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A NOTE ON DISCRETELY COMPACT OPERATORS

HANS-JÜRGEN REINHARDT

Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

It is the aim of this note to prove an important result in the framework of discrete approximations and discrete convergence. The underlying general perturbation theory applies to sequences of linear and nonlinear operators and solutions of operator equations. The theory was originally developed by Stummel [23-31] (1) with contributions by Grigorieff [6-10], Rannacher [18], Wolf [38, 39] and the author [19-22] (2). Similar approaches are used by Vainikko [32-37] and, in special cases, by Anselone [1, 2], Aubin [3], Browder [4, 5], Petryshyn [15-16].

At the time the monograph [22] appeared, the theory of discrete convergence was completed. According to the emphasis of the book [22] on nonlinear mappings and applications, not every aspect of the theory is contained in [22], e.g. not the perturbation of eigenvalue problems, not the perturbation of Sobolev spaces and no results on weak convergence. There are a few newer publications known to the author which use the theory of discrete convergence for special problem settings (see e.g. Niepage [13, 14]). The result of the present contribution is not yet published; it is interesting by itself and has importance with respect to applications.

The main result of this paper, Theorem 3, states the equivalence of weak discrete compactness and discrete compactness of not necessarily linear operatos in a special setting of subspaces. Before, it is shown that discrete compactness implies weak discrete compactness in a general setting (see Theorem 1) and boundedness properties are shown (see Theorem 2). Compactness properties are important for the existence of solutions of operator equations and their convergence - for examples and applications see [1, 2, 7-10, 15, 22, 23, 28, 31, 33, 34, 36].

2. NOTATIONS

Let $E, F, E_n, F_n, n \in \mathbf{N}$, be normed linear spaces which form discrete approximations $\mathscr{A}(E, \prod_n, E_n \lim^E)$ and $\mathscr{A}(F, \prod_n F_n, \lim^F)$ in the sense of [22], 5.1. Discretely convergent sequences are denoted by $\lim^E u_n = u$ or $u_n \to u (n \in N)$; analogous notations are

(1) The references mentioned are important papers of the corresponding authors but give by no means a complete list of their works in that field. A more complete list is contained in the bibliography of [22].

(2) The author is especially grateful to Professor G. Köthe who communicated his first publication [19] in a mathematical journal.

used for sequences $w \in F, w_n \in F_n, n \in \mathbb{N}$. A sequence of continuous linear functionals v_n on E_n is called *weakly discretely convergent* to $v \in E'$ if $\lim_{n\to\infty} \langle v_n, u_n \rangle = \langle v, u \rangle$ for all discretely convergent sequences $u_n \to u(n \in \mathbb{N})$ (compare [23], 2). We write $v_n \rightharpoonup v(n \in \mathbb{N})$ or $w - \lim v_n = v$. A sequence of elements $u_n \in E_n, n \in \mathbb{N}$, is called *discretely compact* if for every subsequence $\mathbb{N}' \subset \mathbb{N}$ there exist an element $u \in E$ and a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that the discrete convergence $u_n \to u(n \in \mathbb{N}'')$ holds. A sequence of continuous linear functionals $v_n \in E_n', n \in \mathbb{N}$, is called *weakly discretely compact* if for every $\mathbb{N}' \subset \mathbb{N}$ there exist a $v \in E'$ and $\mathbb{N}'' \subset \mathbb{N}'$ such that $v_n \rightharpoonup v(n \in \mathbb{N}'')$.

The spaces of continuous linear functionals on E will be denoted by E'; the same notation will be used for E_n, F, F_n. The value of a functional v at a certain element u will be denoted by $\langle u, v \rangle$ instead of writing $v(u)$. The definition of weakly discrete convergence and weakly discrete compactness also applies to functionals on F_n.

Let us further consider not necessarily linear operators $K_n : D(K_n) \subset E_n \to F_n, n \in \mathbb{N}$, with domain of definition $D(K_n)$; the range of the operators lies in F_n. If the domain of definition is all of E_n, we also write $K_n : E_n \to F_n$.

