Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **33** (2013) no. 2, 97–106. doi:10.1285/i15900932v33n2p97

Duals of curves in Hyperbolic space

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Received: 13.3.2013; accepted: 9.8.2013.

Abstract. We define two kinds of duals of a curve in Hyperbolic space and investigate the singularities and the relationship from the view point of Legendrian dualities.

Keywords: Hyperbolic space, Legendrian dualities, curves

MSC 2010 classification: Primary 53A35, Secondary 58C27

1 Introduction

In this paper we investigate curves in Hyperbolic 3-space from the view point of dual relations. For a curve in Hyperbolic space with non-zero hyperbolic curvature, we define a *de Sitter dual surface* of the curve in de Sitter space which is the natural analogue of the dual surface of a curve in Euclidean 3-sphere. We give a classification of the singularities of de Sitter dual surface (§4) and investigate the geometric meanings (§5). On the other hand, there exists another dual surface in the lightcone which is called a *horospherical surface* of the curve [2]. In §3 we give a relationship between those dual surfaces of the curve from the view point of Legendrian dualities which were introduced in [3].

2 Basic notions and results

We adopt the model of the hyperbolic 3-space in the Lorentz-Minkowski space-time. Let $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) \mid x_i \in \mathbb{R}\ (i = 0, 1, 2, 3)\ \}$ be a 4-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, x_2, x_3),\ \boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -x_0y_0 + \sum_{i=1}^{3} x_iy_i$. We call

$(\mathbb{R}^4, \langle , \rangle)$ *Lorentz-Minkowski space-time.* We denote $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. For a vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number c, we define *the hyperplane with pseudo normal* $\boldsymbol{v}$ by $HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}$. We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *Hyperbolic* 3*-space* by

$$H^3_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 \geq 1\},$$

de Sitter 3*-space* by

$$S^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}$$

and a *closed lightcone* with the vertex $\boldsymbol{a}$ by

$$LC_a = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\ \}.$$

We denote that $LC^*_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in LC_0\ | x_0 > 0\ \}$ and we call it *the future lightcone* at the origin. For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x^1_0 & x^1_1 & x^1_2 & x^1_3 \\ x^2_0 & x^2_1 & x^2_2 & x^2_3 \\ x^3_0 & x^3_1 & x^3_2 & x^3_3 \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is the canonical basis of $\mathbb{R}^4_1$. We have three kinds of surfaces in $H^3_+(-1)$ which are given by intersections of $H^3_+(-1)$ and hyperplanes in $\mathbb{R}^4_1$. A surface $H^3_+(-1) \cap HP(\boldsymbol{v}, c)$ is called a *sphere*, an *equidistant surface* or a *horosphere* if $H(\boldsymbol{v}, c)$ is spacelike, timelike or lightlike respectively. We write $SP^2(\boldsymbol{v}, c)$ as a sphere and $ES^2(\boldsymbol{v}, c)$ as an equidistant surface. Especially, $ES^2(\boldsymbol{v}, 0)$ is called a *hyperbolic plane.*

We now construct the explicit differential geometry on curves in $H^3_+(-1)$. Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a regular curve. Since $H^3_+(-1)$ is a Riemannian manifold, we can reparametrise $\boldsymbol{\gamma}$ by the arc-length. Hence, we may assume that $\boldsymbol{\gamma}(s)$ is a unit speed curve. So we have the tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ with $\|\boldsymbol{t}(s)\| = 1$, where $\|\boldsymbol{v}\| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v} \rangle|}$. In the case when $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, then we have a unit vector $\boldsymbol{n}(s) = \dfrac{\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)}{\|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|}$. Moreover, define $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4_1$ along $\boldsymbol{\gamma}$. By standard arguments, under the assumption that $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, we have the following *Frenet-Serret type formula*:

$$\begin{cases} \boldsymbol{\gamma}'(s) & = \boldsymbol{t}(s), \\ \boldsymbol{t}'(s) & = \kappa_h(s)\boldsymbol{n}(s) + \boldsymbol{\gamma}(s), \\ \boldsymbol{n}'(s) & = -\kappa_h(s)\boldsymbol{t}(s) + \tau_h(s)\boldsymbol{e}(s), \\ \boldsymbol{e}'(s) & = -\tau_h(s)\boldsymbol{n}(s), \end{cases}$$

where $\kappa_h(s) = \|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|$ and $\tau_h(s) = -\dfrac{\det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s))}{(\kappa_h(s))^2}$.
Since $\langle \boldsymbol{t}'(s) - \boldsymbol{\gamma}(s), \boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\rangle = \langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle + 1$, the condition $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq -1$ is equivalent to the condition $\kappa_h(s) \neq 0$. We can study all properties of hyperbolic space curves by using this natural equation.

Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. We define a map as follows:

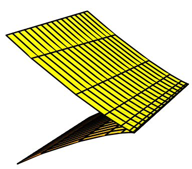
$$DD_\gamma : I \times J \longrightarrow S^3_1;\ DD_\gamma(s, \theta) = \cos\theta\boldsymbol{n}(s) + \sin\theta\boldsymbol{e}(s)$$

where $0 \leq \theta < 2\pi$, which is called a *de Sitter dual surface* of $\boldsymbol{\gamma}$,

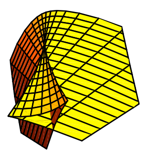
In this paper we give a classification of the singularities of this surface.

Theorem 1. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. Then we have the following*:
(1) *The de Sitter dual surface DD_γ of $\boldsymbol{\gamma}$ is singular at a point (s_0, θ_0) if and only if $\theta_0 = \pi/2$ or $\theta_0 = 3\pi/2$.*
(2) *The de Sitter dual surface DD_γ of $\boldsymbol{\gamma}$is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if $\theta_0 = \pi/2$ or $\theta_0 = 3\pi/2$ and $\tau_h(s_0) \neq 0$.*
(3) *The de Sitter dual surface DD_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallow tail SW at (s_0, θ_0) if $\theta_0 = \pi/2$ or $\theta_0 = 3\pi/2$, $\tau_h(s_0) = 0$ and $\tau_h'(s_0) \neq 0$.*

Here, $C \times \mathbb{R} = \{(x_1, x_2, x_3) | {x_1}^2 = {x_2}^3\}$ is the *cuspidal edge* (cf. Fig.1) and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the *swallow tail* (cf. Fig.2).



cuspidaledge
Fig.1.



swallowtail
Fig. 2.

The geometric meanings of the singularities of DD_γ and τ_h are gvien in §5.

On the other hand, the *horospherical surface* of $\boldsymbol{\gamma}$ is defined as follows [2]:

$$HS_\gamma : I \times J \longrightarrow LC^*;\ HS_\gamma(s,\theta) = \gamma(s) + \cos\theta \boldsymbol{n}(s) + \sin\theta \boldsymbol{e}(s).$$

In order to characterize the singularities of horospherical surface, a hyperbolic invariant $\sigma_h(s)$ is defined to be

$$\sigma_h(s) = ((\kappa_h')^2 - (\kappa_h)^2(\tau_h)^2((\kappa_h)^2 - 1))(s).$$

The singularities of the horospherical surfaces are classified into the following theorem.

Theorem 2. *[[2]] Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. Then we have the following*:
(1) *The horospherical surface HS_γ of $\boldsymbol{\gamma}$ is singular at a point (s_0, θ_0) if and only if $\cos\theta_0 = 1/\kappa_h(s_0)$.*
(2) *The horospherical surface HS_γ of $\boldsymbol{\gamma}$is locally diffeomorphic the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if $\cos\theta_0 = 1/\kappa_h(s_0)$ and $\sigma_h(s_0) \neq 0$.*
(3) *The horospherical surface HS_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallow tail SW at (s_0, θ_0) if $\cos\theta_0 = 1/\kappa_h(s_0)$, $\sigma_h(s_0) = 0$ and $\sigma_h'(s_0) \neq 0$.*

3 Legendrian dualities

In [3] the second author introduced the Legendrian dualities between pseudo-spheres in Lorentz-Minkowski space. We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this paper. Let N be a $(2n+1)$-dimensional smooth manifold and K be a field of tangent hyperplanes on N. Such a field is locally defined by a 1-form α. The tangent hyperplane field K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point on N. The pair (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α a *contact form.* A submanifold $i : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if dim $L = n$ and $di_x(T_xL) \subset K_{i(x)}$ at any $x \in L$. A smooth fibre bundle $\pi : E \to M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and the fibers of π are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset E$, $\pi \circ i : L \to M$ is called a *Legendrian map.* The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i and is denoted by $W(i)$.

