved in a remarkable work by Giambruno and Zaicev in the late 90's (see [17], [18]). The limit is denoted by $\exp(A)$. In their proof, Giambruno and Zaicev, give an interpretation of $\exp(A)$ as the dimension of a certain subalgebra A_0 of A in case the algebra is finite dimensional. In case the algebra is affine and PI one uses Kemer's theory to pass to a finite dimensional algebra whereas in the general case $\exp(A)$ is interpreted as the dimension of a certain $\mathbb{Z}_2$-graded subspace of the finite dimensional $\mathbb{Z}_2$-graded algebra whose existence is assured by Kemer's theorem.

For instance, if $A \cong M_n(F)$, then the subalgebra A_0 is $M_n(F)$ itself and hence $\exp(A) = n^2$. For an arbitrary finite dimensional algebra A we proceed as follows. We write A as a direct sum of $\bar{A}$ (the semisimple part) and J, the Jacobson radical (where the decomposition is a decomposition of vector spaces). Next we decompose the semisimple part $\bar{A}$ into direct sum of matrix algebras $\mathcal{M} = \{A_1, \ldots, A_r\}$. Then we consider *nonzero* products (denoted by α) of the form

$$\alpha : A_{i_1} J A_{i_2} J \cdots J A_{i_k}$$

where $A_{i_j} \in \mathcal{M}$ and to each product of this type we attach an integer n_α which is the sum of the dimensions of the different simple components that participate in the product (i.e. ignore repetitions). Giambruno and Zaicev proved that $\exp(A)$ coincides with the $\max\{n_\alpha\}$ where we run over all possible products α. You see in particular, that if the algebra is simple, the exponent is just its dimension over F.

Polynomial identities were found to be very useful in the construction of the generic division algebra and Azumaya algebras which serve as a representing object with respect to all k-central simple algebras, any k, of degree n. Let us present this classical construction before we turn our attention to G-graded polynomial identities.

4 Polynomial identities and the generic division algebra

Let F be an algebraically closed field of characteristic zero. Let A be the algebra of $n \times n$-matrix algebra over F and consider the k-forms of the algebra A where k is a subfield of F. It is well known that these are precisely the k-central simple algebras of dimension n^2 over their center.

Our goal is to construct an algebra $\mathcal{A}$ such that every k-central simple algebra of degree n over its center is a specialization of $\mathcal{A}$ (this means that for any k and any k-central simple algebra B of degree n, there exists a prime ideal I of $Z(\mathcal{A})$ (the center of $\mathcal{A}$), such that $Z(\mathcal{A})/I$ is a $\mathbb{Q}$-subalgebra of k and such that the algebra $\mathcal{A}/(I\mathcal{A})$ becomes isomorphic to B after extension of scalars to the field k. Conversely, any simple homomorphic image of $\mathcal{A}$ is a form of $M_n(F)$.

In order to construct the algebra $\mathcal{A}$ we first need to find a field k (if exists) which is the minimal subfield k of F over which there is a k-form of $M_n(F)$. Obviously, this is the rational field $\mathbb{Q}$ since there is a form over $\mathbb{Q}$ and it is minimal. This very first step is already problematic for G-simple algebras.

Next we consider the free algebra $\mathbb{Q}\langle X \rangle$ on a countable number of variables and let $I = Id(A)$ be the T-ideal of identities of $A = M_n(F)$. A simple lemma shows that if there is a form over a certain field then the identities are defined over that field and hence the polynomial identities of A are defined over $\mathbb{Q}$. We consider the relatively free algebra $\mathbb{Q}\langle X\rangle/I$ over $\mathbb{Q}$.

Clearly, the algebra $\mathbb{Q}\langle X\rangle/I$ can be mapped "onto" any k-form (after extension of scalars). Indeed, any map from the free algebra $\mathbb{Q}\langle X\rangle$ into $M_n(F)$ factors through its quotient $\mathbb{Q}\langle X\rangle/I$. However it is not difficult to see that we can send (some of) the variables to zero and obtain an homomorphic image which is not a form of A. For instance, one can easily get $\mathbb{Q}$ as an homomorphic image. To "fix" this problem we localize the algebra $\mathbb{Q}\langle X\rangle/I$ by a central element.