The sequence $K_n, n \in \mathbb{N}$, is called *discretely compact* if for every bounded sequence of elements $u_n \in E_n, n \in \mathbb{N}$, the sequence of images $K_n u_n$, $n \in \mathbb{N}$, is discretely compact. Moreover, K_n, $n \in \mathbb{N}$, is called *weakly discretely compact* if for every weakly discretely convergent null sequence of functionals $v_n \in F_n', n \in \mathbb{N}$, i.e. $v_n \rightharpoonup 0(n \in \mathbb{N})$, and for every bounded sequence $u_n \in E_n, n \in \mathbb{N}$, the relation $\lim_{n\to\infty} \langle K_n u_n, v_n \rangle = 0$ holds.

3. PROPERTIES OF DISCRETE COMPACT OPERATORS

In analogy to the case of linear operators (see [23], Thm. 3.1 (1)), the following result holds.

Theorem 1. *Every discretely compact sequence of operators $K_n, n \in \mathbb{N}$, is weakly discretely compact.*

Proof. Let us assume that a discretely compact sequence $K_n, n \in \mathbb{N}$, is not weakly discretely compact. Then there exist a weakly discretely convergent null sequence of functionals $v_n \in F_n', n \in \mathbb{N}, v_n \rightharpoonup 0(n \in \mathbb{N})$, and a bounded sequence $u_n \in E_n, n \in \mathbb{N}$, such that for a positive number $\varepsilon_0 > 0$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ the estimates $|\langle K_n u_n, v_n \rangle| \geq \varepsilon_0, n \in \mathbb{N}'$, hold. According to the assumption, there exists another subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and a $w \in F$ such that $K_n u_n \to w(n \in \mathbb{N}'')$. Together with $v_n \rightharpoonup 0(n \in \mathbb{N})$, we also have $v_n \rightharpoonup 0(n \in \mathbb{N}'')$. The definition of weakly discrete convergence hence implies that $\langle K_n u_n, v_n \rangle \to 0(n \to \infty, n \in \mathbb{N}'')$ which contradicts $|\langle K_n u_n, v_n \rangle| \geq \varepsilon_0, n \in \mathbb{N}'$. ∎

For a sequence of bounded linear operators K_n, the weakly discrete compactness is obviously equivalent to the fact that the sequence of adjoint operators $K_n' : F_n' \to E_n', n \in \mathbb{N}$, satisfies $||K'_n v_n||_{E'_n} \to 0 (n \to \infty)$ for every discretely convergent null sequence $v_n \to 0 (n \in \mathbb{N}), v_n \in F'_n$.

Also in the case of not necessarily linear operators, weakly discretely compact sequences have the following property.

Theorem 2. *For any weakly discretely compact sequence of operators $K_n : D(K_n) \subset E_n \to F_n, n \in \mathbb{N}$, and every bounded sequence of elements $u_n \in E_n, n \in \mathbb{N}$, the sequence of images $K_n u_n, n \in \mathbb{N}$, is bounded.*

Proof. Let us assume that the sequence of images $K_n u_n, n \in \mathbb{N}$, is not bounded. Then there exists a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $0 < \alpha_n = ||K_n u_n||_{F_n} \to \infty (n \in \mathbb{N}')$. According to the theorem on the existence of sufficiently many functionals (see e.g. [11], V. 7), there exist $v_n \in F'_n, n \in \mathbb{N}'$, such that

$$||v_n||_{F'_n} = 1 \qquad \text{and} \qquad \langle K_n u_n, v_n \rangle = ||K_n u_n||_{F_n}, n \in \mathbb{N}'.$$

The sequence $u_n \in E_n, n \in \mathbb{N}$, is assumed to be bounded. We define

$$\overline{v}_n = \frac{1}{\sqrt{\alpha_n}} v_n, n \in \mathbb{N}', \qquad \overline{v}_n = 0, n \in \mathbb{N} - \mathbb{N}'.$$

