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FIRST-ORDER QUOJECTIONS

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION AND NOTATION

A *quojection* is a Fréchet space F which is the projective limit of a sequence of Banach spaces X_n and surjective maps $R_n : X_{n+1} \to X_n$.

We write $F = \text{quoj}_n(X_n, R_n)$. Quojections have been introduced in [1] to characterize those Fréchet spaces admitting a nuclear Köthe quotient and, since then, they have been extensively studied by several authors (cf. our survey [9]). Here we just recall that every quojection is a quotient of a countable product of Banach spaces by a quojection subspace (cf. [3] and [10]).

A quojection is *twisted* if it is not isomorphic to a (countable) product of Banach spaces. Examples of twisted quojections were firstly exhibited in [12] by the second author and we term *standard* those quojections constructed by using the method in [12]. Standard quojections have interesting properties (cf. [8] and [10]) and, apart from products, have in some way the simplest topological structure among quojections: they are quotients of products by Banach subspaces. Motivated by this situation we introduce here what we call *quojection of the first order,* that is quojections which can be represented as quotients of countable products of Banach spaces by Banach subspaces. We show that most of the properties of standard quojections still hold for first-order quojections.

In section 2 we characterize first-order quojections as those quojections admitting a continuous seminorm p such that ker p is a product; we give also some examples of first-order and non-first-order quojections.

In section 3 we study when a first-order quojection is twisted and we find that such a quojection $F = \frac{P}{B}$, where P is a product and B Banach, is a product if and only if there exists a Banach subspace Z of P which contains B and is complemented in P.

In section 4 we prove that twisted first-order quojections are never complemented in any product of Banach spaces and we give new examples of twisted quojections. We study also a special exact sequence of quojections.

Finally, in section 5, we deal with prequojections and we show that the results of [11] for prequojections associated to standard quojections are still true for prequojections associated to first-order quojections.

Our notation is standard (see e.g. [5]) and, throughout this paper, product will always mean «countable product of Banach spaces». We recall the construction of standard quojections

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(cf. [8] or [12]). Let X_n be Banach spaces, let Y_n be closed subspaces of X_n and let L be a normal Banach sequence space. We form the step-spaces

$$F_1 = \left(\bigoplus \frac{X_n}{Y_n}\right)_L \quad \text{and} \quad F_{k+1} = \left[\left(\bigoplus_{n\leq k} X_n\right) \bigoplus \left(\bigoplus_{n>k} \frac{X_n}{Y_n}\right)\right]_L ;$$

if $R_k : F_{k+1} \to F_k$ is the canonical quotient map, then the *standard quojection* $Q((X_n), (\frac{X_n}{Y_n}); L)$ is the space $\text{quoj}_k(F_k, R_k)$ and we know by [9] or [12] that such a quojection is twisted if and only if Y_n is not complemented in X_n for infinitely many n. A *twisted* (LB)-space is a strict (LB)-space which is not isomorphic to a countable direct sum of Banach spaces.

2. FIRST-ORDER QUOJECTIONS

Definition 2.1. *F is a first-order quojection if there exists a product P and a Banach subspace B of P such that F is isomorphic to $\frac{P}{B}$.*

As shown in [10], Corollary 3.1, every standard quojection is a first-order quojection. The following theorem clarifies the topological structure of first-order quojections.

Theorem 2.2. *Let F be a quojection. The following assertions are equivalent:*

(i) *F is a first-order quojection.*

(ii) *There exists an exact sequence $0 \to P \to F \to B \to 0$, where P is a product and B is a Banach space.*

(iii) *There exists a continuous seminorm p on F such that* ker p *is a product.*

(iv) *There exists a fundamental family (p_n) of seminorms on F such that* ker p_n *is a product for evey n.*

Proof. Using the fact that every quotient with a continuous norm of a quojection is a Banach space it is easy to see that (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv).

Let's suppose that (iii) holds and put $P = \ker p$. Then $Y = \frac{F}{P}$ is a Banach space when endowed with the quotient topology from F. Let $q : F :\to Y$ be the quotient map and let C be a closed, bounded, convex body in F such that $q(C)$ contains the ball of Y.

Let $X =$ span C with the Banach space topology generated by the Minkowski functional of C. Then we have a map:

$$r : X \times P \to F, \qquad r(x, z) = x - z,$$

which is continuous and surjective. Since $\ker r = \{(x, x) : x \in X \bigcap P\} = B \subset X \times P$ and since the topology of X is stronger then the topology induced by F, it follows that B is Banach and $F = \frac{X \times P}{B}$ is of the first-order.

Conversely, if F is isomorphic to $\frac{P}{B}$, with P a product and B Banach, let us take a continuous seminorm p of P which induces on B its Banach space topology. Of course, ker $p \bigcap B = \{0\}$; we prove that ker $p + B$ is closed in P. If $z_n = x_n + b_n$ converges to zero with $p(x_n) = 0$ and $b_n \in B$, we have $0 = \lim_n p(z_n) = \lim_n p(b_n)$. But then b_n converges to 0 in P and hence so does (x_n). This shows that ker $p + B$ is closed in P so that the quotient map $q : P \to \frac{P}{B}$ is an isomorphism on ker p.

Let $\overline{p}$ be the quotient seminorm induced by p on $\frac{P}{B}$. If $z \in$ ker $\overline{p}$, there exists a sequence $(u_n) \subset P$ such that $q(u_n) = z$ and $p(u_n) < \frac{1}{n}$. Then $u_n - u_m \in B$ and $p(u_n - u_m) < \frac{1}{n} + \frac{1}{m}$. Since B is Banach and p induces on B its own topology, it follows that (u_n) converges to a point u of P so that $p(u) = 0$ and $q(u) = 0$. This shows that ker $\overline{p} = q(\ker p)$ and the proof is complete, since ker p is a product (because P is so) and q is an isomorphism from ker p onto ker $\overline{p}$.