The duality concepts we use in this paper are those introduced in [3], where four Legendrian double fibrations are considered on the subsets Δ_i, $i = 1, \ldots, 4$ of the product of two of the pseudo spheres $H^n(-1)$, S^n_1 and LC^*. In this paper we need the following three Legendrian double fibrations:

(1) (a) $H^3(-1) \times S^3_1 \supset \Delta_1 = \{(v,w) \mid \langle v,w\rangle = 0\}$,
(b) $\pi_{11} : \Delta_1 \to H^3(-1), \quad \pi_{12} : \Delta_1 \to S^3_1$,
(c) $\theta_{11} = \langle dv, w\rangle|\Delta_1,\ \theta_{12} = \langle v, dw\rangle|\Delta_1$.

(2) (a) $H^3(-1) \times LC^* \supset \Delta_2 = \{(v,w) \mid \langle v,w\rangle = -1\ \}$,
(b) $\pi_{21} : \Delta_2 \to H^3(-1), \pi_{22} : \Delta_2 \to LC^*$,
(c) $\theta_{21} = \langle dv, w\rangle|\Delta_2,\ \theta_{22} = \langle v, dw\rangle|\Delta_2$.

(3) (a) $LC^* \times S^3_1 \supset \Delta_3 = \{(v,w) \mid \langle v,w\rangle = 1\ \}$,
(b) $\pi_{31} : \Delta_3 \to LC^*, \pi_{32} : \Delta_3 \to S^3_1$,
(c) $\theta_{31} = \langle dv, w\rangle|\Delta_3,\ \theta_{32} = \langle v, dw\rangle|\Delta_3$.

Above, $\pi_{i1}(v,w) = v$ and $\pi_{i2}(v,w) = w$ for $i = 1,2,3$, $\langle dv, w\rangle = -w_0dv_0 + \sum_{i=1}^3 w_idv_i$ and $\langle v, dw\rangle = -v_0dw_0 + \sum_{i=1}^3 v_idw_i$. The 1-forms θ_{i1} and θ_{i2}, $i = 1,2,3$, define the same tangent hyperplane field over Δ_i which is denoted by K_i. We have the following duality theorem on the above spaces.

Theorem 3. *[[3]] The pairs (Δ_i, K_i), $i = 1,2,3$, are contact manifolds and π_{i1} and π_{i2} are Legendrian fibrations.*

Given a Legendrian submanifold $i : L \to \Delta_i$, $i = 1,2,3$, We say that $\pi_{i1}(i(L))$ is the Δ_i*-dual of* $\pi_{i2}(i(L))$ and vice-versa. Then we have the following dual relations on de Sitter duals and horospherical surfaces.

Theorem 4. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. Then we have the following*:
(1) *$\boldsymbol{\gamma}$ is the Δ_1-dual of DD_γ.*
(2) *$\boldsymbol{\gamma}$ is the Δ_2-dual of HS_γ.*
(3) *HS_γ is the Δ_3-dual of DD_γ.*

Proof. (1) Consider a mapping $\mathcal{L}_1 : I \times J \longrightarrow \Delta_1$ defined by $\mathcal{L}_1(s,\theta) = (\boldsymbol{\gamma}(s), DD_\gamma(s,\theta))$. Then we have $\langle \boldsymbol{\gamma}(s), DD_\gamma(s,\theta)\rangle = 0$, so that the mapping is well-defined. Since we have

$$\frac{\partial \mathcal{L}_1}{\partial s}(s,\theta) = (\boldsymbol{t}(s), \cos\theta \boldsymbol{n}'(s) + \sin\theta \boldsymbol{e}'(s)),\ \frac{\partial \mathcal{L}_1}{\partial \theta}(s,\theta) = (\boldsymbol{0}, -\sin\theta \boldsymbol{n}(s) + \cos\theta \boldsymbol{e}(s)),$$

$\mathcal{L}_1$ is an immersion. Moreover, we have $\mathcal{L}_1^*\theta_{11} = \langle \boldsymbol{t}(s), \cos\theta \boldsymbol{n}(s) + \sin\theta \boldsymbol{e}(s)\rangle = 0$. Therefore, $\mathcal{L}_1(I \times J)$ is a Legendrian submanifold in Δ_1.

