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ON THE CONVEX COMPACTNESS PROPERTY FOR THE STRONG OPERATOR TOPOLOGY

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Dedicated to the memory of Professor Gottfried Köthe

In the strong operator topology, the space $K(X,Y)$ of compact operators between two Banach spaces X,Y is not complete, not even sequentially complete. It is, however, Mackey complete, i.e., every bounded closed absolutely convex subset is a Banach disk (cf. [4]). In this paper we show that $K(X,Y)$, with the strong operator topology, has a stronger completeness property, namely the convex compactness property (see the definition below). This property is also true for the space of weakly compact operators ([9]).

These considerations concerning the convex compactness property of $K(X,Y)$ and of other subspaces of $L(X,Y)$ (the space of all continuous linear operators) in the strong operator topology were motivated by the paper of Weis [11]. They originated from the context of the perturbation theory of C_0-semigroups, in particular from the application to the neutron transport equation. We refer to [11] as well as to the references quoted there for motivation.

In section 1 we show the convex compactness property for $K(X,Y)$. In fact, we show the «strong convex compactness property» which is, at least formally, slightly stronger.

In section 2 we indicate several other subspaces of $L(X,Y)$ which have this property, for instance the space of weakly compact operators.

In section 3 it is shown that, under certain additional assumptions, the strong convex compactness property is implied by the convex compactness property. In order to prove this we establish a refined version of Carathéodory's theorem on the equivalence of separable probability spaces and the unit interval, which should be of independent interest (Theorem 3.5).

In section 4 we discuss the relations between different completeness properties, and we show by an example that not every closed subspace of $L(X,Y)$ has the convex compactness property in the strong operator topology.

Concluding this introduction we recall the convex compactness property. For a locally convex space E the *[metric] convex compactness property* is defined as follows: for each [metrizable] compact subset $C \subset E$ the closed convex hull $\overline{\text{co}}(C)$ is compact (cf. [7], [13; Definition 9.2.8], [10; p. 92]).

The following equivalent formulation of these properties was pointed out to the author by H. Pfister (München, 1981).

Theorem 0.1. *Let E be a Hausdorff locally convex space. Then the following conditions are equivalent:*

(a) E has the [metric] convex compactness property.

(b) If Ω is a compact [metric] space, μ a (positive) Borel measure on Ω and $f : \Omega \to E$ continuous, then f is μ-Pettis integrable.

Proof. $(a) \Rightarrow (b)$. This is an immediate consequence of [3; chap. III, § 3, n. 2, Proposition 5]. (For the «metric» case, note that the continuous image of a compact metric space is metrizable; cf. [2; chap. IX, § 2, n. 10, Proposition 17].)

$(b) \Rightarrow (a)$. Let C be compact [and metrizable]. Then $\overline{\mathrm{co}(C)}^{\widetilde{E}}$ is compact, where $\widetilde{E}$ denotes the completion of E. By [3; chap. IV, § 7, n. 1, Proposition 1] every point of $\overline{\mathrm{co}(C)}^{\widetilde{E}}$ is the barycenter of a probability measure on C. Now the hypothesis implies $\overline{\mathrm{co}(C)}^{\widetilde{E}} \subset E$. ∎

1. THE STRONG CONVEX COMPACTNESS PROPERTY FOR THE SPACE OF COMPACT OPERATORS

The properties stated in Theorem 0.1 should serve as a motivation for the following definition concerning subspaces of $L(X,Y)$, where X, Y are Banach spaces.

Definition 1.1. *Let E be a closed (with respect to the operator norm) subspace of $L(X,Y)$. E is defined to have the* strong convex compactness property *of the following holds: for any finite measure space $(\Omega, \mathcal{A}, \mu)$ and any bounded function $U : \Omega \to E$ which is strongly measurable (i.e., $U(\cdot)x$ is measurable for all $x \in X$) the strong integral $\int U \mathrm{d}\mu$ $(\in L(X,Y))$, defined by*

$$\int_\Omega U \mathrm{d}\mu x := \int_\Omega U(w) x \mathrm{d}\mu(w) \qquad (x \in X),$$

belongs to E.

Remarks 1.2. (a) If E in the situation of the preceding definition has the strong convex compactness property then $(E, \mathcal{T}_s)$, where $\mathcal{T}_s$ denotes the strong operator topology, has the convex compactness property, by Theorem 0.1.