Corollary 2.3. *A quotient of a first-order quojection by a Banach subspace is again a first-order quojection. The bidual of a first-order quojection is a first-order quojection.*

Examples 2.4.

1) As already mentioned before, every standard quojection is a first-order quojection.

2) Let (F_k) be a sequence of standard quojections such that F_k is twisted for infinitely many k. $\prod_k F_k$ is not a first-order quojection. To see this let us suppose, arguing by contradiction, that there exists a seminorm p on $\prod_k F_k$ such that ker p is a product. Then, there exists n such that $\prod_{k>n} F_k \subset$ ker p and hence is complemented in ker p. But, in this case, we can find a twisted F_m, $m > n$, complemented in ker p. This is impossible since no standard twisted quojection can be complemented in a product (cf. [8]). The contradiction shows that $\prod_k F_k$ is not of the first order.

3) We recall the construction of twisted quojections given in [7]. Let X be a Banach space and Y a closed, non-pseudo-complemented subspace of X. If (Y_n) is an increasing sequence of subspaces of X containing Y and such that dim $\frac{Y_n}{Y_{n-1}} = \infty$ then $E = s-$ ind$_n Y_n$ is a twisted (LB)-space. If X is reflexive, $F = E'$ is a twisted quojection. Now we show with an example that such twisted quojections can be of the first order. Let $X = L^r$, $r > 2$, and let $q : L^r \to l^p$ be a quotient map, $2 < p < r$ (see [6]). By a classical result of Kadec and Pełczyński q is strictly singular and hence $Y =$ ker q is not pseudo-complemented in X. Write $l^p = (\bigoplus_n l^p)_p$ and let $r_k : (\bigoplus_n l^p)_p \to (\bigoplus_{n\geq k} l^p)_p$ be the canonical projection. If $Y_k =$ ker $r_k q$, then we have that $Y \subset Y_k$, dim $\frac{Y_{k+1}}{Y_k} = \infty$ and $\frac{Y_k}{Y}$ is complemented in $\frac{X}{Y}$, hence in $\frac{Y_{k+1}}{Y}$. Let $E = s-$ ind$_n Y_n$. E is twisted and contains B Banach such that

$$\frac{E}{B} = s- \text{ ind}_n \frac{Y_n}{B} \simeq (l^p)^{(N)} .$$

Since E is reflexive this means, by duality and using Theorem 2.2, that $F = E'$ is a twisted first-order quojection.

4) Let (p_n) be a strictly increasing sequence of real numbers with $p_1 > 2$ and let $R_n : L^{p_{n+1}}(0,1) \to L^{p_n}(0,1)$ be quotient maps (cf. [6]). The quojection $F = \operatorname{quoj}_n(L^{p_n}(0,1), R_n)$ is twisted and if it contains a Banach subspace, this can only be isomorphic to l^2 ([8], Prop. 2.1). We show now that F is not a first-order quojection. In fact, let us suppose by contradiction that F is of the first order. Then Theorem 2.2 ensures the existence of a continuous seminorm p on F such that ker p is a product and hence ker $p \simeq (l^2)^N$. Let $S_n : F \to L^{p_n}(0,1)$ be the quotient map. ker S_n is a quojection for every n and there exists k such that ker $S_k \subset$ ker p. But then ker S_k is a quojection subspace of $(l^2)^N$ and it is not difficult to see that in fact ker S_k must be isomorphic to $(l^2)^N$. Let $Z = \ker R_k \subset \subset L^{p_{k+1}}(0,1)$. Z is a quotient of ker S_k and hence Z must be isomorphic to l^2. The classical result of Kadec and Pełczyński about l^2-copies in L^p spaces implies that Z is complemented in $L^{p_{k+1}}(0,1)$ and so we may write, since R_k is surjective, $L^{p_{k+1}}(0,1) = Z \bigoplus L^{p_k}(0,1)$. Since the latter space is isomorphic to $L^{p_k}(0,1)$ we have a contradiction and hence F cannot be a first-order quojection.

Other examples of first-order quojections will be given in the next section.

Remark 2.5. With reference to Corollary 2.3 we note that the bidual F'' of a quojection F may be of the first order without F being so. For example, let E be a standard twisted quojection such that E'' is a product (such an E exists by [8]). If $F = E^N$, then F is not first-order by Example 2.4.2 while, of course, F'' is.

3. TWISTED FIRST-ORDER QUOJECTIONS

We start this section with a lemma which describes Banach subspaces and complemented Banach subspaces of products.

Lemma 3.1. *Let $P = \prod_n X_n$ be a product and B be a Banach subspace of P. Then there exists $k \in \mathbf{N}$, a closed subspace Y of $\prod_{i=1}^k X_i$ and a sequence $(T_n : n > k)$ of continuous linear maps $T_n : Y \to X_n$ such that*

$$(*) \qquad B = \{(y_n) : y_n = T_n(y_1, \ldots, y_k) \quad \text{for} \quad n > k \quad \text{and} \quad (y_1, \ldots, y_k) \in Y\}.$$

If, in addition, B is complemented in P we can choose $k \in \mathbf{N}$ such that Y is complemented in $\prod_{i=1}^k X_i$.