Suppose we can find a central (nonidentity) polynomial $s(x_1,\dots,x_n)\in\mathbb{Q}\langle X\rangle$ of $M_n(F)$, that belongs to every T-ideal J that strictly contains $I = Id(A)$. Then, if we invert $s(x_1,\dots,x_n)$ in the relatively free algebra $\mathbb{Q}\langle X\rangle/I$, we see that $s^{-1}\mathbb{Q}\langle X\rangle/I$ cannot have nonzero homomorphic images which strictly satisfy more identities than A and in particular the identities of $(n-1)\times(n-1)$-matrices. On the other hand, the algebra $s^{-1}\mathbb{Q}\langle X\rangle/I$ satisfies the identities of $n\times n$-matrices and so invoking a fundamental theorem of Artin and Procesi we obtain that the algebra $\mathcal{A} = s^{-1}\mathbb{Q}\langle X\rangle/I$ is an Azumaya algebra of rank n^2 over its center (see [24]). Recall that an algebra $\mathcal{A}$ is said to be Azumaya of rank n^2 if it is faithful, finitely generated projective over its center and such that modulo maximal ideals we obtain central simple algebras.

How to construct such a central polynomial? As mentioned above this a nontrivial problem which was solved independently by Formanek and Razmyslov (see [15], [25]). Here is an explicit polynomial R_n (called Regev polynomial) which is of degree $2n^2$ and is known to be central for the algebra A:

$$R_n = \sum_{\sigma\in Sym(n^2)}^{\tau\in Sym(n^2)} sgn(\sigma)sgn(\tau)x_{\sigma(1)}y_{\tau(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(4)}y_{\tau(2)}y_{\tau(3)}y_{\sigma(4)}x_{\sigma(5)}\cdots x_{\sigma(9)}$$
$$y_{\tau(5)}\cdots y_{\tau(9)}\cdots x_{n^2-(2n-1)}\cdots x_{n^2}y_{n^2-(2n-1)}\cdots y_{n^2}$$

Note that the polynomial R_n is alternating on sets of cardinality n^2 and hence it must be an identity on an algebra of dimension $< n^2$ and in particular on the algebra of $(n-1)\times(n-1)$-matrices.

If we extend scalars of $\mathcal{A}$ to the field of quotients of $Z(\mathcal{A})$, we get the "famous" generic division algebra of degree n (over $\mathbb{Q}$). This algebra is of great importance in Brauer theory. For instance, using that construction Amitsur proved in 72 the existence of noncrossed products for any degree divisible by 8 or by p^2 (where p is an odd prime). To this end (say in the case where p is odd) he showed the existence of crossed products with the group $\mathbb{Z}_{p^2}$ which *are not crossed products* with the elementary abelian group $\mathbb{Z}_p\times\mathbb{Z}_p$, and on the other hand the existence of crossed products with the elementary abelian group $\mathbb{Z}_p\times\mathbb{Z}_p$ which are noncyclic algebras (that is, are not crossed products with the cyclic group of order p^2). Then the generic division algebra cannot be a crossed product since if it were, say with a group G, then every specialization would have to be a crossed product with the same group.

Much attention is devoted to *rationality* questions of the center of the generic division algebra. Let me only say that part of the interest on this question came from the fact that its positive solution would lead to a rather easy proof of the result we mentioned above of Merkurjev and Suslin (on generation of the Brauer group by symbol algebras).

My goal for the rest of these lectures is to present generalizations of several of the results mentioned above to the context of G-graded algebras and at the end present an application of graded PI theory for the solution of a conjecture of Bahturin and Regev on regular gradings of algebras.

5 G-graded polynomial identities

We start with the the definition of graded identities. Let G be any group and $X_G = \cup_{g\in G} X_g$ be a set of noncommuting graded variables, where $X_g = \{x_{g,1}, x_{g,2}, \ldots\}$ is a countable set of variables *of degree* $g \in G$. We consider the free G-graded algebra $F\langle X_G\rangle$, where the homogeneous degree of a monomial $x_{g_1}x_{g_2}\cdots x_{g_n}$ is $g_1g_2\cdots g_n \in G$. Suppose A is a G-graded (associative) algebra. We say that a polynomial in $F\langle X_G\rangle$ is a G-graded identity of A if it vanishes on any admissible evaluation, that is, graded variables are evaluated only by homogeneous elements of A of the same degree.