(2) We define a mapping $\mathcal{L}_2 : I \times J \longrightarrow \Delta_2$ by $\mathcal{L}_2(s,\theta) = (\boldsymbol{\gamma}(s), HS_\gamma(s,\theta))$. We also define a mapping $\Psi_{12} : \Delta_1 \longrightarrow \Delta_2$ by $\Psi_{12}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{w})$. We can easily show that this mapping is well-defined. Moreover, we have $(\Psi_{12})^*\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{w}\rangle = \langle d\boldsymbol{v}, \boldsymbol{w}\rangle = \theta_{11}$. We have the inverse mapping $\Psi_{21} : \Delta_3 \longrightarrow \Delta_1$ defined by $\Psi_{21}(\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{w}-\boldsymbol{v})$. Thus, Ψ_{12} is a contact diffeomorphism from

Δ_1 to Δ_2. By definition, we have $\Psi_{12}\circ\mathcal{L}_1 = \mathcal{L}_2$, so that $\mathcal{L}_2(I\times J)$ is a Legendrian submanifold in Δ_2.

(3) We define a mapping $\Psi_{13} : \Delta_1 \longrightarrow \Delta_3$ by $\Psi_{13}(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}+\boldsymbol{w}, \boldsymbol{w})$. By the similar calculation to the case (2), we can show that Ψ_{13} is a contact diffeomorphism from Δ_1 to Δ_3. By definition, we have $\Psi_{13}\circ\mathcal{L}_1 = (HS_\gamma, DD_\gamma)$, so that $(HS_\gamma, DD_\gamma)(I\times J)$ is a Legendrian submanifold in Δ_3. This completes the proof. QED

4 De Sitter height functions

In this section we introduce a family of functions on a curve which is useful for the study of invariants of hyperbolic space curves. For a hyperbolic space curve $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$, we define a function $D : I\times S^3_1 \longrightarrow \mathbb{R}$ by $D(s, \boldsymbol{v}) = \langle\boldsymbol{\gamma}(s), \boldsymbol{v}\rangle$. We call D a *de Sitter height function* on $\boldsymbol{\gamma}$. We denote that $d_{v_0}(s) = D(s, \boldsymbol{v}_0)$ for any $\boldsymbol{v}_0 \in S^3_1$. Then we have the following proposition.

Proposition 1. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$. Then we have the following*:
(1) $d_{v_0}(s_0) = 0$ *if and only if there exist* $\lambda, \mu, \nu \in \mathbb{R}$ *such that* $\boldsymbol{v_0} = \lambda\boldsymbol{t}(s_0) + \mu\boldsymbol{n}(s_0) + \nu\boldsymbol{e}(s_0)$.
(2) $d_{v_0}(s_0) = d'_{v_0}(s_0) = 0$ *if and only if* $\boldsymbol{v_0} = \cos\theta\boldsymbol{n}(s_0) + \sin\theta\boldsymbol{e}(s_0)$, *where* $\theta \in [0, 2\pi)$.
(3) $d_{v_0}(s_0) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = 0$ *if and only if* $\boldsymbol{v_0} = \pm\boldsymbol{e}(s_0)$.
(4) $d_{v_0}(s_0) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''(s_0) = 0$ *if and only if* $\tau_h(s_0) = 0$ *and* $\boldsymbol{v_0} = \pm\boldsymbol{e}(s_0)$.
(5) $d_{v_0}(s_0) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = d^{(4)}_{v_0}(s_0) = 0$ *if and only if* $\tau_h(s_0) = \tau'_h(s_0) = 0$ *and* $\boldsymbol{v_0} = \pm\boldsymbol{e}(s_0)$.

Proof. Since $d_{v_0}(s) = \langle\boldsymbol{\gamma}(s), \boldsymbol{v}_0\rangle$, we have the following calculations:

(a) $d'_{v_0}(s) = \langle\boldsymbol{t}(s), \boldsymbol{v}_0\rangle$,

(b) $d''_{v_0}(s) = \langle\kappa_h(s)\boldsymbol{n}(s) + \boldsymbol{\gamma}(s), \boldsymbol{v}_0\rangle$,

(c) $d'''_{v_0}(s) = \langle(1-(\kappa_h)^2(s))\boldsymbol{t}(s) + \kappa'_h(s)\boldsymbol{n}(s) + \kappa_h(s)\tau_h(s)\boldsymbol{e}(s), \boldsymbol{v}_0\rangle$,

(d) $d^{(4)}_{v_0}(s) = \langle(1-(\kappa_h)^2(s))\boldsymbol{\gamma}(s) - 3\kappa_h(s)\kappa'_h(s)\boldsymbol{t}(s) + (\kappa_h(s) - \kappa^3_h(s)$
$\quad - \kappa_h(s)(\tau_h)^2(s) + \kappa''_h(s))\boldsymbol{n}(s) + (2\kappa'_h(s)\tau_h(s) + \kappa_h(s)\tau'_h(s)\boldsymbol{e}(s), \boldsymbol{v}_0\rangle$.