Proof. Since B is Banach, there exists $k \in \mathbf{N}$ such that

$$\sum_{j=1}^{n} \| x_j \| \leq c_n \sum_{j=1}^{k} \| x_j \|, \quad c_n > 0, \quad \text{for all} \quad n > k \quad \text{and} \quad x = (x_j) \in B.$$

Let $r : P \to \prod_{i=1}^{k} X_i$ be the canonical projection and $Y = r(B)$. Y is Banach and for $(x_j) \in B$ we have

$$\| x_n \| = \| r_n\left(x_j\right) \| \leq c_n \| r\left(x_j\right) \| = c_n \| y \|$$

where $r_n : P \to X_n$ is the canonical projection and $y = r(x_j)$. This yields the existence of linear maps $T_n : Y \to X_n$ with $\| T_n y \| \leq c_n \| y \|$ so that B is of the form indicated above.

If B is complemented in P, let $S : P \to B$ be a projection. There exists $k \in \mathbb{N}$ such that we have the following factorization

$$\begin{array}{ccc} P & \xrightarrow{S} & B \\ & \searrow r \quad \nearrow \widetilde{S} & \\ & \prod_{i=1}^{k} X_i & \end{array}$$

from which we obtain again that $Y = r(B)$, that $r\widetilde{S}$ is a projection from $\prod_{i=1}^{k} X_i$ onto Y and that B is in the form (*), the existence of the maps T_n being proved as in the first part.

It is clear that the conditions of the above lemma are also sufficient, that is a subspace B of the form (*) is a Banach subspace of P (isomorphic to Y), complemented if Y is so. Now we can state the main result of this section:

Theorem 3.2. *A first-order quojection $F = \frac{P}{B}$ is isomorphic to a product if and only if there exists a Banach subspace Z of P which is complemented in P and contains B.*

Proof. Sufficiency: if such a subspace Z exists we can write $P = Z \bigoplus G$ and, since $Z = \frac{P}{G}$, by [10] Prop. 3.5 we obtain that G is a product. But then it follows that

$$F = \frac{P}{B} = \frac{Z}{B} \bigoplus G \quad \text{is a product}.$$

Necessity: let's write $P = \prod_i X_i$ and B in the form (*). We may suppose that $Y \subset X_1$ and, for $j > 1$, we introduce the spaces $Y_j \subset \prod_{i=1}^{j} X_i$ defined by $Y_j = \{(x_1, \ldots, x_j) : x_i = T_i x_1, \text{ for } 1 < i \leq j,\ x_1 \in Y\}$.

Let

$$R_n : \frac{\prod_{i=1}^{n+1} X_i}{Y_{n+1}} \to \frac{\prod_{i=1}^{n} X_i}{Y_n}$$

be the quotient map induced by the canonical projection of $\prod_{i=1}^{n+1} X_i$ onto $\prod_{i=1}^{n} X_i$ so that we have

$$F = \frac{P}{B} = \mathrm{quoj}_n\left(\frac{\prod_{i=1}^{n} X_i}{Y_n}, R_n\right).$$

Let $P_n : \prod_i X_i \to \prod_{i=1}^{n} X_i$ be the canonical projection and $Q_n : \frac{P}{B} \to \frac{\prod_{i=1}^{n} X_i}{Y_n}$ the quotient map induced by P_n. If F is not twisted we have also a representation $F =$ $= \mathrm{quoj}_m(H_m, S_m)$ where $S_m : H_{m+1} \to H_m$ is surjective, H_m is Banach and ker S_m is complemented in H_{m+1} for every m. We may assume that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\frac{X_1}{Y} & & \overset{R_{1j}}{\longleftarrow} & & \frac{\prod_{i=1}^{j} X_i}{Y_j} & & \overset{Q_j}{\longleftarrow} & & F \\
& \nwarrow & & L \swarrow & & \nwarrow N & & \swarrow \tilde{Q}_2 & \\
& & H_1 & & \overset{S_1}{\longleftarrow} & & H_2 & &
\end{array}$$

where all the maps are surjective, $R_{1j} = R_1 R_2 \ldots R_{j-1}$, $\tilde{Q}_2$ is the quotient map of F onto H_2 and j is a suitable integer, $j > 1$. The main point of the proof is to show that Q_j has a right-inverse; i.e. ker Q_j is complemented in F.

From the diagram it follows that there is a Banach space M with $Y_j \subset M \subset Y \times \prod_{i=2}^{j} X_i$ such that $H_1 = \frac{\prod_{i=1}^{j} X_i}{M}$. Since $Y_j = \{(y, T_2 y, \ldots, T_j y) : y \in Y\}$ it is easy to see that $M = Y_j \bigoplus U$ with $U \subset \prod_{i=1}^{j} X_i$. Let $u : H_1 \to H_2$ be a right-inverse of S_1. Then $A = Nu : H_1 \to \frac{\prod_{i=2}^{j} X_i}{Y_j}$ is a right-inverse of L and so we have

$$\frac{\prod_{i=1}^{j} X_i}{Y_j} = A(H_1) \bigoplus \ker L = A(H_1) \bigoplus \frac{M}{Y_j}.$$

Let's consider now $U \subset \prod_{i=2}^{j} X_i \subset \prod_i X_i$. We have that $U \bigcap B = \{0\}$ and $U + B$ is closed in $\prod_i X_i$ as is easily seen recalling the form (*) of B. In the same way $U \bigcap Y_j = \{0\}$ and $U + Y_j$ is closed in $\prod_{i=1}^{j} X_i$. This means exactly that the quotient map $q : P \to \frac{P}{B}$ is an isomorphism of U onto $\frac{U+B}{B}$ and the quotient map $q_j : \prod_{i=1}^{j} X_i \to \frac{\prod_{i=1}^{j} X_i}{Y_j}$ is an isomorphism from U onto $\frac{U+Y_j}{Y_j}$.

In turn this implies that the quotient map $Q_j : \frac{P}{B} \to \frac{\prod_{i=1}^{j} X_i}{Y_j}$ is an isomorphism of $\frac{U+B}{B}$ onto $\frac{U+Y_j}{Y_j} = \frac{M}{Y_j}$ so that there exists a right-inverse of Q_j, say L_j, defined on $\frac{M}{Y_j}$.