As in the ungraded case, the set $Id_G(A)$ of G-graded identities of a G-graded algebra is a G-graded ideal of $F\langle X_G\rangle$. Furthermore, it is a G-graded T-ideal, that is, invariant by all G-graded endomorphisms of $F\langle X_G\rangle$. Concretely, an homogeneous variable of degree g can be replaced by a polynomial p whose monomials are of degree g. See [1] for more details on graded polynomial identities.

Example 5.1. Consider $M_n(F)$, the algebra of all $n \times n$ matrices with the crossed product grading mentioned above. It was proved by Di Vincenzo for $G \cong Z_2$ ([13]), by Vasilovsky for $G \cong Z_n$ ([33]) and by Bahturin and Drensky for any group ([9]), that the identities are generated as a G-graded T-ideal by

(1) $x_e y_e - y_e x_e$

(2) $x_\sigma y_{\sigma^{-1}} z_\sigma - z_\sigma y_{\sigma^{-1}} x_\sigma$ for every $\sigma \in G$.

As one can see, graded identities are much easier to describe since polynomials need not vanish on all evaluations but only on special ones. Nevertheless, the graded identities "tell the whole story" in the sense that two algebras that satisfy the same G-graded identities they satisfy the same ordinary identities. This basic fact will be used later.

As in the ungraded case there is a "Representability Theorem" for G-graded algebras where G is a finite group (see [1]). The theorem says that if W is a PI algebra over an algebraically closed field F of characteristic zero, which is G-graded (G finite), then there exists a finite dimensional $\mathbb{Z}_2 \times G$-graded algebra A over F such that its Grassmann envelope is G-graded PI equivalent to W. In order to state the theorem precisely, recall that given any $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$ we may consider its Grassmann envelope $E(A) = E_0 \otimes A_0 \oplus E_1 \otimes A_1$ as an ungraded algebra. Thus, if $A = (\oplus_{g\in G} A_{0,g}) \oplus (\oplus_{g\in G} A_{1,g})$ is a $\mathbb{Z}_2 \times G$-graded algebra, we may consider its Grassmann envelope $E(A) = E_0 \otimes (\oplus_{g\in G} A_{0,g}) \oplus E_1 \otimes (\oplus_{g\in G} A_{1,g})$ as a G-graded algebra where $E(A)_g = E_0 \otimes A_{0,g} \oplus E_1 \otimes A_{1,g}$. The representability theorem for PI algebras which are G-graded can be stated as follows.

Theorem 5.2. *Let G be a finite group and let W be a PI algebra and G-graded. Then there exists a $\mathbb{Z}_2 \times G$-graded finite dimensional algebra A such that $Id_G(W) = Id_G(E(A))$.*

From these results one can deduce (with some efforts) the positive solution of the Specht problem for G-graded algebras which are PI.

Remark 5.3. In case the group is abelian, the representability theorem and the solution of the Specht problem was obtained independently by Irina Sviridova (see [32]).

Remark 5.4. Note that an algebra W may be G-graded PI and non-PI (as an ungraded algebra). Of course, the representability theorem for G-graded algebras cannot hold for such algebras. This follows from the following two facts: (1) any two algebras that are G-graded PI-equivalent, are also PI-equivalent as ungraded algebras (2) the Grassmann envelope of a finite dimensional algebra is (ungraded) PI. Nevertheless one may ask and indeed it is an open problem whether the Specht problem holds for G-graded PI non-PI algebras.

As mentioned in the beginning of these lectures, also the asymptotic PI-theory was developed in the G-graded case. Let W be a PI algebra which is G-graded (G-finite). It was proved by Antonio Giambruno, Daniela La Mattina and the author of these notes that $\lim_{n\to\infty} \sqrt[n]{(c_n^G(W))}$ exists and is equal to a nonnegative integer denoted by $\exp_G(W)$ (see [4], [16], [3]). The sequence $c_n^G(W)$ is determined as follows. Consider the $n!\,\mathrm{ord}(G)^n$-dimensional vector space spanned by all multilinear G-graded monomials of degree n

$$P_n^G = span_F\{x_{\sigma(1)}^{g_{i_1}} x_{\sigma(2)}^{g_{i_2}} \cdots x_{\sigma(n)}^{g_{i_n}} : \sigma \in Sym(n), g_{i_j} \in G\}.$$