(b) The strong convex compactness property of $E \subset L(X,Y)$ is obviously equivalent to the following property: for any measure space $(\Omega, \mathcal{A}, \mu)$ and any function $U : \Omega \to E$ which is strongly measurable and for which the upper integral $\overline{\int}_\Omega \| U(w) \| \mathrm{d}\mu(w)$ is finite, the strong integral $\int_\Omega U \mathrm{d}\mu$ belongs to E. It was in this form that the strong convex compactness property was stated for various subspaces of $L(X,Y)$, in [11].

Theorem 1.3. *The space $K(X,Y)$ of compact operators has the strong convex compactness property.*

Remark 1.4. Theorem 1.3 was stated by Weis (cf. [11; Corollary 2.3]). There is, however, a gap in the proof of [11; Proposition 2.2] which we shall explain subsequently.

In [11, upper part of p. 9] it is claimed that, under the assumption that X is separable, there exists a sequence $M_1 \supset M_2 \supset \dots$ of finite codimensional closed subspaces of X such that $\| S|M_n \| \to 0$ for all $S \in K(X,Y)$. We now demonstrate that this statement is erroneous if X' is not separable, and for $Y = \mathbb{K}$ (the scalars). Let $M_1 \supset M_2 \supset \dots$ be a sequence of finite codimensional closed subspaces of X. Without restriction codim $M_n = n$ for all $n \in \mathbb{N}$. Then there exists a sequence $(x'_n) \subset X'$ such that $M_n = \bigcap_{j=1}^{n} {x'_j}^{-1}(\{0\})$ for all $n \in \mathbb{N}$. Since X' is not separable there exists $x' \in X' \setminus \overline{\text{lin}}\{x'_n; n \in \mathbb{N}\}$. It follows from the Hahn-Banach theorem that

$$\| x'|M_n \| = \text{dist}\left(x', \text{ lin }\left\{x'_j; 1 \le j \le n\right\}\right) \ge$$

$$\ge \text{dist}\left\{x', \text{ lin }\left\{x'_j; j \in \mathbb{N}\right\}\right) > 0.$$

Proof of Theorem 1.3. (i) An operator in $L(X,Y)$ is compact if and only if its restriction to any separable subspace is compact. Therefore we may assume without restriction that X is separable.

(ii) Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, $U : \Omega \to K(X,Y)$ bounded and strongly measurable. Since X is separable it follows that there exists a μ-null set $N \subset \Omega$ such that

$$\{U(w)x; w \in \Omega \setminus N, x \in X\}$$

is contained in a separable subspace of Y. Therefore we may assume without restriction that Y is separable.

(iii) As a separable Banach space, Y can be embedded isomorphically into $C[0,1]$ (cf. [1; chap. XI, § 8 Thèorème 9]). Since enlarging the range space does not affect the compactness of operators we may assume without restriction $Y = C[0,1]$. As a consequence, since $C[0,1]$ has a Schauder basis (cf. [6; p. 3]), we may assume that there exists a sequence $(P_n) \subset L(Y)$ of finite dimensional projections converging strongly to the identity on Y. Then, an operator $T \in L(X,Y)$ is compact if and only if $\| (I - P_n)T \| \to 0 \ (n \to \infty)$.

(iv) Let $(\Omega, \mathcal{A}, \mu)$ and $U(\cdot)$ be as in (ii). Then

$$\| (I - P_n) \int_\Omega U \mathrm{d}\mu \| \le \int_\Omega \| (I - P_n) U(w) \| \mathrm{d}\mu(w) \to 0$$

$(n \to \infty)$, by the dominated convergence theorem (note that the measurability of $\| (I - P_n)U(\cdot) \|$ follows from the separability of X). ∎

Remark 1.5. Assume that the dual X' of X is separable. If Y is finite dimensional, then any strongly measurable function $U : \Omega \to L(X,Y)$ is already Bochner measurable as an $L(X,Y)$-valued function.

Therefore the proof of Theorem 1.3 shows that any bounded strongly measurable function $U : \Omega \to K(X,Y)$ is already Bochner measurable (and therefore $\int U \mathrm{d}\mu \in K(X,Y)$) (the author is endebted to G. Godefroy for pointing out this fact).

2. THE STRONG CONVEX COMPACTNESS PROPERTY FOR OTHER SUBSPACES OF $L(X,Y)$

As in section 1, let X,Y be Banach spaces.

Proposition 2.1. *The space $V(X,Y)$ of completely continuous operators has the strong convex compactness property (T completely continuous means: for each weak null sequence $(x_n) \subseteq X$ the sequence (Tx_n) is a null sequence).*

The proof consists in a straightforward application of the dominated convergence theorem.