Let $w : H_2 \to F$ be a right-inverse of $\tilde{Q}_2$. It is easy to see that $A = Q_j wu$ and for $x = A(h) \in A(H_1) \subset \frac{\prod_{i=1}^j X_i}{Y_j}$ we have $Q_j wuL(x) = Q_j wu(h) = A(h) = x$ which shows that $wuL : A(H_1) \to F$ is a right-inverse of Q_j.

Since

$$\frac{\prod_{i=1}^j X_i}{Y_j} = A\,(H_1) \bigoplus \frac{M}{Y_j}$$

and Q_j has right-inverses on $A(H_1)$ and $\frac{M}{Y_j}$, it follows that the whole map Q_j has a right-inverse $A_j : \frac{\prod_{i=1}^j X_i}{Y_j} \to F$. Now we put

$$Z = q^{-1} A_j \left(\frac{\prod_{i=1}^j X_i}{Y_j} \right) \subset P.$$

Of course Z contains B and, since $\frac{Z}{B} = A_j(\frac{\prod_{i=1}^j X_i}{Y_j})$, which is a Banach subspace of P, Z is Banach (recall that «being a Banach space» is a three-space property in the class of locally convex spaces). We show now that Z is complemented in P and this will conclude the proof.

We may write

$$\frac{P}{B} = A_j \left(\frac{\prod_{i=1}^j X_i}{Y_j} \right) \bigoplus \ker Q_j.$$

Since $\ker Q_j = \frac{Y_j \bigoplus \prod_{i>j} X_i}{B}$ it follows that $q^{-1}(\ker Q_j) = Y_j + B + \prod_{i>j} X_i$. We have $B = Z \bigcap q^{-1}(\ker Q_j)$ and hence

$$Z \bigcap \prod_{i>j} X_i = Z \bigcap q^{-1} \left(\ker Q_j\right) \bigcap \prod_{i>j} X_i = B \bigcap \prod_{i>j} X_i = \{0\}.$$

Since $P = Z + q^{-1}(\ker Q_j)$ and $B \subset Z$ it is easy to see, recalling the definition of Y_j, that $P = Z + \prod_{i>j} X_i$ and hence $P = Z \bigoplus \prod_{i>j} X_i$ which shows that Z is complemented in P, as claimed (in fact Z is just a «twisted» copy of $\prod_{i=1}^j X_i$ in P, that is there are operators $T_n : \prod_{i=1}^j X_i \to X_n, n > j$, such that $Z = \{(x_i) : x_i = T_i(x_1, \ldots, x_j), i > j\}$; this is not difficult to see).

Remark 3.3. It is easy to see (recalling the proof of the theorem) that the hypothesis that $B \subset \subset Z$, with Z Banach and complemented in P, is equivalent to the existence of an automorphism $T : P \to P$ such that $T(B) \subset \prod_{i=1}^{j} X_i$ for a suitable $j \in \mathbf{N}$.

Remark 3.4. It is also clear from the proof of Theorem 3.2 that all the maps Q_k, $k \leq j$, have right-inverse. In the same way, all the maps R_k, $k \geq j$, have right-inverses too. It is worth mentioning that it is possible to represent a product P in the form $\text{quoj}_n(Z_n, L_n)$ such that no L_n has a right-inverse. Theorem 3.2 shows, however, that such a representation cannot be of the form $\frac{\widetilde{P}}{B}$, with $\widetilde{P}$ a product and B a Banach subspace of $\widetilde{P}$.

4. CONSEQUENCES AND EXAMPLES

To give new examples of twisted first-order quojections we have to know when a given Banach subspace B of P is contained in a bigger Banach subspace Z of P, complemented in P. To handle this situation we must translate such a «geometric» condition into an «analytic» one. For this we introduce some further notation.

Let B be a Banach subspace of P written in the form (*). We recall that $Y_j = \{(x_1, \ldots, x_j) : x_i = T_i(x_1, \ldots, x_k), (x_1, \ldots, x_k) \in Y, k < i \leq j\}$ (Y_j is defined only for $j \geq k$). Let $T_{jn} : Y_j \to X_n, n > j$, be defined by $T_{jn}(y, T_{k+1}y, \ldots, T_j y) = T_n y$ for $y = (y_1, \ldots, y_k) \in \in Y$. We have the following

Proposition 4.1. *B is contained in a complemented Banach subspace Z of P if and only if there exists $j \in \mathbf{N}$ such that every map $T_{jn}(n > j)$ has an extension*

$$\widetilde{T}_{jn} : \prod_{i=1}^{j} X_i \to X_n.$$

Proof. If the $\widetilde{T}_{jn}$ exist, then we may take $Z = \{(x_n) \in \prod_n X_n : x_m = \widetilde{T}_{jm}(x_1, \ldots, x_j), m > j\}$ which is isomorphic to $\prod_{i=1}^{j} X_i$, complemented in P and, of course, contains B.

Let's suppose now that such a subspace Z exists. Then we have, according to Lemma 3.1:

$$B = \{(x_n) : x_n = T_n(x_1, \ldots, x_k),\ n > k \text{ and } (x_1, \ldots, x_k) \in Y\}$$

$$Z = \left\{(x_n) : x_n = S_n\left(x_1, \ldots, x_j\right),\ n > j \text{ and } \left(x_1, \ldots, x_j\right) \in W\right\}$$

where W is complemented in $\prod_{i=1}^{j} X_i$. We may suppose that $j > k$ so that $Y_j \subset W$ and hence $S_n(u) = T_{jn}(u)$ for every $u \in Y_j$, since $B \subset Z$. Since W is complemented in $\prod_{i=1}^{j} X_i$ we can extend the maps S_n to maps $\widetilde{S}_n : \prod_{i=1}^{j} X_i \to X_n$, $n > j$, and it is clear that these maps are the $\widetilde{T}_{jn}$ we were looking for.