Then $\overline{v}_n \to 0 (n \in \mathbb{N})$ since

$$||\overline{v}_n||_{F'_n} \leq \frac{1}{\sqrt{\alpha_n}} ||v_n||_{F'_n} = \frac{1}{\sqrt{\alpha_n}} \to 0 (n \in \mathbb{N}'), \quad ||\overline{v}_n||_{F'_n} = 0, n \in \mathbb{N} - \mathbb{N}'.$$

Furthermore the convergence

$$\langle K_n u_n, \overline{v}_n \rangle = \frac{1}{\sqrt{\alpha_n}} \langle K_n u_n, v_n \rangle = \sqrt{\alpha_n} \to \infty (n \in \mathbb{N}')$$

holds which contradicts the weakly discrete compactness of $K_n, n \in \mathbb{N}$. ∎

4. THE EQUIVALENCE THEOREM

It will now be our aim to prove the converse of Theorem 1. For this purpose, certain requirements have to be posed for the spaces $F, F_n, n \in \mathbb{N}$. We assume in the following that F and $F_n, n \in \mathbb{N}$, are subspaces of a separable Banach space N which form a metric discrete approximation $\mathscr{A}(F, \prod_n, F_n \lim^F)$ with the norm convergence in N as the discrete convergence. The existence of such a discrete approximation is equivalent to the fact that (see [23], 5.1)

$$(1) \qquad |w, F_n| := \inf_{\psi \in F_n} \|w - \psi\|_N \to 0(n \to \infty), \qquad w \in F.$$

Using the notations of [27], 4., relation (1) is further equivalent to

$$(2) \qquad F \subseteq \liminf_{n \in \mathbb{N}} F_n.$$

At this place it should be mentioned that in a discrete approximation of a separable normed space bounded sequences of functionals are always weakly discretely compact (see [23], Thm. 2.3 (1)). Moreover, discrete approximations of separable normed spaces fulfil Property (A4) of [23], 2.1, which means that discrete convergent null sequences can be characterized by the fact that their images under every weakly discretely convergent sequence of functionals converge to zero.

The proof of the equivalence of weakly discrete compactness of a sequence of operators and discrete compactness itself uses the following Isometry Theorem of Banach-Mazur (cf. e.g. [12], 21.3).

Lemma 1. *Each separable normed linear space M is linearly isometric to a linear subset of the space $C[0,1]$ of continuous functions.*

Denoting the isometry by G then $G(M)$ is closed in $C[0,1]$ provided that M is complete in addition.

By means of the isometry G_0 of the Banach space $N, G_0 : N \to C[0,1]$, we define mappings H, H_n by restrictions of G_0 to F and F_n, respectively,

$$(3) \qquad H = G_0|F, \qquad H_n = G_0|F_n, \qquad n \in \mathbb{N}.$$

Obviously, one obtains $\|G_0\| = \|H\| = \|H_n\| = 1, n \in \mathbb{N}$, for the norms of the mappings. Hence, the sequence of mappings $H_n, n \in \mathbb{N}$, is stable and inversely stable (s. [23], 1.2, 1.3). Furthermore, according to (1) the sequence $H, H_n, n \in \mathbb{N}$, is consistent.

In addition to the condition (1) required for the subspaces F, F_n, we assume that every limit of a convergent subsequence of elements $y_n \in F_n, n \in \mathbb{N}' \subset \mathbb{N}$, lies in F, which can be written as

$$(4) \qquad \limsup_{n \in \mathbb{N}} F_n \subset F.$$

Using the closedness of F as a subspace of N, it is not difficult to see that in the present setting of subspaces condition (4) is equivalent to

$$(5) \qquad |y_n, F| \to 0(n \to \infty, n \in \mathbb{N}')$$

for every convergent subsequence of elements $y_n \in F_n, n \in \mathbb{N}' \subset \mathbb{N}$.

To be prepared for the equivalence theorem we prove an additional lemma.

