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An alternate proof of "Tame Fréchet spaces are Quasi-Normable"

Angela A. Albanese
Departimento di Matematica e Fisica "E. De Giorgi"
Università del Salento
Via Per Arnesano
I-73100 Lecce, Italy
angela.albanese@unisalento.it

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Abstract. In [12] Piszczek proved that "tameness always implies quasinormability" in the setting of Fréchet spaces. In this paper we present an alternate proof of this interesting result.

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Introduction

Let E be a Fréchet space and let $(\|\ \|_k)_k$ denote a fundamental system of increasing seminorms defining the topology of E such that the sets $U_k := \{\, x \in E \mid \|x\|_k \leq 1 \,\}$ form a basis of 0–neighbourhoods in E.

A Fréchet space E is called *tame* if there exists an increasing function $S\colon \mathbb{N} \to \mathbb{N}$ such that for every continuous linear operator $T\colon E \to E$ there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there is a constant $C_k > 0$ for which

$$\forall x \in X: \quad \|Tx\|_k \leq C_k \|x\|_{S(k)}.$$

The definition does not depend on the choice of seminorms. The class of tame Fréchet spaces was introduced and studied by Dubinsky and Vogt in [7]. Tameness condition is related to important questions concerning the structure of Fréchet spaces, in particular of infinite/finite type power series spaces and of Köthe sequence spaces, see [7, 10, 11, 12, 13, 14, 16] and the references therein.

A Fréchet space E is called *quasinormable* if there exists a bounded subset B of E such that

$$\forall n\ \exists m > n\ \forall \varepsilon > 0\ \exists \lambda > 0: \quad U_m \subset \lambda B + \varepsilon U_n.$$

The class of quasinormable locally convex spaces was introduced and studied by Grothendieck in [8]. Such a class of spaces has received a lot of attention in the setting of Fréchet spaces and of Köthe sequence spaces, see [1, 2, 3, 4, 5, 6, 9, 11, 12, 15] and the references therein.

Piszczek proved that every tame Fréchet space is quasinormable, [12] (see also [11]). The aim of this note it to present an alternate proof of such a result. The proof is different in spirit and relies on some results established in [1, 2].

1 An alternate proof

In the sequel, given a Fréchet space E we denote by $(\| \|_k)_k$ a fundamental system of increasing seminorms defining the topology of E such that the sets $U_k := \{ x \in E \mid \|x\|_k \leq 1 \}$ form a basis of 0–neighbourhoods in E. The dual seminorms are defined by $\|f\|'_k := \sup\{|f(x)| \mid x \in U_k\}$ for $f \in E'$; hence $\| \|'_k$ is the gauge of $\overset{\circ}{U}_k$. We denote by $E'_k := \{ f \in E' \mid \|f\|'_k < \infty \}$ the linear span of $\overset{\circ}{U}_k$ endowed with the norm topology defined by $\| \|'_k$. Clearly, $(E'_k, \| \|'_k)$ is a Banach space and $E'_k = (E/\ker \| \|_k, \| \|_k)'$.

If E is a Fréchet space with a continuous norm, we may assume that each $\| \|_k$ is a norm on E.

In order to give the proof we recall the following two lemmas. The first one, due to Piszczek [11], gives a necessary condition for the tameness of a Fréchet space E.

Lemma 1. ([11, Lemma 3]) *In every tame Fréchet space E the following condition holds. There exists $\psi\colon \mathbb{N} \to \mathbb{N}$ such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for all $m \geq k$ there are $n \in \mathbb{N}$ and a constant $C_m > 0$ such that*

$$\forall f \in E',\ y \in E: \quad \max_{k \leq l \leq m} \|f\|'_{\psi(l)} \|y\|_l \leq C_m \max_{1 \leq p \leq n} \|f\|'_{\varphi(p)} \|y\|_p. \tag{1.1}$$

An examination of the proofs given in [1, Theorem 1] and in [2, Theorem 3] (respectively, for separable Fréchet spaces and for general Fréchet spaces) shows that the following fact holds.

