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Extremal Tri-Cyclic Graphs with respect to the First and Second Zagreb Indices

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Abstract. In this paper, the first and second maximum values of the first and second Zagreb indices of $n-$vertex tri-cyclic graphs are obtained.

Keywords: First Zagreb index, second Zagreb index, tri-cyclic graph.

MSC 2010 classification: primary 05C07, secondary 05C35

1 Introduction

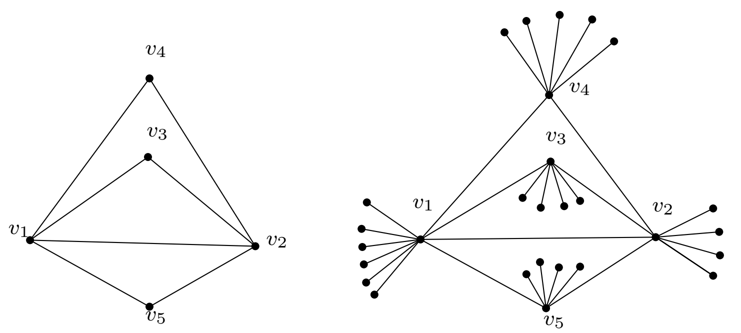
Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The first and second Zagreb indices of G are defined as $M_1(G) = \sum_{v\in V(G)} d^2(v)$ and $M_2(G) = \sum_{e=uv\in E(G)} d(u)d(v)$, respectively, where $d(u)$ is the vertex degree of u [8]. During the last decades, a lot of work was done on this topic. For more results concerning Zagreb group indices see [1, 2, 3, 10, 14]. In [7], a history of these graph parameters as well as their mathematical properties are presented.

If G is a connected graph having n vertices and m edges, then $c = m-n+1$ is called the cyclomatic number of G and conventionally, G is said to be cyclic if $c > 0$. In particular, if $c = 1, 2, 3$ then we call G to be uni-cyclic, bi-cyclic and tri-cyclic graph, respectively. In some research papers [12, 16], the extremal properties of these graph invariants on the set of all bicyclic graphs with a fixed number of pendant vertices and bicyclic graphs with a given matching number are investigated. Finally, in [13, 15], some extremal graphs for Zagreb indices were obtained in the classes of all quasi-tree graphs and polyominochains. We encourage the interested reader to consult [4, 5, 6, 9, 17, 18, 19] and references therein for more information on this topic.

The aim of this paper is to determine the first and second maximum values of M_1 and M_2 in the class of all *tri-cyclic* graphs with $n \geq 6$ vertices. To do this, we introduce the following notations:

(1) A simple graph G with $V(G) = \{v_1, \ldots, v_5\}$ and $E(G) = \{v_1v_i, v_2v_j | 2 \leq i \leq 5, 3 \leq j \leq 5\}$ is denoted by $q^{3,3,3}$. Figure 1(a).

(2) The $q_n(n_1, n_2, n_3, n_4, n_5)$ is resulting graph from $q^{3,3,3}$ by adding $n_i - 1$ pendant vertices to vertex v_i, $1 \leq i \leq n$ such that $n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5$ and $n_i \geq 1$, see Figure 1(b).



(a) The $q^{3,3,3}$ graph. (b) The $q_n(n_1, n_2, n_3, n_4, n_5)$ graph.

Figure 1. **a**) The $q^{3,3,3}$ graph. **b**) The $q_n(n_1, n_2, n_3, n_4, n_5)$ graph.

(3) Consider the cycle graph C_5 with $V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Connect v_1 to vertices v_2 and v_3. This graph is denoted by g_5. We now add $n_i - 1$ pendant vertices to vertex v_i, $1 \leq i \leq 5$, such that $n_i \geq 1$, $n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5$ and $\sum_{i=1}^{5} n_i = n$. The resulting graph is denoted by $g_n(n_1, n_2, n_3, n_4, n_5)$, Figure 2. We denote the set of all such graphs by $\mathfrak{g}_n$.

(4) The $K_n(n_1, n_2, n_3, n_4)$ is a graph obtained from K_4 by adding $n_i - 1$

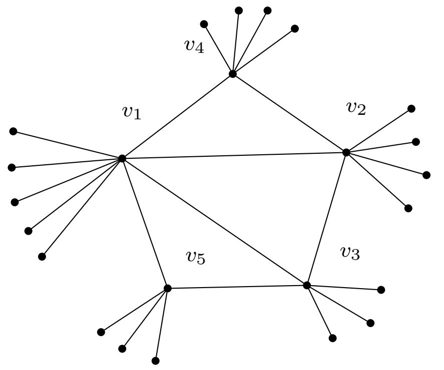


Figure 2. The $g_n(n_1, n_2, n_3, n_4, n_5)$ graph.

pendant vertices to vertex v_i, $1 \leq i \leq 4$, such that $n_i \geq 1$ and $n_1 = \max\{n_1, n_2, n_3, n_4\}$, Figure 3.