Lemma 2. *Let the mappings $H, H_n, n \in \mathbf{N}$, be defined by (3) and let the assumptions (2), (4) hold. Then for every subsequence $\mathbf{N}' \subset \mathbf{N}$ and arbitrary sequences of scalars $(\alpha_n)_{n\in\mathbf{N}'}, (\beta_n)_{n\in\mathbf{N}'}$ from $[0,1]$ fulfilling $|\alpha_n - \beta_n| \to 0 (n \to \infty, n \in \mathbf{N}')$, the sequence of functionals $v_n \in F_n'$, $n \in \mathbf{N}$, defined by*

$$\langle w_n, v_n\rangle = H_n w_n(\alpha_n) - H_n w_n(\beta_n), w_n \in F_n, n \in \mathbf{N}', \tag{6}$$

is a weakly discretely convergent null sequence, $v_n \rightharpoonup 0 (n \in \mathbf{N}')$.

Proof. Let $\mathbf{N}'$ be an arbitrary subset of infinitely many integers of $\mathbf{N}$. The functionals defined in (6) are obviously linear, and they are bounded because of the estimates

$$|\langle v_n, w_n\rangle| \leq |H_n w_n(\alpha_n)| + |H_n w_n(\beta_n)| \leq 2\|w_n\|_N, w_n \in F_n, n \in \mathbf{N}'.$$

For any discretely convergent sequence $w_n \to w (n \in \mathbf{N}), w \in F, w_n \in F_n, n \in \mathbf{N}'$, the following estimates hold,

$$\begin{aligned} |H_n w_n(\alpha_n) - H_n w_n(\beta_n)| &\leq |H_n w_n(\alpha_n) - H w(\alpha_n)| + \\ &+ |H w(\alpha_n) - H w(\beta_n)| + |H w(\beta_n) - H_n w_n(\beta_n)| \\ &\leq 2\|Hw - H_n w_n\|_{C[0,1]} + |Hw(\alpha_n) - Hw(\beta_n)|, n \in \mathbf{N}'. \end{aligned}$$

Since the sequence $w_n \to w (n \in \mathbf{N}')$ converges, and consistency together with stability of $H, H_n, n \in \mathbf{N}$, hold, the discrete convergence $H_n \to H (n \in \mathbf{N}')$ follows. In this special setting, this is equivalent to

$$\|Hw - H_n w_n\|_{C[0,1]} \to 0 (n \to \infty, n \in \mathbf{N}').$$

Using the uniform continuity of $Hw(\cdot)$, one additionally has

$$|Hw(\alpha_n) - Hw(\beta_n)| \to 0 (n \to \infty, n \in \mathbf{N}'),$$

which proves $|\langle w_n, v_n\rangle| \to 0 (n \in \mathbf{N}')$ and $v_n \rightharpoonup 0 (n \in \mathbf{N}')$. ∎

We are now in the position to prove the equivalence theorem already mentioned.

Theorem 3. *A discretely compact sequence of operators $K_n : D(K_n) \subset E_n \to F_n, n \in \mathbb{N}$, is necessarily weakly discretely compact. Conversely, if the $F, F_n, n \in \mathbb{N}$, are subspaces of a separable Banach space N and fulfill the condition*

$$(7) \qquad F = \lim_{n \in \mathbb{N}} F_n, \qquad i.e. \qquad F = \liminf_{n \in \mathbb{N}} F_n = \limsup_{n \in \mathbb{N}} F_n,$$

then the weakly discrete compactness of (K_n) is also sufficient for its discrete compactness.

Proof. (a) The necessity is already proven in Theorem 1.

(b) Sufficiency. Let $u_n \in E_n, n \in \mathbb{N}$, be a bounded sequence. Using $\liminf F_n \subset \limsup F_n$, condition (2) together with (4) is obviously equivalent to (7). A metric discrete approximation $\mathscr{A}(F, \prod_n F_n, \lim^F)$ thus exists and it remains to show that the sequence $K_n u_n, n \in \mathbb{N}$, is compact in N with cluster points in F. According to Theorem 2, $(K_n u_n)$ is bounded.