By the definition of the de Sitter height function, the assertion (1) follows. By the formula (a), $d_{v_0}(s) = d'_{v_0}(s_0) = 0$ if and only if and $\mu^2 + \nu^2 = 1$. It follows that $\mu = \cos\theta, \nu = \sin\theta$, where $0 \leq \theta < 2\pi$. Therefore the assertion (2) holds. By the formula (b), $d_{v_0}(s_0) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = 0$ if and only if

$\kappa_h(s)\cos\theta = 0$. Since $\kappa_h(s_0) \neq 0$, we have $\theta = \pi/2, 3\pi/2$. We have the assertion (3). By the formula (c), $d_{v_0}(s) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = 0$ if and only if $\tau_h(s) = 0$ and $\boldsymbol{v_0} = \pm\boldsymbol{e}(s_0)$. This means that the assertion (4) holds. By the similar arguments to the above, we can show the assertion (5). This completes the proof. $\boxed{QED}$

In order to prove Theorem 1, we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [1]. Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s, x_0)$. We say that f has an *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has an *$A_{\geq k}$-singularity* at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ has an A_k-singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 by $j^{(k-1)}(\partial F/\partial x_i(s, x_0))(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s-s_0)^j$ for $i = 1, \ldots, r$. Then F is called a *versal unfolding* if the $k \times r$ matrix of coefficients (α_{ji}) has rank k ($k \leq r$).

We now introduce an important set concerning the unfoldings relative to the above notions. The *discriminant set* of F is the set

$$\mathcal{D}_F = \left\{ x \in \mathbb{R}^r \;\middle|\; \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x) \right\}.$$

Then we have the following well-known result (cf., [1]).

Theorem 5. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has an A_k singularity at s_0. Suppose that F is a versal unfolding.*

(1) *If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

(2) *If $k = 3$, then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

For the proof of Theorem 1, we have the following proposition.

Proposition 2. *Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \neq 0$ and $D : I \times S^3_1 \longrightarrow \mathbb{R}$ be the de Sitter height function on $\boldsymbol{\gamma}(s)$. If d_{v_0} has an A_k-singularity ($k = 2, 3$) at s_0, then D is a versal unfolding of d_{v_0}.*

Proof. Let us consider the pseudo orthonormal basis $\boldsymbol{e}_0 = \boldsymbol{\gamma}(s_0)$, $\boldsymbol{e}_1 = \boldsymbol{t}(s_0)$, $\boldsymbol{e}_2 = \boldsymbol{n}(s_0)$ and $\boldsymbol{e}_3 = \boldsymbol{e}(s_0)$ instead of the canonical basis of $\mathbb{R}^4_1$. Then

$$D(s, \boldsymbol{v}) = -v_0x_0(s) + v_1x_1(s) + v_2x_2(s) + v_3x_3(s),$$

where v_i and $x_i(s)$ denote respectively the coordinates of $\boldsymbol{v}$ and $\boldsymbol{\gamma}(s)$ with respect to this basis. Since $v_3 = \sqrt{v_0^2 - v_1^2 - v_2^2 + 1}$, we have

$$\frac{\partial D}{\partial v_0}(s,\boldsymbol{v}) = -x_0(s) + \frac{v_0}{v_3}x_3(s),\ \frac{\partial^2 D}{\partial s \partial v_0}(s,\boldsymbol{v}) = -x_0'(s) + \frac{v_0}{v_3}x_3'(s),$$
$$\frac{\partial^3 D}{\partial s^2 \partial v_0}(s,\boldsymbol{v}) = -x_0''(s) + \frac{v_0}{v_3}x_3''(s),$$
$$\frac{\partial D}{\partial v_i}(s,\boldsymbol{v}) = x_i(s) - \frac{v_i}{v_3}x_3(s), \frac{\partial^2 D}{\partial s \partial v_i}(s,\boldsymbol{v}) = x_i'(s) - \frac{v_i}{v_3}x_3'(s),$$
$$\frac{\partial^3 D}{\partial^2 s \partial v_i}(s,\boldsymbol{v}) = x_i''(s) - \frac{v_i}{v_3}x_3''(s), \quad (i = 1, 2),$$