(5) The graph E is constructed from $K_4 - e$ and K_3 by identifying one vertex of degree three in $K_4 - e$ and one vertex of K_3. We label the graph E as Figure 4. Also, we assume that $E_n(n_1, n_2, n_3, n_4, n_5, n_6)$ is the graph formed from E by attaching $n_i - 1$ pendant vertices to the vertex v_i, where $n_i \geq 1$, $i = 1, \cdots, 6$, $n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5 \geq n_6$ and $\sum_{i=1}^{6} n_i = n$.

(6) Let F be the resulting graph from $K_4 - e$ and K_3 by identifying one vertex of degree two in each graph. Label the vertices of F as in Figure 5(a). Also, we assume that $F_n(n_1, n_2, n_3, n_4, n_5, n_6)$ is the graph formed F by attaching $n_i - 1$ pendant vertices to v_i, where $n_i \geq 1$, $i = 1, \cdots, 6$, $n_1 \geq n_2 \geq n_3 \geq n_4 \geq n_5 \geq n_6$ and $\sum_{i=1}^{6} n_i = n$, Fig. 5(b).

A graph G is called a cactus graph if blocks of G are either edges or cycles. The set of cacti of order n with k pendant vertices is denoted by $G_{n,k}$. The graphs with the largest values of M_1 and M_2 in the class of $G_{n,k}$, are determined by Li et al. [11]. For the sake of completeness we mention here the main result of this paper.

Lemma 1. *Let G be a graph in $G_{n,k}$. Then the following statements are satisfied:*

(i). If $n - k \equiv 1 \pmod 2$, then $M_2(G) \leq 2n^2 - (k+2)n - k$, with equality if and only if $G \cong C^1(n,k)$, where $C^1(n,k)$ is depicted in Fig. 6(a).

(ii). If $n - k \equiv 0 \pmod 2$, then $M_2(G) \leq 2n^2 - (k+5)n + 4$, with equality if and only if $G \cong C^2(n,k)$, where $C^2(n,k)$ is depicted in Fig. 6(b).

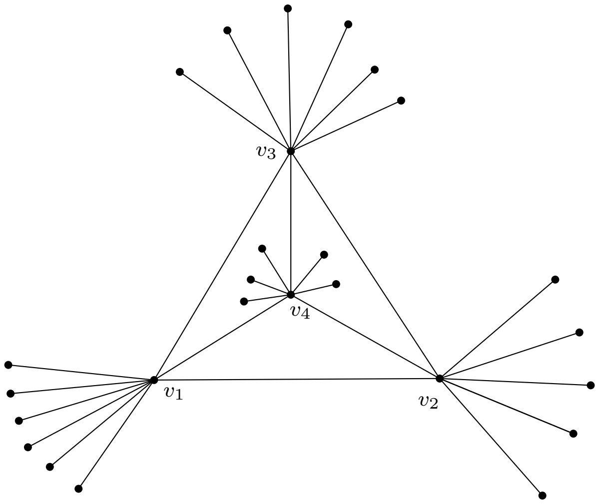


Figure 3. The $K_n(n_1, n_2, n_3, n_4)$ graph.

(iii). If $n-k \equiv 1 \pmod 2$, *then* $M_1(G) \leq n^2+2n-3k-3$, *with equality if and only if* $G \cong C^1(n,k)$, *where* $C^1(n,k)$ *is depicted in Fig.6(a).*

(iv). If $n-k \equiv 0 \pmod 2$, *then* $M_1(G) \leq n^2-3k$, *with equality if and only if* $G \cong C^2(n,k)$ *or* $C^3(n,k)$, *where* $C^2(n,k)$ *and* $C^3(n,k)$ *are depicted in Fig. 6.*

Throughout this paper K_n, C_n and P_n denote the complete, cycle and path graphs on n vertices, respectively. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$. R_n is the set of all tri-cyclic graphs with n vertices and its subset containing tri-cyclic graphs with p pendant vertices is denoted by $R_{n,p}$. Our other notations are standard and taken from the standard book on graph theory.

2 Main Results

In this section, the tri-cyclic $n-$vertex graphs, $n \geq 5$, with the first and second Zagreb indices are determined. Suppose that G is a simple $n-$vertex tri-cyclic graph containing $p \geq 0$ pendant vertices. Choose a non-pendant edge

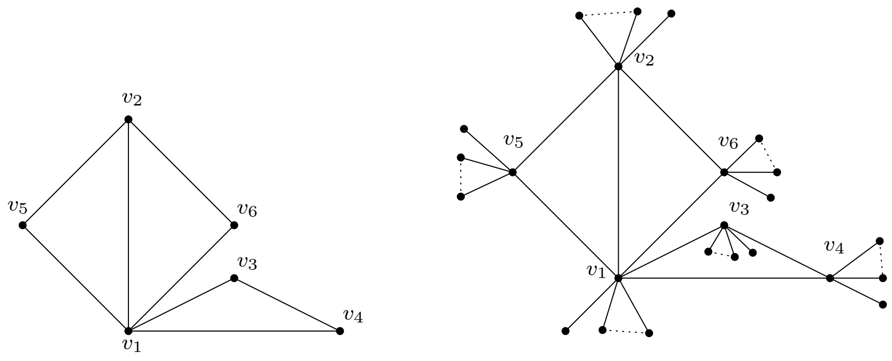


Figure 4. $E_n(n_1, n_2, n_3, n_4, n_5, n_6)$.