We let $P_n^G/(P_n^G \cap Id_G(W))$ be the space of all G-graded functions on W represented by a multilinear polynomial of degree n and let $c_n^G(W)$ be its dimension over

F. We refer to the integer $c_n^G(W)$ as the nth term of the G-graded codimension sequence of W.

In case the algebra W is finite dimensional, the integer $\exp_G(W)$ is interpreted as the dimension of a certain G-graded subalgebra of $W/J(W)$. If W is an affine G-graded algebras we "pass" to a finite dimensional G-graded algebra via the representability theorem whereas if W is nonaffine we "reduce the calculation" to finding the exponent of the finite dimensional $\mathbb{Z}_2 \times G$-graded algebra which appears in the representability theorem for G-graded algebras. In particular, in case A is a finite dimensional G-simple algebra, then the G-exponent is just the dimension of the algebra A.

6 PI and the Generic crossed product algebra

In this "tiny" paragraph we present briefly a natural extension of Amitsur's construction of the generic division algebra. Consider the matrix algebra $M_n(F)$ with the crossed product G-grading where G is of order n. Applying G-graded polynomial identities we construct the relatively free algebra over $\mathbb{Q}$

$$\mathbb{Q}\langle X_G\rangle / Id_G(M_n(\mathbb{Q})).$$

This algebra can be localized by a "central polynomial" and we obtain a G-graded Azumaya algebra which specializes precisely to all G-crossed product algebras. Taking the field of fractions of the center we obtained the generic G-crossed product. It should be noted that one can find in the literature different ways to construct the "generic crossed product"(see [27], [31] and [28]). Generic constructions were obtained for other G-gradings (e.g. twisted group algebras) and also for certain type of H-comodule algebras (see [6], [7]). Applying the construction of a generic crossed products and the corresponding G-graded Azumaya algebra mentioned above it is not hard to prove the following result.

Theorem 6.1. *Let G be a finite group and let $\mathfrak{B}_G$ be the family of all G-crossed product algebras over a field of characteristic zero. Suppose every G-crossed product A is also an $H = H_A$ crossed product ($ord(G) = ord(H)$ but nonisomorphic). Then there exists a group S such that any G-crossed product is also an S-crossed product. A group G satisfying the condition above is said to be "nonrigid".*

7 Regular G-gradings and PI-asymptotics

We close this series of lectures by presenting a rather different connection between (asymptotic) PI theory and G-gradings. In particular I'll present a positive solution of a conjecture posed by Bahturin and Regev on regular gradings (see [11]). This is joint work with Ofir David. We start with the definition of regular gradings (see [26]).

Definition 7.1. Let A be an associative algebra over a field F (algebraically closed of characteristic zero) and let G be a finite abelian group. Suppose A is G-graded. We say that the G-grading on A is regular if there is a commutation function $\vartheta : G \times G \to F^{\times}$ such that

(1) For every integer $n \geq 1$ and every n-tuple $(g_1, g_2, \ldots, g_n) \in G^n$, there are elements $a_i \in A_{g_i}$, $i = 1, \ldots, n$, such that $\prod_1^n a_i \neq 0$.

(2) For every $g, h \in G$ and for every $a_g \in A_g, b_h \in A_h$, we have $a_g b_h = \vartheta_{g,h} b_h a_g$.

Let me say right away that the definition above can be extended to nonabelian groups, but in these lectures I will restrict myself to abelian groups.

Clearly, any G-grading on an algebra A induces a natural G/N-grading on A where N is a normal subgroup of G. Indeed, we let the gN-component A_{gN} to be the sum of all components A_{gn}, $n \in N$. We say that a regular G-grading is minimal if for any normal subgroup N of G, the induced G/N-grading on A is not regular. It is easy to show that any regular G-grading on A yields (via a homomorphic image of G) a minimal regular grading (Remark: this particular fact is false in case G is nonabelian).

Let me start with some examples.

The following example corresponds to the grading determined by the symbol algebra $(1,1)_n$ over F.