Lemma 2. *Let E be a Fréchet space with a continuous norm. If E is not quasinormable, then there exist sequences $(x_{jk})_{j,k\in\mathbb{N}} \subset E$ and $(f_{jk})_{j,k\in\mathbb{N}} \subset E'$, and there exist a decreasing sequence $(\beta_k)_{k\in\mathbb{N}} \subset]0, 1[$ and an increasing sequence $(\alpha_k)_{k\in\mathbb{N}} \subset]1, +\infty[$ satisfying the following properties.*

(1) $0 < \alpha \leq \inf_{k\in\mathbb{N}} \alpha_k\beta_k < \sup_{k\in\mathbb{N}} \alpha_k\beta_k \leq \beta < \infty$.

(2) $(x_{jk}, f_{jk})_{j,k\in\mathbb{N}}$ *is a biorthogonal system.*

(3) $\|f_{jk}\|'_1 \leq 1$ *for all* $j, k \in \mathbb{N}$.

(4) $\sup_{j\in\mathbb{N}} \|x_{jk}\|_k \leq \alpha_k$ *for all* $k \in \mathbb{N}$.

(5) $\inf_{k\in\mathbb{N}} \|f_{jk}\|'_k \geq \beta_k$ *for all* $k \in \mathbb{N}$.

(6) $\|f_{jk}\|'_{k+1} \leq k^{-j}$ *for all* $j, k \in \mathbb{N}$.

We can now present an alternate proof of the following result which was already established in [12, §3] (see also [11, Theorem 6]).

Theorem 1. *Every tame Fréchet space is quasinormable.*

Proof. Let E be a tame Fréchet space. Then by [11, Proposition 5] we may assume that E has a continuous norm. So, if suppose that E is not quasi-normable, Lemma 2 yields that there exist two sequences $(x_{jk})_{j,k\in\mathbb{N}} \subset E$ and $(f_{jk})_{j,k\in\mathbb{N}} \subset E'$, and there exist a decreasing sequence $(\beta_k)_{k\in\mathbb{N}} \subset]0,1[$ and an increasing sequence $(\alpha_k)_{k\in\mathbb{N}} \subset]1,+\infty[$ satisfying all the properties (1)÷(6) in Lemma 2.

Since E is tame, by Lemma 1 we have that there exists $\psi\colon \mathbb{N} \to \mathbb{N}$ such that for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for all $m \geq k$ there are $n \in \mathbb{N}$ and a constant $C_m > 0$ such that inequality (1.1) holds. So, setting $m = k$, the following holds.

$$\forall j,\, h \in \mathbb{N}: \quad \|f_{h\varphi(k-1)}\|'_{\psi(k)}\|z_{jk-1}\|_k \leq C_k \max_{1\leq p\leq n} \|f_{h\varphi(k-1)}\|'_{\varphi(p)}\|z_{jk-1}\|_p. \tag{1.2}$$

Without loss of generality we may assume that $n \geq k$ in (1.2).

As φ is arbitrary, we may choose the function φ such that $\varphi(k-1) = \psi(k)$ for all $k \geq 1$. Therefore, by Lemma 2(5) we have, for every $h \in \mathbb{N}$, that

$$\|f_{h\varphi(k-1)}\|'_{\psi(k)} = \|f_{h\varphi(k-1)}\|'_{\varphi(k-1)} \geq \beta_{\varphi(k-1)},$$

and so

$$\frac{1}{\|f_{h\varphi(k-1)}\|'_{\psi(k)}} \leq \beta^{-1}_{\varphi(k-1)}. \tag{1.3}$$

On the other hand, we have that

$$\lim_{j\to\infty} \|x_{jk-1}\|_k = +\infty. \tag{1.4}$$

Indeed, Lemma 2(2) implies, for every $j \in \mathbb{N}$, that

$$1 = \langle x_{jk-1}, f_{jk-1}\rangle \leq \|x_{jk-1}\|_k \|f_{jk-1}\|'_k.$$

Combining this inequality with property (6) in Lemma 2, we obtain, for every $j \in \mathbb{N}$, that

$$k^j \leq \frac{1}{\|f_{jk-1}\|'_k} \leq \|x_{jk-1}\|_k$$

from which (1.4) clearly follows.