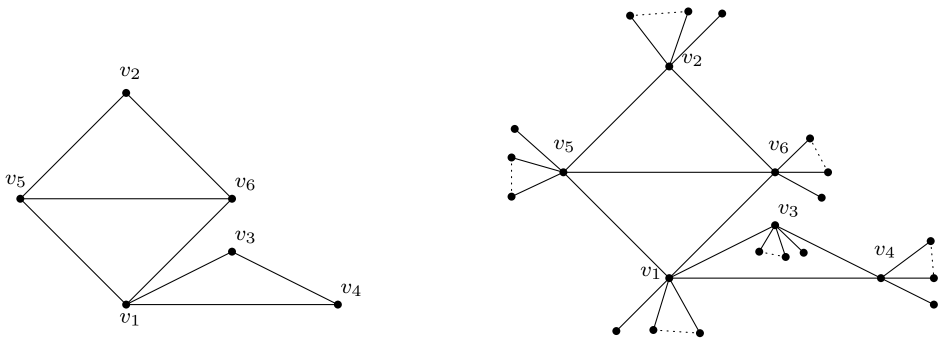
$e = uv$ that does not belong to a cycle of length 3. Suppose A is a graph constructed from G by contraction and then deleting the edge $e = uv$, and H is another graph constructed from A by adding a new vertex to H and connecting it to the contracted vertices u and v. The resulting graph G' is a simple tri-cyclic $n-$vertex graph containing $p + 1$ pendant vertices.

It should be noted that this procedure is decreasing the length of at least one cycle. In what follows, we prove that $M_1(G) < M_1(G')$ and $M_2(G) < M_2(G')$. To prove the statement, we assume that $d(u) = s \geq 2$, $d(v) = r \geq 2$, $N_G(u)-\{v\} = \{x_1, \cdots, x_{s-1}\}$ and $N_G(v) - \{u\} = \{y_1, \cdots, y_{r-1}\}$. Therefore,

$$\begin{aligned} M_1(G) - M_1(G') &= r^2 + s^2 - (r + s - 1)^2 - 1 \\ &= -2rs - 2 + 2r + 2s < 0\,, \end{aligned}$$

and so $M_1(G) < M_1(G')$. On the other hand,

$$\begin{aligned} M_2(G) \,-\, M_2(G') &= \sum_{i=1}^{s-1} sd(x_i) + \sum_{i=1}^{r-1} rd(y_i) \\ &+ rs - \left(\sum_{i=1}^{s-1}(r + s - 1)d(x_i) + \sum_{i=1}^{r-1}(r + s - 1)d(y_i) + (r + s - 1) \right) \\ &= -\sum_{i=1}^{s-1}(r - 1)d(x_i) - \sum_{i=1}^{r-1}(s - 1)d(y_i) \\ &- (r + s - 1) + rs < 0. \end{aligned}$$



(a) (b)

Figure 5. $F_n(n_1, n_2, n_3, n_4, n_5, n_6)$

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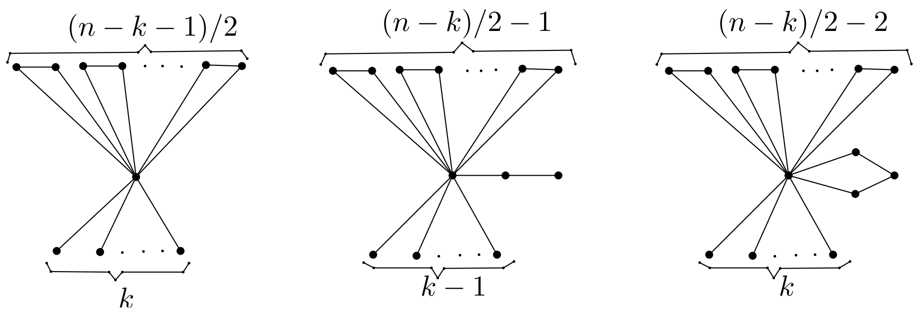
Therefore, we conclude that if the number of non-pendant vertices decreases then the first and second Zagreb indices of the graph under consideration will increase. This implies that the maximum of Zagreb indices among all tri-cyclic graphs will be occurred in graphs with a few number of non-pendant vertices. Therefore, the maximum of Zagreb indices will occur when each chordless cycle has length 3. The set of all tri-cyclic cactus graphs is denoted by Λ. We have following simple and useful Lemma:

Lemma 2. *Let G be a graph in Λ.*

(i). If $n-k \equiv 1 \pmod 2$, then $M_1(G) \le n^2 - n + 18$, with equality if and only if $G \cong B^1(n,k)$, where $B^1(n,k)$ is depicted in Fig. 7(a).