It is easy to see that in the case of a standard quojection

$$F = Q\left((X_n)\,,\left(\frac{X_n}{Y_n}\right);L\right),$$

Theorem 3.2 (together with Proposition 4.1) says exactly that F is twisted if and only if Y_n is not complemented in X_n for infinitely many n, which is a well-known result for standard quojections (cf. [10]).

Corollary 4.2. *Every quojection isomorphic to*

$$\frac{X \times (l^\infty)^{\mathbf{N}}}{B} \quad or \quad \frac{W \times (c_0)^{\mathbf{N}}}{B},$$

with B, X, W Banach and W separable, is a product.

Proof. The maps T_k (which define B) can be extended to maps $\widetilde{T}_k$ since l^∞ and c_0 have the extension property (the second one only for separable spaces but this is just the case). Proposition 4.1 and Theorem 3.2 give the result.

Now we give some examples of twisted first-order quojections, constructed using Theorem 3.2 and Proposition 4.1 and different from standard quojections.

Examples 4.3.

(i) Let $X_1 = l^\infty$, $X_n = l^{p_n}$ for $n \geq 2$ with $p_n < 2$. We take Y isomorphic to l^1, $Y \subset X_1$ and quotient operators $T_n : Y \to l^{p_n}$. Let $B = \{(x_n) : x_n = T_n x_1$ for $n \geq 2$ and $x_1 \in Y\}$. The quojection $F = \frac{\prod_n X_n}{B}$ is twisted. In fact, if F were not there would exist, by Theorem 3.2 and Proposition 4.1, a natural number j such that every map $T_{jn} : Y_j \to l^{p_n}$ would have an extension $\widetilde{T}_{jn} : l^\infty \times l^{p_1} \times \ldots \times l^{p_j} \to l^{p_n}$. Since T_{jn} is surjective, $\widetilde{T}_{jn}$ is surjective too. But this is impossible, since every operator from l^∞ to $l^p (p < 2)$ is compact and every operator from l^q to l^r, $q \neq r$, is strictly singular. So, $\widetilde{T}_{jn}$ cannot exist and F is twisted.

(ii) In the above example we can take also $X_1 = C([0,1])$ and get, as in point (i), a separable twisted first-order quojection. It is also clear that we can substitute the space l^1 with any big space W (big with respect to quotients) such that there are quotient maps $T_n : W \to l^{p_n}$. For example we can take $W = (\bigoplus_n l^{p_n})_2$.

(iii) Let $X_1 = L^p(0,1), 1 < p < 2, X_n = l^{p_n}$ for $n \geq 2$ with $p < p_n < 2$. Let $Y = (\bigoplus_n l^{p_n})_p$. Y is a subspace of $L^p(0,1)$ (cf. [6]) and there exist quotient maps $T_n : Y \to l^{p_n}$. Let $B = \{(x_n) : x_n = T_n x_1, n \geq 2$ and $x_1 \in Y\}$, as usually $\frac{\prod_n X_n}{B}$ is twisted. For this it is sufficient to note that every operator from $L^p(0,1)$ to l^q, $q<2$, is strictly cosingular (recall that every quotient of l^q contains l^q as a complemented subspace) and every operator from l^q to l^r is strictly cosingular too, which is a well-known fact. As in point (i) we cannot have extension operators $\widetilde{T}_{jn}$ and so $\frac{\prod_n X_n}{B}$ is a twisted reflexive first-order quojection.

If X_i are Banach spaces and L is a normal Banach sequence space, $L(X_i)$ is the Banach space of all sequences $(x_i)\in\prod_i X_i$ such that $(\|x_i\|)\in L$.

The following lemma, which is in the spirit of the construction of standard quojections, shows how to obtain twisted first-order quojections from special products.

Lemma 4.4. *Let X_i, Y_i, Z_i be Banach spaces with $Y_i\subset Z_i\subset X_i$ and let L be a normal Banach sequence space. Let $T_i: Y_i\to Z_i$ be continuous linear maps and $B=\{(y,w): y\in L(Y_i),\ w\in\prod_i X_i,\ w_j=T_jy_j \text{ for all } j\}\subset L(X_i)\times\prod_i Z_i$.*

$$\frac{L(X_i)\times\prod_i Z_i}{B}$$

is twisted if and only if, for infinitely many i, T_i has no extension defined on X_i.

Proof. Use Theorem 3.2 and Proposition 4.1.

Note that the well-known criterion for standard quojections is obtained by the above Lemma taking $Y_i=Z_i$ and $T_i=I_{Y_i}$.

Examples 4.5.

(i) With reference to Lemma 4.4 let $X_i=Z_i=X=l^p$, $1<p<\infty$, $p\neq 2$ and let $Y_i=Y$ be an uncomplemented copy of l^p (see [2] and [14]). Take $L=l^p$ and $T:Y\to X$ an isomorphic map which is onto X. Since Y is not complemented in X, T cannot be extended to X and so we obtain that $\frac{(l^p)^{\mathbb{N}}}{B}$ is twisted with $B\simeq l^p$. In this case $\frac{(l^p)^{\mathbb{N}}}{B}=Q(l^p,\frac{l^p}{Y};l^p)$ as it is easy to verify.

(ii) Proceeding as in point (i) we can find a twisted first-order quojection of the form $\frac{(L^p(0,1))^{\mathbb{N}}}{B}$, $1<p<\infty$, $p\neq 2$, with $B\simeq L^p(0,1)$ (see again [2] and [14] for uncomplemented copies of $L^p(0,1)$ in $L^p(0,1)$).