so that we consider the following matrix:

$$A = \begin{pmatrix} -x_0(s_0) + \frac{v_0}{v_3}x_3(s_0) & x_1(s_0) - \frac{v_1}{v_3}x_3(s_0) & x_2(s_0) - \frac{v_2}{v_3}x_3(s_0) \\ -x_0'(s_0) + \frac{v_0}{v_3}x_3'(s_0) & x_1'(s_0) - \frac{v_1}{v_3}x_3'(s_0) & x_2'(s_0) - \frac{v_2}{v_3}x_3'(s_0) \\ -x_0''(s_0) + \frac{v_0}{v_3}x_3''(s_0) & x_1''(s_0) - \frac{v_1}{v_3}x_3''(s_0) & x_2''(s_0) - \frac{v_2}{v_3}x_3''(s_0) \end{pmatrix}.$$

We denote that

$$\boldsymbol{a}_i = \begin{pmatrix} x_i(s_0) \\ x_i'(s_0) \\ x_i''(s_0) \end{pmatrix}, \ (i = 0, 1, 2, 3).$$

Then we have

$$\det A = \frac{v_0}{v_3}\det(\boldsymbol{a}_3\ \boldsymbol{a}_1\ \boldsymbol{a}_2) + \frac{v_1}{v_3}\det(\boldsymbol{a}_0\ \boldsymbol{a}_3\ \boldsymbol{a}_2) + \frac{v_2}{v_3}\det(\boldsymbol{a}_0\ \boldsymbol{a}_1\ \boldsymbol{a}_3) - \frac{v_3}{v_3}\det(\boldsymbol{a}_0\ \boldsymbol{a}_1\ \boldsymbol{a}_2)$$
$$= \frac{v_0}{v_3}\det(\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3) - \frac{v_1}{v_3}\det(\boldsymbol{a}_0\ \boldsymbol{a}_2\ \boldsymbol{a}_3) + \frac{v_2}{v_3}\det(\boldsymbol{a}_0\ \boldsymbol{a}_1\ \boldsymbol{a}_3) - \frac{v_3}{v_3}\det(\boldsymbol{a}_0\ \boldsymbol{a}_1\ \boldsymbol{a}_2).$$

Since we have

$$\boldsymbol{\gamma} \wedge \boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}'' = (-\det(\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3), -\det(\boldsymbol{a}_0\ \boldsymbol{a}_2\ \boldsymbol{a}_3), +\det(\boldsymbol{a}_0\ \boldsymbol{a}_1\ \boldsymbol{a}_3), -\det(\boldsymbol{a}_0\ \boldsymbol{a}_1\ \boldsymbol{a}_2))$$

at $s = s_0$, $\det A = \langle (\frac{v_0}{v_3}, \frac{v_1}{v_3}, \frac{v_2}{v_3}, \frac{v_3}{v_3}), (\boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}'' \wedge \boldsymbol{\gamma}''') \rangle = \langle \frac{1}{v_3}\boldsymbol{e}(s_0), \kappa_h(s_0)\boldsymbol{e}(s_0) \rangle = \frac{\kappa_h(s_0)}{v_3} \neq 0$. Thus, we have rank, $A = 3$.

If we consider the matrix

$$B = \begin{pmatrix} -x_0(s_0) + \frac{v_0}{v_3}x_3(s_0) & x_1(s_0) - \frac{v_1}{v_3}x_3(s_0) & x_2(s_0) - \frac{v_2}{v_3}x_3(s_0) \\ -x_0'(s_0) + \frac{v_0}{v_3}x_3'(s_0) & x_1'(s_0) - \frac{v_1}{v_3}x_3'(s_0) & x_2'(s_0) - \frac{v_2}{v_3}x_3'(s_0) \end{pmatrix},$$

this consists of the first and the second columns of the matrix A, so that the rank of B is two. If d_{v_0} has an A_k-singularity (k=2,3) at s_0, then D is a versal unfolding of d_{v_0}. This completes the proof. QED