(a) $C^1(n,k)$ (b) $C^2(n,k)$ (c) $C^3(n,k)$

Figure 6. The graphs $C^1(n,k)$, $C^2(n,k)$ and $C^3(n,k)$.

(ii). *If* $n-k \equiv 0 \pmod 2$*, then* $M_1(G) \leq n^2 - 3n + 24$*, with equality if and only if* $G \cong B^2(n,k)$ *where* $B^2(n,k)$ *is depicted in Fig. 7(b).*

(iii). *If* $n-k \equiv 0 \pmod 2$*, then* $M_1(G) \leq n^2 - 3n + 24$*, with equality if and only if* $G \cong B^3(n,k)$ *where* $B^3(n,k)$ *is depicted in Fig. 7(c).*

(iv). *If* $n-k \equiv 1 \pmod 2$*, then* $M_2(G) \leq n^2 + 4n + 7$*, with equality if and only if* $G \cong B^1(n,k)$*, where* $B^1(n,k)$ *is depicted in Fig. 7(a).*

(v). *If* $n-k \equiv 0 \pmod 2$*, then* $M_2(G) \leq n^2 + 3n + 4$*, with equality if and only if* $G \cong B^2(n,k)$*, where* $B^2(n,k)$ *is depicted in Fig. 7(b).*

Proof. We can demonstrate the proof by putting $k = n-7$ and $n-8$ in Lemma 1. QED

Lemma 3. *If* $G = K_n(n_1, n_2, n_3, n_4)$ *with* $n_i \geq n_j \geq 2$, $1 \leq i \neq j \leq 4$ *and* b *is a pendant vertex of* v_j*. Then*

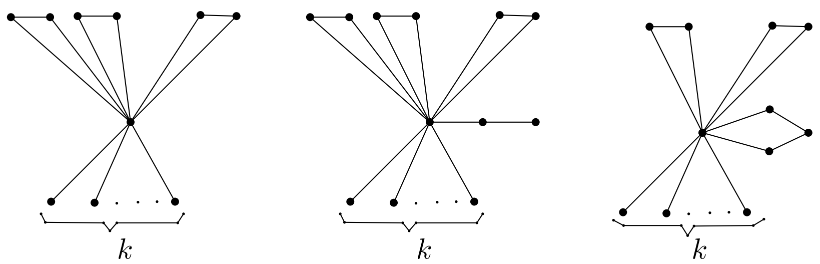
$$M_1(G - v_jb + v_ib) > M_1(G) \quad \text{and} \quad M_2(G - v_jb + v_ib) > M_2(G)\,.$$

Proof. Without loss of generality, we can assume that $i = 1$ and $j = 2$. Then

$$\begin{aligned} M_1(G - v_2b + v_1b) - M_1(G) &= n_1 + (n_1+3)^2 + (n_2-2) + (n_2+1)^2 \\ &\quad -(n_1-1) - (n_1+2)^2 - (n_2-1) - (n_2+2)^2 \\ &= 2n_1 - 2n_2 + 2 > 0\,. \end{aligned}$$

Also, we have :

$$\begin{aligned} M_2(G - v_2b + v_1b) - M_2(G) &= [n_1(n_1+3) - (n_1-1)(n_1+2)] \\ &\quad +[(n_2-2)(n_2+1) - (n_2-1)(n_2+2)] \end{aligned}$$



(a) $B^1(n,k)$ (b) $B^2(n,k)$ (c) $B^3(n,k)$

Figure 7. The graphs $B^1(n,k)$, $B^2(n,k)$ and $B^3(n,k)$.

$$
\begin{aligned}
&+[(n_2+1)(n_1+3)-(n_1+2)(n_2+2)] \\
&= n_1 - n_2 + 1 > 0\,,
\end{aligned}
$$

which completes the proof. QED

Lemma 4. *Let $G = K_n(n_1, n_2, n_3, n_4)$, where $n_1, n_2 \geq 2$. Then*

a) $M_1(G) \leq M_1(K_n(n-3, 1, 1, 1))$,

b) $M_2(G) \leq M_2(K_n(n-3, 1, 1, 1))$.