(iii) Let $X_i=Z_i=C([0,1])$ and $Y_i=Y$ be a subspace of $C([0,1])$ such that there exists an operator $T:Y\to C([0,1])$ which cannot be extended to all of $C([0,1])$. Such a couple (Y,T) exists since $C([0,1])$ is not separably injective and since $C([0,1])$ is universal for separable spaces. Arguing as in (i) and taking $L=c_0$ we obtain a twisted first-order quojection of the form $\frac{C([0,1])^{\mathbb{N}}}{B}$.

(iv) Let $X_i=Z_i=L^1(0,1)$ and $Y_i=R=\overline{sp}\{r_n\}$, where (r_n) are the Rademacher functions on $(0,1)$. Since $L^\infty(0,1)$ has not finite cotype (hence not cotype 2) there exists a (compact) operator $T : R \to L^1(0,1)$ which cannot be extended to $L^1(0,1)$ (cf. [13] Ch. 6). Taking $L = l^1$ we obtain that $\frac{L^1(0,1)^{\mathbb{N}}}{B}$ is twisted with $B \simeq l^1(R) \simeq l^1(l^2)$.

(v) Let $X_i = Z_i = l^1$. We produce now a subspace Y of l^1 and an operator $T : Y \to l^1$ which cannot be extended to l^1, thereby obtaining a twisted first-order quojection $\frac{(l^1)^{\mathbb{N}}}{B}$ (taking $L = l^1$).

Since $C_2(l^\infty_n) = \sqrt{n}$, where $C_2(X)$ is the cotype constant of order 2 of X, we have by [13], Ch. 6, that for every n there exists $u_n : sp\{r_1, \ldots, r_n\} \to l^1_n$, $\| u_n \| = 1$ such that $\| \tilde{u}_n \| \geq \sqrt{n}$ for every extension $\tilde{u}_n : L^1(0,1) \to l^1_n$. Let $R_n = sp\{r_1, \ldots, r_n\}$. By the Kintchine inequality $d(R_n, l^2_n) \leq C$ for every n, where «d» is the Banach-Mazur distance. Since R_n is a finite-dimensional subspace of $L^1(0,1)$ by well-known properties of $\mathcal{L}_1$-spaces we can find $F_n \supset R_n$ such that dim $F_n < \infty$, $d(F_n, l^1_{\dim F_n}) \leq 2$ and projections $P_n : L^1(0,1) \to F_n$ with $\sup_n \| P_n \| < \infty$. From this last property we deduce that $\| \tilde{u}_n \| \geq C'\sqrt{n}$, for a suitable $C' > 0$, for any $\tilde{u}_n : F_n \to l^1_n$ which extends u_n.

Let $W = l^1(F_n)$ and $Y = l^1(R_n) \subset W$. Of course $W \simeq l^1$. Let $T : Y \to l^1 = l^1(l^1_n)$ defined by $T(x_n) = (u_n(x_n))$ for $(x_n) \in l^1(R_n)$. Of course T cannot be extended to $W \simeq l^1$ and so the proof is complete.

We can obtain other examples of twisted first-order quojections combining the Examples in 4.3 and 4.5.

It is known that twisted standard quojections cannot appear as complemented subspaces of products. Now we show that the same holds for twisted first-order quojections. The proof relies upon the following lemma, communicated to us by P. Domański, which can be deduced from Lemma 10.21 of [4].

Lemma 4.6. *Let $F = \mathrm{proj}_n(X_n, R_n)$ be a Fréchet space and let the projective limit be reduced. If F is complemented in a product then the fundamental exact sequence*

$$0 \to F \to \prod_n X_n \to \prod_n X_n \to 0$$

(cf. [15]) splits, that is F is complemented in $\prod_n X_n$ and has a complement isomorphic to $\prod_n X_n$.

Theorem 4.7. *Let F be a twisted first-order quojection. Then F is not complemented in any product.*

Proof. We write $F = \frac{\prod_n X_n}{B}$, where B is in the form (*). Let's suppose that F is twisted but complemented in a suitable product. By the above lemma there are Banach spaces Z_n such that $E = F \bigoplus \prod_n Z_n$ is again a product.

We define $W_n = X_n \bigoplus Z_n$ and $S_n : Y \to W_n$ by $S_n(y) = (T_n(y), 0)$.

Let $\tilde{B} = \{(w_n) : w_n = S_n(w_1, \dots, w_k)$ for $n > k$ and $(w_1, \dots, w_k) \in Y\}$.

Of course $\tilde{B}$ is isomorphic to B and we have $E = \frac{\prod_n W_n}{\tilde{B}}$. Since by assumption E is a product we can find $j \in \mathbb{N}$ and extension operators $\tilde{S}_{jn} : \prod_{i=1}^{j} W_i \to W_n, n > j$, according to Theorem 3.2 and Proposition 4.1. But, if $r_n : W_n \to X_n$ is the canonical projection, it is immediate to see that

$$\tilde{T}_{jn} = r_n \tilde{S}_{jn}\Big|_{\prod_{i=1}^{j} X_i} : \prod_{i=1}^{j} X_i \to X_n$$

are extensions of T_{jn}, so that F should be a product. Since this is a contradiction, the theorem follows.

Remark 4.8. Note that Theorem 4.7 yields also the following result: if F is a Fréchet space such that $F^{(2n)}$ is a twisted first-order quojection for some n, then F is not complemented in any product.