Example 7.2. Let $M_n(F)$ be the matrix algebra over the field F, and let $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. For ζ a primitive n-th root of 1 we define

$$X = diag(1, \zeta, ..., \zeta^{n-1}) = \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & \zeta & 0 & & \vdots \\ & 0 & \zeta^2 & \ddots & \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & \zeta^{n-1} \end{bmatrix}$$

$$Y = E_{n,1} + \sum_1^{n-1} E_{i,i+1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Note that $\zeta XY = YX$. Furthermore, the elements $\{X^iY^j \mid 0 \leq i,j \leq n-1\}$ form a basis of $M_n(F)$, and so we can define a G-grading on $M_n(F)$ by $(M_n(F))_{(i,j)} = FX^iY^j$. Let us check the G-grading is regular. For any two basis elements we have that

$$\begin{aligned}(X^{i_1}Y^{j_1})(X^{i_2}Y^{j_2}) &= \zeta^{i_2j_1}X^{i_1}X^{i_2}Y^{j_1}Y^{j_2} = \zeta^{i_2j_1}X^{i_2}X^{i_1}Y^{j_2}Y^{j_1}\\ &= \zeta^{i_2j_1-i_1j_2}\left(X^{i_2}Y^{j_2}\right)\left(X^{i_1}Y^{j_1}\right)\\ &\Rightarrow \vartheta_{(i_1,j_1)(i_2,j_2)} = \zeta^{i_2j_1-i_1j_2}\end{aligned}$$

and hence the second condition in the definition of a regular grading is satisfied. The first condition in the definition follows at once from the fact that the elements X and Y are invertible. Finally we note that since ζ is a primitive n-th root of unity, the regular grading is in fact minimal.

Next we present an example of a different nature.

Example 7.3. Let E be the Grassmann algebra considered above with the usual $\mathbb{Z}/2\mathbb{Z}$-grading. The commutation function is given by $\tau_{0,0} = \tau_{0,1} = \tau_{1,0} = 1$ and $\tau_{1,1} = -1$. It is easy to see that this grading regular and minimal.

Now it is clear that an algebra A may admit nonisomorphic regular gradings and even nonisomorphic minimal regular gradings. In fact, it is easy to show that more is true, that is, an algebra A may admit minimal regular gradings with nonisomorphic groups. For instance, consider the following two (minimal) regular gradings on $M_4(F)$: (1) with the group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (as in the example above) (2) with the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (here we grade two copies of $M_2(F)$, each with the Klein 4-group, and then we take their tensor product over F).

Bahturin and Regev conjectured however that the order of the group is invariant. More precisely they conjectured that if an algebra A admits minimal regular gradings with finite abelian groups G_1 and G_2, then $ord(G_1) = ord(G_2)$ (see [11]). In addition, Bahturin and Regev made a conjecture which concerns with the "commutation matrix" of a minimal regular grading: Let A be an associative algebra and suppose it is regularly graded with the group $G = \{g_1, \ldots, g_n\}$. Consider the $n \times n$-matrix Θ (the commutation matrix) whose entry (i,j) is given by $\Theta_{i,j} = \vartheta_{g_i,g_j}$ where ϑ is the commutation function. It is not difficult to show that a regular grading is minimal if and only if the commutation matrix is invertible.

Conjecture 7.4. Let A be an associative algebra over a field F of characteristic zero and suppose it is regularly graded by groups G_1 and G_2. Suppose the gradings are minimal and let Θ_{G_1}, Θ_{G_2} be the corresponding commutation matrices. Then $\det(\Theta_{G_1}) = \det(\Theta_{G_2})$.

In a joint work with Ofir David we prove these conjectures (see [2]). We show that if an associative algebra admits a G-grading which is minimal and regular, then $ord(G) = \exp(A)$. In particular the order of the group is invariant. For the second conjecture, we show that the determinant of the commutation matrix is equal to $\exp(A)^{\exp(A)/2}$. We close these notes explaining roughly the idea of the proof.

Suppose that A is G-graded and let $Id_G(A)$ be the corresponding T-ideal of G-graded identities. In case the G-grading is regular one can write down explicitly G-graded polynomials which generate $Id_G(A)$. Then, we construct a "model algebra" B, which admits a regular G-grading and such that is G-graded PI equivalent to A, namely $Id_G(A) = Id_G(B)$. The point here is that two algebras that are G-graded PI equivalent are also (ordinary) PI-equivalent and hence, in particular, they have the same exponent. The final step is to realize that the order of G coincides with $\exp(B)$.

For the second conjecture we prove that two commutation matrices of two minimal regular gradings are conjugate to each other and hence have the same characteristic values. In particular the commutation matrices arising from minimal regular gradings on an associative algebra A over F have the same determinant.

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