Proof. Without loss of generality let $n_1 \geq \ldots \geq n_4$. By Lemma 3 and putting $n_3 = n_4 = 1$, one can see that with deletion of any pendant edge from the vertex v_2 and adding it to vertex v_1, the first and second Zagreb indices will be increased. Therefore, applying Lemma 3, $n_2 + n_3 + n_4 - 3$ times in a row we find $M_i(K_n(n_1, n_2, n_3, n_4)) \leq M_i(K_n(n_1 + n_2 + n_3 + n_4 - 3, 1, 1, 1))$, for $i = 1, 2$. QED

Theorem 1. *Suppose $G \in R_{n,n-4}$ with $n \geq 4$. Then*

a) $M_1(G) \leq n^2 - n + 24\,,$

b) $M_2(G) \leq n^2 + 4n + 22\,.$

The equalities hold if and only if $G \cong K_n(n-3, 1, 1, 1)$.

Proof. **a)** Suppose $G \in R_{n,n-4}$ has maximum of first Zagreb index. By Lemma 3, one can find another graph H in R_n such that $M_1(H) > M_1(G)$. Without loss of generality, we assume that $G = K_n(n_1, n_2, 1, 1)$. Then by Lemma 4, $G = K_n(n_1 + n_2 - 1, 1, 1, 1)$. Equality holds if and only if $G \cong K_n(n-3, 1, 1, 1)$. The proof of **b)** is similar. QED

Lemma 5. *Let $G = q_n(n_1, n_2, n_3, n_4, n_5)$, $1 \leq i \leq 5$, $n_i \geq 2$. Then*

$$
\begin{array}{lrcl}
1) & M_1(G) & < & M_1(q_n(n_1+1, n_2-1, n_3, n_4, n_5)), \\
2) & M_2(G) & < & M_2(q_n(n_1+1, n_2-1, n_3, n_4, n_5)), \\
3) & M_1(G) & < & M_1(q_n(n_1, n_2, n_3+1, n_4-1, n_5)), \\
4) & M_2(G) & < & M_2(q_n(n_1, n_2, n_3+1, n_4-1, n_5)), \\
5) & M_1(G) & < & M_1(q_n(n_1, n_2, n_3+1, n_4, n_5-1)), \\
6) & M_2(G) & < & M_2(q_n(n_1, n_2, n_3+1, n_4, n_5-1)), \\
7) & M_1(G) & < & M_1(q_n(n_1+1, n_2, n_3-1, n_4, n_5)), \\
8) & M_2(G) & < & M_2(q_n(n_1+1, n_2, n_3-1, n_4, n_5)).
\end{array}
$$

Proof. Suppose $G_1 = q_n(n_1+1, n_2-1, n_3, n_4, n_5)$. Then

$$\begin{aligned} M_1(G_1) - M_1(G) &= n_1 + (n_1+4)^2 + (n_2+2)^2 + (n_2-2) - (n_1-1) \\ &- (n_1+3)^2 - (n_2+3)^2 - (n_2-1) \\ &= 2n_1 - 2n_2 + 2 > 0. \end{aligned}$$

Also,

$$\begin{aligned} M_2(G_1) - M_2(G) &= [n_1(n_1+4) - (n_1-1)(n_1+3)] \\ &+ [(n_2-2)(n_2+2) - (n_2-1)(n_2+3)] \\ &+ [(n_2+2)(n_1+4) - (n_2+3)(n_1+3)] \\ &= n_1 - n_2 + 1 > 0. \end{aligned}$$

Other cases are similar and so they are omitted. QED

Lemma 6. *Let $G = q_n(n_1, n_2, 1, 1, 1)$ with $n_1, n_2 \geq 2$. Then*

a) $M_1(G) < M_1(q_n(n_1+n_2-1, 1, 1, 1))$,

b) $M_2(G) < M_2(q_n(n_1+n_2-1, 1, 1, 1))$.

Proof. Suppose $G_1 = q_n(n_1+n_2-1, 1, 1, 1)$, where $n_1, n_2 \geq 2$. Then

$$\begin{aligned} M_1(G_1) - M_1(G) &= n_1 + n_2 - 2 + (n_1+n_2+2)^2 + 28 \\ &- (n_1-1) - (n_2-1) - (n_1+3)^2 - (n_2+3)^2 - 12 \\ &= 2n_1n_2 + 2 - 2n_1 - 2n_2 > 0, \end{aligned}$$

which completes the proof of part (a). To prove (b), we notice that

$$\begin{aligned} M_2(G_1) - M_2(G) &= (n_1+n_2-2)(n_1+n_2+2) + 6(n_1+n_2+2) + 24 \\ &+ 4(n_2+n_1+2) - (n_1-1)(n_1+3) - (n_2-1)(n_2+3) \\ &- 6(n_2+3) - 6(n_1+3) + (n_2+3)(n_1+3) \\ &= n_1n_2 + 1 - n_1 - n_2 > 0\,. \end{aligned}$$