By means of the theorem of Arzelà and Ascoli, we now prove that the sequence $\varphi_n = H_n K_n u_n, n \in \mathbb{N}$, of continuous functions is compact in $C[0,1]$. The boundedness of (φ_n) follows from the boundedness of $(K_n u_n)$ and $\|H_n\| = 1$. Let us assume that the functions $\varphi_n, n \in \mathbb{N}$, are not equicontinuous. Then there exists a number $\varepsilon_0 > 0$ such that for every null sequence of positive numbers $\varepsilon_n, n \in \mathbb{N}$, and any $n \in \mathbb{N}$ an index $\nu = \nu_n \in \mathbb{N}$ and numbers $\alpha_n, \beta_n \in [0,1]$ exist such that

$$|\alpha_n - \beta_n| \le \varepsilon_n \qquad |H_\nu K_\nu u_\nu(\alpha_n) - H_\nu K_\nu u_\nu(\beta_n)| \ge \varepsilon_0 .$$

The set $\{\nu_n, n \in \mathbb{N}\}$ may consist of finitely many indices or, otherwise, it contains infinitely many pairwise distinct positive integers. In the first case, there exists an index $\nu_0 \in \mathbb{N}$ and a subset of countably many integers $I \subseteq \mathbb{N}$ such that $\nu_n = \nu_0, n \in I$. According to $|\alpha_n - \beta_n| \to 0(n \to \infty, n \in I)$ and the uniform continuity of $\varphi_{\nu_0} = H_{\nu_0} K_{\nu_0} u_{\nu_0}$, the convergence

$$|\varphi_{\nu_0}(\alpha_n) - \varphi_{\nu_0}(\beta_n)| \to 0(n \to \infty, n \in I)$$

holds in contraction to the inequality above. In the alternating case, there exist a subset I' of countably many integers such that $\mathbb{N}' = \{\nu_n, n \in I'\}$ is a subset of infinitely many elements of $\mathbb{N}$. Setting $\alpha_n = \alpha_{\nu_n}, \beta_n = \beta_{\nu_n}, n \in I'$, then Lemma 2 assures that the sequence of functionals $v_k \in F_k', k \in \mathbb{N}'$, defined by

$$\langle w_k, v_k \rangle = H_k w_k(\alpha_k) - H_k w_k(\beta_k), k \in \mathbb{N}',$$

is a weakly discretely convergent null sequence, $v_k \rightharpoonup 0(k \in \mathbb{N}')$. Using the assumption of discrete compactness of (K_n), the convergence $\langle K_k u_k, v_k \rangle \to 0(k \in \mathbb{N}')$ holds. Finally, by the definition of v_k, this means that

$$|\langle K_k u_k, v_k \rangle| = |H_k K_k u_k(\alpha_k) - H_k K_k u_k(\beta_k)| \to 0(k \to \infty, k \in \mathbb{N}')$$

contracting the inequality

$$|H_k K_k u_k(\alpha_k) - H_k K_k u_k(\beta_k)| \geq \varepsilon_0 , k \in \mathbf{N}',$$

and thus proving the equicontinuity of $\varphi_n = H_n K_n u_n , n \in \mathbf{N}$.

The compactness of (φ_n) in $C[0,1]$ is therefore assured. Using the completeness of $N, G_0(N)$ is a closed subspace of $C[0,1]$ and the cluster points of (φ_n) necessarily lie in $G_0(N)$. Hence, also $K_n u_n , n \in \mathbf{N}$, is a compact sequence in N. The cluster points of $(K_n u_n)$ belong to F as a consequence of (4) which completes the proof of the discrete compactness of (K_n). ∎

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H.-J. Reinhardt
Fachbereich 6 Mathematik
Universität - Gesamthochschule - Siegen
5900 Siegen
Germany