Remarks 2.2. (a) It was shown by G. Schlüchtermann [9] that the space $W(X,Y)$ of weakly compact operators has the strong convex compactness property.
(b) In [11; Proposition 2.4] it is proved that the subspace of $L(X,Y)$ consisting of the strictly singular operators has the strong convex compactness property if X is $L_p(\mu)$, $1 \le p \le \infty$, or $C(K)$ where K is compact and metric or compact and extremely disconnected (note that the proof in [11] uses the contents of our Theorem 1.3).
(c) For completeness we mention two further subspaces of $L(X,Y)$ possessing the strong convex compactness property:

(i) The space of unconditionally summing operators, $\{T \in L(X,Y)$; for all sequences $(x_n) \subset X$ such that $\sum_n | < x_n, x' > | < \infty$ for all $x' \in X'$ the series $\sum T x_n$ is convergent$\}$;

(ii) the space of Dieudonné operators, $\{T \in L(X,Y)$; for all weak Cauchy sequences $(x_n) \subset X$ the sequence (Tx_n) is weakly convergent$\}$.

Again, the proof follows from the dominated convergence theorem.

3. EQUIVALENCE OF CONVEX COMPACTNESS PROPERTY AND STRONG CONVEX COMPACTNESS PROPERTY

Throughout this section let X,Y be Banach spaces.

Remark 3.1. Let X be separable and let $C \subset L(X,Y)$ be compact with respect to the strong operator topology. Then C is metrizable.

In fact, let $D \subset X$ be countable and dense. The countable family of seminorms

$$T \mapsto \| Tx \| \ (x \in D)$$

separates the points of $L(X,Y)$ and therefore generates a metric which is coarser than $\mathcal{T}_s$. Since C is $\mathcal{T}_s$-compact this metric and $\mathcal{T}_s$ coincide on C.

Thus, for a closed subspace E of $L(X,Y)$ the convex compactness property and the metric convex compactness property with respect to $\mathcal{T}_s$ are equivalent.

Definition 3.2. *Let $E \subset L(X,Y)$ be a closed subspace. We say that E has the* measurability property *if the following is true: if $U : [0,1] \to L(X,Y)$ is bounded and strongly measurable (with respect to the Borel σ-algebra on $[0,1]$) then the set $\{t \in [0,1]; U(t) \in E\}$ is a Borel set.*

Lemma 3.3. *If X is separable then $K(X,Y)$ has the measurability property.*

Proof. Let $U : [0,1] \to L(X,Y)$ be bounded and strongly measurable. As in the proof of Theorem 1.3 we conclude that we may assume without restriction that there exists a sequence $(P_n) \subset L(Y)$ of finite dimensional projections converging strongly to the identity. This implies

$$\{t \in [0,1]; U(t) \in K(X,Y)\} = \left\{t \in [0,1]; \| \left(I - P_n\right) U(t) \| \to 0\right\},$$

and the right hand side clearly defines a Borel subset of $[0,1]$. ∎

The following theorem contains the main result of this section.

Theorem 3.4. *Let X be separable, and let $E \subset L(X,Y)$ be a closed subspace which has the measurability property. Then E has the strong convex compactness property if and only if $E, \mathcal{T}_s)$ has the (metric) convex compactness property.*

In order to prove this theorem we need a refinement of Carathéodory's theorem concerning the isomorphism of separable probability spaces to the Borel sets of $[0,1]$; which we state and prove next.

In the following, we denote by $\mathcal{B}$ the σ-algebra of Borel subsets of $[0,1]$, and by λ the Borel-Lebesgue measure on $[0,1]$.

Theorem 3.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, and let $F \subset \mathscr{L}_\infty(\Omega, \mathcal{A}; Y)$ be a separable subspace (here, $\mathscr{L}_\infty(\Omega, \mathcal{A}; Y)$ denotes the smallest subspace of the bounded Y-valued functions which contains the $\mathcal{A}$-simple Y-valued functions and with each pointwise convergent sequence contains the limit). Assume that μ restricted to the smallest σ-algebra for which all $f \in F$ are measurable is atom free.*

(a) Then there exists a linear mapping

$$\varphi : F \to \mathscr{L}_\infty([0,1], \mathcal{B}; Y)$$

with the following properties:

$$\| \varphi(f) \|_{\sup} \leq \| f \|_{\sup},$$
$$\| \varphi(f) \|_{\text{ess sup}} = \| f \|_{\text{ess sup}},$$
$$\int_{[0,1]} \varphi(f) \, d\lambda = \int_{\Omega} f \, d\mu$$

for all $f \in F$.