Hence the result follows. QED

Lemma 7. *Suppose* $G = g_n(n_1, n_2, n_3, n_4, n_5)$*, where* $n_i \geq 2$*,* $1 \leq i \leq 5$*. Then we have:*

$$\begin{array}{ll}
1) & M_1(G) < M_1(g_n(n_1+1, n_2-1, n_3, n_4, n_5)),\\
2) & M_2(G) < M_2(g_n(n_1+1, n_2-1, n_3, n_4, n_5)),\\
3) & M_1(G) < M_1(g_n(n_1+1, n_2, n_3-1, n_4, n_5)),\\
4) & M_2(G) < M_2(g_n(n_1+1, n_2, n_3-1, n_4, n_5)),\\
5) & M_1(G) < M_1(g_n(n_1+1, n_2, n_3, n_4-1, n_5)),\\
6) & M_2(G) < M_2(g_n(n_1+1, n_2, n_3, n_4-1, n_5)),\\
7) & M_1(G) < M_1(g_n(n_1+1, n_2, n_3, n_4, n_5-1)),\\
8) & M_2(G) < M_2(g_n(n_1+1, n_2, n_3, n_4, n_5-1)).
\end{array}$$

Proof. Suppose $G_1 = g_n(n_1+1, n_2-1, n_3, n_4, n_5)$. Then,

$$\begin{array}{rcl}
M_1(G_1) - M_1(G) & = & n_1 + (n_1+4)^2 + (n_2-2) + (n_2+1)^2\\
& - & (n_1-1) - (n_1+3)^2 - (n_2+2)^2 - (n_2-1)\\
& = & 2n_1 + 4 - 2n_2 > 0.
\end{array}$$

which completes the proof of part (1). To prove (2), we notice that

$$\begin{array}{rcl}
M_2(G_1) - M_2(G) & = & [n_1(n_1+4) - (n_1-1)(n_1+3)]\\
& + & [(n_2-2)(n_2+1) - (n_2-1)(n_2+2)]\\
& + & [(n_2+1)(n_1+4) - (n_2+2)(n_1+3)]\\
& + & [(n_1+4)(n_4+1) - (n_1+3)(n_4+1)]\\
& + & [(n_5+1)(n_1+4) - (n_5+1)(n_1+3)]\\
& + & [(n_1+4)(n_3+2) - (n_1+3)(n_3+2)]\\
& + & [(n_2+1)(n_3+2) - (n_2+2)(n_3+2)]\\
& + & [(n_2+1)(n_5+1) - (n_2+2)(n_5+1)]\\
& = & n_1 + n_2 + 2 > 0.
\end{array}$$

Other cases are similar. QED

Lemma 8. *Suppose* $G = g_n(n_1, n_2, 1, 1, 1)$*, where* $n_1 \geq n_2 \geq 2$*. Then*

$$M_1(G) < M_1(g_n(n_1+n_2-1, 1, 1, 1, 1)),\ \ M_2(G) < M_2(g_n(n_1+n_2-1, 1, 1, 1, 1)).$$

Proof. By Lemma 7 and putting $n_3 = n_4 = n_5 = 1$, one can see that by removing any pendant edge from vertex v_2 and adding it to the vertex v_1, the first Zagreb index will increase. Therefore,

$$M_1(g_n(n_1, n_2, 1, 1, 1)) < M_1(g_n((n_1+1, n_2-1, 1, 1, 1))$$

$$\begin{aligned} &< \cdots \\ &< M_1(g_n(n_1 + n_2 - 2, 2, 1, 1, 1)) \\ &< M_1(g_n(n_1 + n_2 - 1, 1, 1, 1, 1))\,. \end{aligned}$$

The second part is similar and so omitted. QED

Theorem 2. *Suppose $G \in \mathfrak{g}_n$ where $n \geq 5$. Then*

$$M_1(G) \leq n^2 - n + 22,\ M_2(G) \leq n^2 + 4n + 16\,.$$

The equality holds if and only if $G \cong g_n(n-4,1,1,1,1)$.

Proof. Suppose $H = g_n(n_1, n_2, n_3, n_4, n_5) \in \mathfrak{g}_n$ has maximum of the first Zagreb index. By Lemma 7 one can find another graph in R_n with greater first Zagreb index. Suppose $G = g_n(n_1, n_2, 1, 1, 1)$. Then by Lemma 8, we get $M_1(G) \leq M_1(g_n(n_1+n_2-1, 1, 1, 1, 1))$. Equality holds if and only if $G \cong g_n(n-4, 1, 1, 1, 1)$. The proof of the second part is similar. QED

Consider the complete graph K_4 with vertex set $\{v_1, v_2, v_3, v_4\}$. Insert a vertex v_5 into an edge of K_4 and name the resulting graph Y_5. Define the graph $Y_5(n_1, n_2, n_3, n_4, n_5)$ to be constructed from Y_5 by attaching n_i edges to the vertex v_i, $1 \leq i \leq 5$. It is not so difficult to prove that $Y_5(n_1, n_2, n_3, n_4, n_5)$ is tri-cyclic such that its Zagreb indices are less than $g_n(n_1, n_2, n_3, n_4, n_5)$.