Corollary 4.9. *Let F be a twisted first-order quojection and G be a (first-order) quojection. Then $F \times G$ is a twisted (first-order) quojection and $F \tilde{\otimes}_\pi G$ is a twisted quojection.*

Corollary 4.10. *Let (F_k) be a sequence of first-order quojection such that F_k is twisted for infinitely many k. Then $\prod_k F_k$ is not a first-order quojection.*

Proof. The proof is analogous to that of the second example in 2.4 using, at this time, Theorem 4.7.

We note that first-order quojections do not have the three-space property that is, if

$$0 \to G \to F \to E \to 0$$

is an exact sequence with G and E first-order quojections, F is not a first-order quojection, in general. To see this let F be a countable product of twisted first-order quojections. By the above corollary F is not of the first-order, but it is easy to see that $F = \frac{P_1}{P_2}$ with P_1 and P_2 products (and hence of first-order).

The following proposition is a «weak» form of the three-space property for first-order quojections.

Proposition 4.11. *Let X, Y be Banach spaces and let E, G be first-order quojections. Then we have:*

(i) If $0 \to G \to F \to Y \to 0$ is exact, then F is a first-order quojection,

(ii) If $0 \to X \to F \to E \to 0$ is exact, then F is a first-order quojection.

Proof. (i) There exists a continuous seminorm p on G such that ker p is a product. Since

$$Y = \frac{F}{G} = \frac{\frac{F}{\ker p}}{\frac{G}{\ker p}}$$

and $\frac{G}{\ker p}$ is a Banach space, we have that $\frac{F}{\ker p}$ is Banach and hence F is a product by Theorem 2.2.

(ii) There exists a continuous seminorm p on E such that $P = \ker p$ is a product. Let $q : F \to E$ be the quotient map and $H = q^{-1}(P)$. We have the exact sequence

$$0 \to X \to H \xrightarrow{q} P \to 0 .$$

Let r be a continuous seminorm of H such that it induces on X its Banach space topology. We have that $X \bigcap \ker r = \{0\}$, $X + \ker r$ is closed in H and q is an isomorphism from ker r onto ker $\overline{r}$, where $\overline{r}$ is the seminorm on P induced by r (recall the proof of Theorem 2.2). But then ker r is a product and hence H is a first-order quojection. Since $\frac{F}{H} = \frac{E}{P} = \frac{E}{\ker p}$, $\frac{F}{H}$ is a Banach space, and F is of the first-order by point (i).

From Proposition 4.11 and Theorem 3.2 we obtain the following result for general quojections.

Theorem 4.12. *Let F be a twisted quojection and let X be a Banach subspace of F. Then $\frac{F}{X}$ is twisted.*

Proof. Let's suppose that $\frac{F}{X}$ is a product. By Proposition 4.11 (ii), F is a first-order quojection and so we may write $F = \frac{P}{B}$ with P a product and B a Banach subspace of P. Then $X = \frac{Y}{B}$ with Y a Banach subspace of P. Since F is twisted, B is not contained in any Banach complemented subspace of P, by Theorem 3.2 and hence the same holds for Y. But then $\frac{F}{X} = \frac{P}{Y}$ is twisted by Theorem 3.2 again.

5. ASSOCIATED PREQUOJECTIONS

The last part of this paper deals with prequojections related to first-order quojections. We recall that a prequojection is a Fréchet space F whole bidual F'' is a quojection. By [11] we know that, given a separable quojection E such that $\frac{E''}{E}$ is not a subspace of $\omega \times Z$, Z Banach, there exists a separable countably normed prequojection F such that $F' = E'$. Of course such a prequojection F is non-trivial, that is, it is not a quojection (it has a continuous norm!).

Let $E = \frac{P}{B}$ be a separable first-order quojection with $\frac{E''}{E} \not\subset \omega \times Z$. Write $P = \prod_i X_i$ and B in the form (*) of Lemma 3.1 with $Y \subset \prod_{i=1}^k X_i$. For $j \geq k$, $Y_j \subset \prod_{i=1}^j X_i$ is defined by $Y_j = \{(x_1, \ldots, x_j) : x_m = T_m(x_1, \ldots, x_k)$ for $k \leq m \leq j$ and $(x_1, \ldots, x_k) \in Y\}$.

The following theorem is the analogue of Theorem 4.2 of [11] stated in the case of standard quojections.

Theorem 5.1. *Let E be a separable first-order quojection with $\frac{E''}{E} \not\subset \omega \times Z$, Z Banach. There exists a separable countably normed prequojection F such that $F' = E'$ and we have the following exact sequences:*

$$0 \to B \to H \to F \to 0$$

and

$$0 \to G \to F \to \frac{\prod_{i=1}^{k} X_i}{Y} \to 0$$

where H and G are separable countably normed prequojections such that $H' = P'$ and $G' = \bigoplus_{i>k} X_i'$.

Proof. We may write, with the notation of Theorem 3.2

$$E = \frac{P}{B} = \operatorname{quoj}_{n \geq k} \left(\frac{\prod_{i=1}^{n} X_i}{Y_n}, R_n \right)$$

and we have $\ker R_n = \frac{Y_n \times X_{n+1}}{Y_{n+1}}$. Let $q_n : \prod_{i=1}^{n} X_i \to \frac{\prod_{i=1}^{n} X_i}{Y_n}$ be the quotient map. Then it is immediate that q_n is an isomorphism from X_n onto $\frac{Y_{n-1} \times X_n}{Y_n}$. The hypothesis $\frac{E''}{E} \not\subset \omega \times Z$ means exactly that $X_n \simeq \ker R_{n-1}$ is a non-quasi-reflexive Banach space for infinitely many $n \geq k$ (cf. [11]) and so we may suppose that X_n is non-quasi-reflexive for all $n \geq k$. Since E is separable, X_n is separable too (for $n \geq k$). By Lemma 2 of [11], for every $n \geq k$ we can find sequences $(z_k^n) \subset X''_n$ and $(x_k'^n) \subset X_n'$ so that:

(i) $\| z_k^n \| = 1$, (z_k^n) is a basic sequence in X''_n and $X_n \bigcap \overline{span}(z_k^n) = \{0\}$;

(ii) $\| x_k'^n \| \leq 4$, $(x_k'^n)$ is total over $X_n + \overline{span}(z_k^n)$ and $\langle x_k'^n, z_h^n \rangle = \frac{\delta_{kh}}{k}$. Let $W_n = \overline{sp}\{z_k^n\} \subset X''_n$ and $u_k'^n$ be Hahn-Banach extensions of $x_k'^n$ to $\frac{\prod_{i=1}^{n} X_i}{Y_n}$ (here we consider $X_n \simeq \frac{Y_{n-1} \times X_n}{Y_n}$ via q_n).

Let $y_k'^n = q_n'(u_k'^n) \in (\prod_{i=1}^{n} X_i)'$. Of course we have that $\| y_k'^n \| \leq 4$, $y_k'^n(x) = 0$ for every $x \in Y_n$ and $(y_k'^n)$ is total over $X_n + W_n$.

Let $J_n = \sum_k \varepsilon_k y_k'^{n+1} \otimes z_k^n : \prod_{i=1}^{n+1} X_i \times W_{n+1} \to W_n$, where $\varepsilon_k > 0$ and $\sum_k \varepsilon_k \leq \frac{1}{8}$ so that $\| J_n \| \leq \frac{1}{2}$. J_n is injective on $X_{n+1} + W_{n+1}$.

Let $S_n : \prod_{i=1}^{n+1} X_i \times W_{n+1} \to \prod_{i=1}^{n} X_i \times W_n$ be defined by $S_n(x, w) = (P_n x, J_n(x, w))$ for $x \in \prod_{i=1}^{n+1} X_i$, $w \in W_{n+1}$ (P_n is the canonical projection of $\prod_{i=1}^{n+1} X_i$ onto $\prod_{i=1}^{n} X_i$).

By the main theorem of [11], $H = \operatorname{proj}_{n\geq k}(\prod_{i=1}^{n} X_i \times W_n, S_n)$ is a separable countably normed prequojection such that $H' = P'$.

If $\widetilde{J}_n : \frac{\prod_{i=1}^{n+1} X_i}{Y_{n+1}} \times W_{n+1} \to W_n$ is defined by $\widetilde{J}_n = \sum_k \varepsilon_k x_k'^{n+1} \otimes z_k^n$ and $\widetilde{S}_n : \frac{\prod_{i=1}^{n+1} X_i}{Y_{n+1}} \times$ $\times W_{n+1} \to \frac{\prod_{i=1}^{n} X_i}{Y_n} \times W_n$ is defined by $\widetilde{S}_n(x, w) = (R_n x, \widetilde{J}_n(x, w))$ for $x \in \frac{\prod_{i=1}^{n+1} X_i}{Y_{n+1}}$ and $w \in W_{n+1}$, then we have, again by the main theorem of [11], that

$$F = \operatorname{proj}_{n\geq k}\left(\frac{\prod_{i=1}^{n} X_i}{Y_n}, \widetilde{S}_n\right)$$

is a separable countably normed prequojection such that $F' = E'$.

Let $r_n : \prod_{i=1}^{n} X_i \times W_n \to \frac{\prod_{i=1}^{n} X_i}{Y_n} \times W_n$ be the quotient map whose kernel is Y_n. By construction of J_n and $\widetilde{J}_n$ it follows that $\widetilde{S}_{n-1} r_n = r_{n-1} S_{n-1}$ and so we have a map $r = (r_n) : H \to F$.

Since $\ker r_n = Y_n = S_{n+1}(Y_{n+1}) = S_{n+1}(\ker r_{n+1}) = P_{n+1}(Y_{n+1})$ it follows that r is surjective and that $\ker r = \operatorname{proj}_n(\ker r_n, S_n) = \operatorname{proj}_n(Y_n, P_n) = B$ and this yields the first exact sequence. Since r_n (for $n \geq k$) is an isomorphism from $X_{k+1} \times \ldots \times X_n \times W_n$ onto $\frac{Y_n + X_{k+1} \times \ldots \times X_n}{Y_n} \times W_n$ and $\widetilde{S}_{n-1} r_n = r_{n-1} S_{n-1}$ it follows that $G = \operatorname{proj}_{n\geq k}(\prod_{i=k+1}^{n} X_i \times$ $\times W_n, S_n)$ is a subspace of F. Again by [11] we have that G is a separable countably normed prequojection and $G' = \bigoplus_{i>k} X_i'$. Since G, considered as a subspace of F, has the reduced projective representation

$$G = \operatorname{proj}_{n\geq k}\left(\frac{Y_n + \prod_{i=k+1}^{n} X_i}{Y_n} \times W_n, \widetilde{S}_n\right),$$

it follows that

$$\frac{F}{G} = \operatorname{proj}_{n\geq k}\left(\frac{\prod_{i=1}^{n} X_i}{Y_n + \prod_{i=k+1}^{n} X_i}, L_n\right),$$

where the L_n are the maps induced by the $\widetilde{S}_n$ on the quotient spaces.

But it is easily seen that $\frac{\prod_{i=1}^{n} X_i}{Y_n + \prod_{i=k+1}^{n} X_i}$ is isomorphic to $\frac{\prod_{i=1}^{k} X_i}{Y}$ and that L_n is an isomorphism, so that $\frac{F}{G} \simeq \frac{\prod_{i=1}^{k} X_i}{Y}$, which establishes the second exact sequence of the theorem.

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