(b) There exist a set $D \subset [0,1]$ *of full outer Lebesgue measure* ($\lambda^*(D) = 1$), *and a mapping* $\psi : D \to \Omega$ *such that*

$$\varphi(f)|D = f \circ \psi \quad \lambda^* - \text{ a.e. on } D$$

for all $f \in F$.

We recall that, if $D \subset [0,1]$ satisfies $\lambda^*(D) = 1$, then the outer measure λ^* restricted to the σ-algebra $\mathcal{B} \cap D$ is a measure.

Proof of Theorem 3.5. The separability of F implies that, for each $\varepsilon > 0$, there exists a set $\Omega_\varepsilon \in \mathcal{A}$, $\mu(\Omega \setminus \Omega_\varepsilon) < \varepsilon$, such that the range $f(\Omega_\varepsilon)$ of f on Ω_ε is relatively compact for all $f \in F$. This implies that Ω is the disjoint union $\Omega = N \cup \bigcup_{n \in \mathbb{N}} \Omega_n$ of sets in $\mathcal{A}$, $\mu(N) = 0$, and $f(\Omega_n)$ relatively compact for all $f \in F$ and all $n \in \mathbb{N}$. Using this fact it is easy to see that it is sufficient to prove the theorem under the additional assumption that the range $f(\Omega)$ of f is relatively compact for all $f \in F$.

From the additional assumption together with the separability of F it follows that there exists a countable subalgebra $\mathcal{A}_F$ of $\mathcal{A}$ such that $F \subset \overline{S(\mathcal{A}_F;Y)}^{\|\cdot\|_{\sup}}$, where $S(\mathcal{A}_F;Y)$ denotes the $\mathcal{A}_F$-simple Y-valued functions. Let N be the union of the μ-null sets in $\mathcal{A}_F$, and define $\Omega' := \Omega \setminus N$, $\mathcal{A}'_F := \mathcal{A}_F \cap \Omega'$.

It then follows that there exist a countable set $M \subset [0,1]$, $1 \in M$, and a family $(A_t; t \in M) \subset \mathcal{A}'_F$ such that

(i) $A_t \subset A_s$ for $t, s \in M$, $t \leq s$;

(ii) $\mathcal{A}'_F$ is the algebra generated by $\{A_t; t \in M\}$;

(iii) $\mu(A_t) = t$ for all $t \in M$.

This can be seen by looking at the proof of Carathèodory's theorem in [5; sec. 41, Theorem C, p. 173] or in [8; chap. 15, sec. 2, Theorem 2, p. 321].

Let $\mathcal{B}_1 \subset \mathcal{B}$ be the algebra generated by the intervals $\{[0,t]; t \in M \setminus \{0\}\}$. Then we obtain mappings

$$\varphi' : \overline{S(\mathcal{A}'_F;Y)}^{\|\cdot\|_{\sup}} \to \overline{S(\mathcal{B}_1;Y)}^{\|\cdot\|_{\sup}},$$
$$\varphi'_{\mathbb{R}} : \overline{S(\mathcal{A}'_F;\mathbb{R})}^{\|\cdot\|_{\sup}} \to \overline{S(\mathcal{B}_1;\mathbb{R})}^{\|\cdot\|_{\sup}},$$

which are defined by $\varphi'(y\chi_{A_t}) = y\chi_{[0,t]}$, $\varphi'_{\mathbb{R}}(\chi_{A_t}) = \chi_{[0,t]}$ respectively, for $t \in M \setminus \{0\}$, $y \in Y$, and extension by linearity and continuity. We note that φ'_Y and $\varphi'_{\mathbf{R}}$ are isometric with respect to the sup norm and the essential sup norm, and that integrals are preserved. It is then easy to see that $\varphi : F \to \mathscr{L}_\infty([0,1], \mathcal{B}; Y)$ defined by $\varphi(f) := \varphi'(f|\Omega')$ has the desired properties.