Theorem 3. *Suppose $G \in R_{n,n-5}$ and $n \geq 5$. Then*

a) $M_1(G) \leq n^2 - n + 24\,,$

b) $M_2(G) \leq n^2 + 4n + 19\,.$

The equality is satisfied if and only if $G \cong q_n(n-4,1,1,1,1)$.

Proof. By Lemma 6 and Theorem 2, the maximum of the first and second Zagreb indices are occurred in $q_n(n-4,1,1,1,1)$ and $g_n(n-4,1,1,1,1)$, respectively. So,

$$\begin{aligned} M_1(g_n(n-4,1,1,1,1)) &< M_1(q_n(n-4,1,1,1,1)), \\ M_2(g_n(n-4,1,1,1,1)) &< M_2(q_n(n-4,1,1,1,1)), \end{aligned}$$

which completes our argument. QED

Lemma 9. *Suppose* $G = E_n(n_1, n_2, n_3, n_4, n_5, n_6)$*, where* $n_i \geq 2$ *for* $1 \leq i \leq 6$*. Then*

1) $M_1(G) < M_1(E_n(n_1+1, n_2-1, n_3, n_4, n_5, n_6))$,
2) $M_2(G) < M_2(E_n(n_1+1, n_2-1, n_3, n_4, n_5, n_6))$,
3) $M_1(G) < M_1(E_n(n_1+1, n_2, n_3-1, n_4, n_5, n_6))$,
4) $M_2(G) < M_2(E_n(n_1+1, n_2, n_3-1, n_4, n_5, n_6))$,
5) $M_1(G) < M_1(E_n(n_1+1, n_2, n_3, n_4-1, n_5, n_6))$,
6) $M_2(G) < M_2(E_n(n_1+1, n_2, n_3, n_4-1, n_5, n_6))$.
7) $M_1(G) < M_1(E_n(n_1+1, n_2, n_3, n_4, n_5-1, n_6))$,
8) $M_2(G) < M_2(E_n(n_1+1, n_2, n_3, n_4, n_5-1, n_6))$,
9) $M_1(G) < M_1(E_n(n_1+1, n_2, n_3, n_4, n_5, n_6-1))$,
10) $M_2(G) < M_2(E_n(n_1+1, n_2, n_3, n_4, n_5, n_6-1))$.

Proof. Suppose $G_1 = E_n(n_1+1, n_2-1, n_3, n_4, n_5, n_6)$. Then

$$\begin{aligned} M_1(G_1) - M_1(G) &= n_1 + (n_1+5)^2 + (n_2-2) + (n_2+1)^2 \\ &- (n_1-1) - (n_1+4)^2 - (n_2+2)^2 - (n_2-1) \\ &= 2n_1 - 2n_2 + 6 > 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} M_2(G_1) - M_2(G) &= [n_1(n_1+5) - (n_1-1)(n_1+4)] \\ &+ [(n_2-2)(n_2+1) - (n_2-1)(n_2+2)] \\ &+ [(n_2+1)(n_1+5) - (n_2+2)(n_1+4)] \\ &+ [(n_1+5)(n_3+1) - (n_1+4)(n_3+1)] \\ &+ [(n_4+1)(n_1+5) - (n_4+1)(n_1+4)] \\ &+ [(n_1+5)(n_5+1) - (n_1+4)(n_5+1)] \\ &+ [(n_1+5)(n_6+1) - (n_1+4)(n_6+1)] \\ &+ [(n_2+1)(n_5+1) - (n_2+2)(n_5+1)] \\ &+ [(n_2+1)(n_6+1) - (n_2+2)(n_6+1)] \\ &= n_1 + n_2 + n_3 + n_4 + 3 > 0. \end{aligned}$$

Other cases are similar. QED

Lemma 10. *Suppose* $G = E_n(n_1, n_2, 1, 1, 1, 1)$*, where* $n_1 \geq n_2 \geq 2$*. Then*

$$\begin{aligned} M_1(G) &< M_1(E_n(n_1+n_2-1, 1, 1, 1, 1, 1)), \\ M_2(G) &< M_2(E_n(n_1+n_2-1, 1, 1, 1, 1, 1)). \end{aligned}$$

Lemma 11. *Suppose* $G = F_n(n_1, n_2, n_3, n_4, n_5, n_6)$*, where* $n_i \geq 2$ *for* $1 \leq i \leq 6$*. Then*

$$\begin{array}{rl}
1) & M_1(G) < M_1(F_n(n_1+1, n_2-1, n_3, n_4, n_5, n_6)),\\
2) & M_2(G) < M_2(F_n(n_1+1, n_2-1, n_3, n_4, n_5, n_6)),\\
3) & M_1(G) < M_1(F_n(n_1+1, n_2, n_3-1, n_4, n_5, n_6)),\\
4) & M_2(G) < M_2(F_n(n_1+1, n_2, n_3-1, n_4, n_5, n_6)),\\
5) & M_1(G) < M_1(F_n(n_1+1, n_2, n_3, n_4-1, n_5, n_6)),\\
6) & M_2(G) < M_2(F_n(n_1+1, n_2, n_3, n_4-1, n_5, n_6)).\\
7) & M_1(G) < M_1(F_n(n_1+1, n_2, n_3, n_4, n_5-1, n_6)),\\
8) & M_2(G) < M_2(F_n(n_1+1, n_2, n_3, n_4, n_5-1, n_6)),\\
9) & M_1(G) < M_1(F_n(n_1+1, n_2, n_3, n_4, n_5, n_6-1)),\\
10) & M_2(G) < M_2(F_n(n_1+1, n_2, n_3, n_4, n_5, n_6-1)).
\end{array}$$