To estimate the right hand side of (1.2) we proceed as it follows.

If $1 \leq p \leq k-1$, then by Lemma 2, (3)-(4), we have that

$$\|x_{jk-1}\|_p \leq \|x_{jk-1}\|_{k-1} \leq \alpha_{k-1} \tag{1.5}$$

and that

$$\|f_{h\varphi(k-1)}\|'_{\varphi(p)} \leq \|f_{h\varphi(k-1)}\|'_1 \leq 1 \tag{1.6}$$

for all $j,\ h \in \mathbb{N}$.

If $k \leq p \leq n$, then Lemma 2(6) implies that

$$\|f_{h\varphi(k-1)}\|'_{\varphi(p)} \leq \|f_{h\varphi(k-1)}\|'_{\varphi(k-1)+1} \leq \varphi(k-1)^{-h}$$

for all $h \in \mathbb{N}$. Consequently, there exists an increasing sequence $(h_j)_{j\in\mathbb{N}}$ of positive integers such that

$$\forall j \in \mathbb{N}: \quad \max_{k\leq p\leq n} \|f_{h_j\varphi(k-1)}\|'_{\varphi(p)}\|x_{jk-1}\|_p \leq 1. \tag{1.7}$$

Indeed, set $c_1 = \max_{k\leq p\leq n} \|x_{jk-1}\|_p$, there exists $h_1 \geq 1$ such that $c_1 \cdot \varphi(k-1)^{-h_1} \leq 1$ and hence, $\max_{k\leq p\leq n} \|f_{h_1\varphi(k-1)}\|'_{\varphi(p)}\|x_{jk-1}\|_p \leq c_1 \cdot \varphi(k-1)^{-h_1} \leq 1$.

Assume we have determined $h_1 < h_2 < \ldots < h_r$ such that (1.7) is satisfied for $j = 1, \ldots, r$. Next, set $c_{r+1} = \max_{k\leq p\leq n} \|x_{r+1k-1}\|_p$, there exists $h_{r+1} > h_r$ such that $c_{r+1} \cdot \varphi(k-1)^{-h_{r+1}} \leq 1$ and hence, $\max_{k\leq p\leq n} \|f_{h_{r+1}\varphi(k-1)}\|'_{\varphi(p)}\|x_{r+1k-1}\|_p \leq c_{r+1} \cdot \varphi(k-1)^{-h_{r+1}} \leq 1$.

Thus, combining inequalities (1.5), (1.6) and (1.7) we obtain that

$$\forall j \in \mathbb{N}: \quad \max_{1\leq p\leq n} \|f_{h_j\varphi(k-1)}\|'_{\varphi(p)}\|x_{jk-1}\|_p \leq \max\{1, \alpha_{k-1}\}. \tag{1.8}$$

Now, from (1.2), (1.3) and (1.8) it follows that

$$\begin{aligned} \forall j \in \mathbb{N}: \quad \|x_{jk-1}\|_k &\leq \frac{1}{\|f_{h_j\varphi(k-1)}}\|_{\varphi(k-1)} C_k \max_{1\leq p\leq n} \|f_{h_j\varphi(k-1)}\|'_{\varphi(p)}\|x_{jk-1}\|_p \\ &\leq \beta^{-1}_{\varphi(k-1)} C_k \max\{1, \alpha_{k-1}\}. \end{aligned}$$

But, by (1.4), $\|x_{jk-1}\|_k \to +\infty$ as $j \to \infty$. Thus, we obtain a contradiction. $\boxed{QED}$

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