In order to prove (b) we define

$$A'_t := \bigcup_{s\in M, s<t} A_s, \quad A''_t := \bigcap_{s\in M, s>t} A_s,$$

for $t \in [0,1]$. We then define

$$D := \{t \in [0,1]; A'_t \neq A''_t\},$$

and proceed to show $\lambda^*(D) = 1$. We first note the easy equality $\Omega' = \bigcup_{t\in[0,1]} A''_t \setminus A'_t$. Further, the properties (i), (ii), (iii) together with the assumption that the σ-algebra generated by F is atom free implies that M is dense in $[0,1]$, and this in turn implies that $C[0,1]$ is in the range of $\varphi'_{\mathbf{R}}$. Moreover, it is not difficult to show that for $g \in C[0,1]$ the function ${\varphi'}_{\mathbb{R}}^{-1}(g)$ can be obtained as follows: for $w \in \Omega'$ there exists a unique $t \in D$ such that $w \in A''_t \setminus A'_t$, and with this t we have ${\varphi'}_{\mathbb{R}}^{-1}(g)(w) = g(t)$.

In order to show $\lambda^*(D) = 1$ we have to show $\lambda(K) = 0$ for any compact $K \subset [0,1] \setminus D$ Now, given a compact $K \subset [0,1] \setminus D$, there exists a sequence $(g_n) \subset C[0,1]$ such that $g_n \downarrow \chi_K$ pointwise. Then ${\varphi'}_{\mathbf{R}}^{-1}(g_n) \downarrow 0$ everywhere, by the previous paragraph. This implies $\int g_n \mathrm{d}\lambda = \int {\varphi'}_{\mathbb{R}}^{-1}(g_n)\mathrm{d}\mu \to 0$, $\lambda(K) = 0$.

We now define $\psi : D \to \Omega'$ by choosing $\psi(t) \in A''_t \setminus A'_t$, for $t \in D$, and we assert $\varphi(f) = f \circ \psi$ λ^*-a.e. on D. For all $t \in M \setminus \{0\}$ we have $\varphi'_{\mathbf{R}}(\chi_{A_t}) = \chi_{[0,t]}$, and

$$\chi_{A_t} \circ \psi(s) = \begin{cases} 1 & \text{for } s \in D \cap [0,t), \\ 0 & \text{for } s \in D \cap (t,1], \end{cases}$$

which implies $\varphi'_{\mathbb{R}}(\chi_{A_t}) = \chi_{A_t} \circ \psi$ λ^*-a.e. on D. This latter property extends to $S(A'_F; Y)$, and therefore

$$\varphi'(f|\Omega') = f \circ \psi \quad \lambda^* - \text{a.e. on } D$$

for all $f \in \overline{S(A_F; Y)}^{\|\cdot\|_{\sup}} \supset F$. ∎

In the following proposition we single out a further detail of the proof of Theorem 3.4.

Proposition 3.6. *Let X be separable, $E \subset L(X,Y)$ a closed subspace having the convex compactness property for $\mathcal{T}_s$. Let Ω be a compact space and μ a probability Borel measure on Ω. Let $U : \Omega \to E$ be bounded and strongly Borel measurable. Then $\int U \mathrm{d}\mu \in E$.*

Proof. Let $\varepsilon > 0$. Lusin's criterion for measurability together with the separability of X implies the existence of a compact subset $\Omega_\varepsilon \subset \Omega$ such that $\mu(\Omega \setminus \Omega_\varepsilon) \leq \varepsilon$ and such that $U : \Omega_\varepsilon \to E$ is strongly continuous. Then Theorem 0.1 implies $\int_{\Omega_\varepsilon} U \mathrm{d}\mu \in E$. This clearly implies $\int_\Omega U \mathrm{d}\mu \in E$. ∎

Proof of Theorem 3.4. We only have to show that the convex compactness property implies the strong convex compactness property.

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, $U : \Omega \to E$ bounded and strongly measurable. Then $F := \{U(\cdot)x; x \in X\}$ is a separable subspace of $\mathscr{L}_\infty(\Omega, \mathcal{A}; Y)$. Further we may assume that $\mathcal{A}$ is the smallest σ-algebra making all $f \in F$ measurable. Let $\mu = \mu_{\mathrm{d}} + \mu_c$ be the decomposition of μ into its discrete and continuous parts. The fact that E is closed implies $\int U \mathrm{d}\mu_{\mathrm{d}} \in E$, and therefore it remains to show $\int U \mathrm{d}\mu_c \in E$. For the remainder of the proof we may therefore assume $\mu = \mu_c$, i.e., μ is atom free, and we are therefore under the hypotheses of Theorem 3.5.