Proof. Suppose $G_1 = F_n(n_1+1, n_2-1, n_3, n_4, n_5, n_6)$. Then

$$\begin{array}{rcl}
M_1(G_1) - M_1(G) & = & n_1 + (n_1+4)^2 + (n_2-2) + (n_2+1)^2\\
& - & (n_1-1) - (n_1+3)^2 - (n_2+2)^2 - (n_2-1)\\
& = & 2n_1 - 2n_2 + 4 > 0.
\end{array}$$

On the other hand,

$$\begin{array}{rcl}
M_2(G_1) - M_2(G) & = & [n_1(n_1+4) - (n_1-1)(n_1+3)]\\
& + & [(n_2-2)(n_2+1) - (n_2-1)(n_2+2)]\\
& + & [(n_2+1)(n_1+4) - (n_2+2)(n_1+3)]\\
& + & [(n_1+4)(n_3+2) - (n_1+3)(n_3+2)]\\
& + & [(n_1+4)(n_5+1) - (n_1+3)(n_5+1)]\\
& + & [(n_1+4)(n_6+1) - (n_1+3)(n_6+1)]\\
& + & [(n_2+1)(n_3+2) - (n_2+2)(n_3+2)]\\
& + & [(n_2+1)(n_4+1) - (n_2+2)(n_4+1)]\\
& = & n_1 + n_5 + n_6 - n_4 + 2 > 0.
\end{array}$$

Other cases are similar. QED

Lemma 12. *Suppose* $G = F_n(n_1, n_2, 1, 1, 1, 1)$*, where* $n_1 \geq n_2 \geq 2$*. Then*

$$M_1(G) < M_1(F_n(n_1+n_2-1, 1, 1, 1, 1, 1)),$$
$$M_2(G) < M_2(F_n(n_1+n_2-1, 1, 1, 1, 1, 1)).$$

No.	Graph	M_1	M_2	n
1	$K_n(n-3,1,1,1)$	n^2-n+24	$n^2+4n+22$	$n\geq 5$
2	$K_n(n-4,2,1,1)$	$n^2-3n+34$	$n^2+3n+27$	$n\geq 6$
3	$q_n(n-4,1,1,1,1)$	n^2-n+24	$n^2+4n+19$	$n\geq 5$
4	$q_n(n-5,2,1,1,1)$	$n^2-3n+36$	$n^2+3n+25$	$n\geq 7$
5	$g_n(n-4,1,1,1,1)$	n^2-n+22	$n^2+4n+16$	$n\geq 5$
6	$E_n(n-5,1,1,1,1,1)$	n^2-n+20	$n^2+4n+11$	$n\geq 6$
7	$F_n(n-5,1,1,1,1,1)$	$n^2-3n+28$	$n^2+2n+17$	$n\geq 6$
8	$B^1(n,k)$	n^2-n+18	n^2+4n+7	$n\geq 7$
9	$B^2(n,k)$	$n^2-3n+24$	n^2+3n+4	$n\geq 9$
10	$B^3(n,k)$	$n^2-3n+24$	n^2+2n+8	$n\geq 8$

Table 1. The First and Second Maximum of M_1 and M_2 in the Class of Tri−Cyclic Graphs.

Proof. The proof is similar to Lemma 4 and so it is omitted. QED

Theorem 4. *Among all graphs in R_n with $n\geq 5$ vertices,*

1. *$K_n(n-3,1,1,1)$ and $q_n(n-4,1,1,1,1)$ have the maximum values of first Zagreb index.*
2. *If $n=6,7$ then $K_6(2,2,1,1)$ and $q_7(2,2,1,1,1)$ have second maximum of the first Zagreb index, respectively. If $n\geq 5$ then $g_n(n-4,1,1,1,1)$ have second maximum of the first Zagreb index.*
3. *The graph $K_n(n-3,1,1,1)$ has maximum value of the second Zagreb index.*
4. *For $n=6,7,8$, the graph $K_n(n-4,2,1,1)$ and for cases $n=5$ and $n\geq 9$ the graph $q_n(n-4,1,1,1,1)$ have second maximum of the second Zagreb index.*

Proof. We record in Table 1, the maximum values of the first Zagreb index among of tri−cyclic graphs. The result follows easily from this table. QED

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