Let φ, D, ψ be as in the conclusion of Theorem 3.5. We define functions

$$\widetilde{U} : D \to E,$$
$$\widehat{U} : [0,1] \to L(X,Y)$$

as follows

$$\widetilde{U}(t) := U(\psi(t)) \qquad (t \in D),$$
$$\widehat{U}(\cdot)x := \varphi(U(\cdot)x) \qquad (x \in X).$$

Then $\widetilde{U}(t) \in E$ for all $t \in D$. Also, $\widehat{U}(t) \in L(X,Y)$, $\| \widehat{U}(t) \| \leq \sup_{w \in \Omega} \| U(w) \|$ for all $t \in [0,1]$ (but not necessarily $\widehat{U}(t) \in E$). Since $\widetilde{U}(t)x = \widehat{U}(t)x$ λ^*-a.e. on D for all $x \in X$, from Theorem 3.5, the separability of X implies $\widetilde{U}(t) = \widehat{U}(t)$ λ^*-a.e. on D. This shows that $\{t \in [0,1]; \widehat{U}(t) \in E\}$ is a set of full outer measure. This set is also measurable, by the measurability property of E. Replacing $\widehat{U}(t)$ by 0 on the complement of this set has no effect on the strong measurability and on the integral $\int \widehat{U}(t)\mathrm{d}t$. Now Theorem 3.5 implies $\int U \mathrm{d}\mu = \int \widehat{U}(t)\mathrm{d}t$, and the latter belongs to E by Proposition 3.6. ∎

4. ADDITIONAL REMARKS AND EXAMPLES

Remark 4.1. Here we recall the relations between the (metric) convex compactness property and other completeness properties for a locally convex space E.

(a) There are the implications
quasi-complete (= boundedly complete)
$\Rightarrow$ convex compactness property $\Rightarrow$
metric convex compactness property $\Rightarrow$
Mackey complete (= locally complete).

The last implication follows from the fact that Mackey completeness is equivalent to the property that the closed convex hull of any convergent sequence is compact (cf. [4; Théorème 1]).

All the implications are strict. For the third this follows from [13; Example 4.6.110, p. 244). For the second see (b) below.

(b) One also has the implications
quasi-complete $\Rightarrow$ sequentially complete
$\Rightarrow$ metric convex compactness property.

The last implication is a consequence of Theorem 0.1. Also, the convex compactness property and sequential completeness are incomparable (cf. [13]). This implies in particular that the last implication above and the second implication of (a) are strict.

(c) In connection with Theorem 0.1 we mention that the metric convex compactness property is also equivalent to every continuous function $f : [0,1] \to E$ is Pettis-integrable for the Lebesgue measure (cf. [12]).

Example 4.2. We present an example showing that there are closed subspaces of $L(X, Y)$ not possessing the convex compactness property or the measurability property, respectively.

For $t \in [0,1]$ let $U(t) \in L(L_1(\mathbb{R}))$ be defined by

$$U(t)f(x) = f(x-t).$$

By $M([0,1])$ we denote the (signed) Borel measures on $[0,1]$. We define a mapping $V : M([0,1]) \to L(L_1(\mathbb{R}))$ by

$$V(\mu) := \int U(t) \, \mathrm{d}\mu(t) \quad \text{(strong integral)}.$$

Then V is isometric: (if $\mu \in M([0,1])$, and $(f_n) \subset L_1(\mathbb{R})$ satisfies $f_n \geq 0$, $\operatorname{supp} f_n \subset (-1/n, 1/n)$, $\| f_n \| = 1$, then, for all $g \in C_c(\mathbb{R})$,

$$\int \left(\int U(t) f_n \mathrm{d}\mu(t) \right) (x) g(x) \, \mathrm{d}x \to \int g(x) \, \mathrm{d}\mu(x).$$

(a) Let $E := V(\ell_1([0,1]))$, where we identify $\ell_1([0,1])$ with the discrete measures on $[0,1]$. Then $U : [0,1] \to E$ defined above is strongly continuous, but $\int U(t)\,dt = V(\lambda)$ (λ Lebesgue measure on $[0,1]$) has distance one from E. Therefore E does not have the metric convex compactness property.
(b) For $A \subset [0,1]$ let

$$E_A := \{V(\alpha);\, \alpha \in \ell_1([0,1]), \alpha_t = 0 \quad \text{for all} \quad t \in [0,1] \setminus A\}.$$

Then E_A is a closed subspace of $L(L_1(\mathbb{R}))$. If A is not a measurable subset of $[0,1]$, then the mapping U shows that E_A does not have the measurability property (cf. Definition 3.2).

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