Proof. (*Proof of Theorem 1.)* By Proposition 1, (2), the discriminant set $\mathcal{D}_D$ of the de Sitter height function D of $\boldsymbol{\gamma}$ is the image of the de Sitter dual surface of $\boldsymbol{\gamma}$. The singularities of the discriminant set are corresponding to the points of Proposition 1, (3), so that the assertion (1) holds. It also follows from Proposition 1, (4) and (5), that d_{v_0} has the A_2-type singularity (respectively, the A_3-type singularity) at $s = s_0$ if and only if $\theta_0 = 2/\pi, 3\pi/2$ and $\tau_h(s_0) \neq 0$.(respectively, $\theta_0 = \pi/2, 3\pi/2$ and $\tau_h(s_0) = 0, \tau_h'(s_0) \neq 0$). By Theorem 5 and Proposition 2, we have the assertions (2) and (3). This completes the proof. QED

5 Invariants of hyperbolic space curves

In this section we investigate the geometric properties of the singularities of DD_γ by using the invariant τ_h of $\boldsymbol{\gamma}$. At first, we consider the case when $\tau_h \equiv 0$.

Proposition 3. *Let* $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ *be a unit speed hyperbolic space curve with* $\kappa_h \neq 0$. *For the de Sitter dual suface* $DD_\gamma(s,\theta) = \cos\theta\boldsymbol{n}(s) + \sin\theta\boldsymbol{e}(s)$ *of* $\boldsymbol{\gamma}$ *and* $\theta_0 = \pi/2, 3\pi/2$, *the following conditions are equivalent*:

(*a*) $DD_\gamma(s,\theta_0)$ *is a constant vector,*

(*b*) $\tau_h(s) \equiv 0$,

(*c*) $\mathrm{Im}(\gamma) \subset ES^2(\boldsymbol{v}, 0)$ *for a spacelike vector* $\boldsymbol{v}$.

Proof. Suppose that $\theta_0 = \pi/2, 3\pi/2$. Then we have $DD_\gamma(s,\theta_0) = \pm\boldsymbol{e}(s)$ and $\dfrac{\partial DD_\gamma(s,\theta_0)}{\partial s} = \mp\tau_h(s)\boldsymbol{e}(s)$, so that $\dfrac{dDD_\gamma(s,\theta_0)}{ds}(s) \equiv 0$ if and only if $\tau_h(s) \equiv 0$. This means that the condition (a) is equivalent to the condition (b). Suppose that $\tau_h(s) \equiv 0$. Then $DD_\gamma(s,\theta_0) = \pm\boldsymbol{e}(s) = \pm\boldsymbol{v}$ are constant. Since $\langle\gamma(s), \pm\boldsymbol{e}(s)\rangle = 0$, $\mathrm{Im}(\gamma) \subset H^3_+(-1) \cap HP(\boldsymbol{v}, 0)$. Here, $\boldsymbol{e}(s) = \boldsymbol{v}$ is spacelike, so that $HP(\boldsymbol{v}, 0)$ is timelike.

On the other hand, suppose that $\mathrm{Im}(\gamma) \subset H^3_+(-1) \cap HP(\boldsymbol{v}, 0)$ and $\boldsymbol{v}$ is spacelike. Then we have $h_v(s) = \langle\gamma(s), \boldsymbol{v}\rangle = 0$. By Proposition 1, (4), $\tau_h(s) \equiv 0$. This completes the proof QED

The above proposition asserts that the degeneracy of singularities of DD_γ might relates how the curve contact with a hyperbolic plane. Let $F : H^3_+(-1) \longrightarrow \mathbb{R}$ be a submersion and $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a spacelike curve. We say that $\boldsymbol{\gamma}$ has *k-point contact* with $F^{-1}(0)$ at $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0,\ g^{(k)}(t_0) \neq 0$. We also say that $\boldsymbol{\gamma}$ has *at least k-point contact* with $F^{-1}(0)$ at $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$. We now consider a function $\mathcal{D} : H^3_+(-1) \times S^3_1 \longrightarrow \mathbb{R}$ defined by $\mathcal{D}(\boldsymbol{x}, \boldsymbol{v}) = \langle\boldsymbol{x}, \boldsymbol{v}\rangle$. Then we have

$D(s, \boldsymbol{v}) = \mathcal{D} \circ (\boldsymbol{\gamma} \times 1_{S_1^3})$. Thus, we have the following proposition as a corollary of Proposition 1.

Proposition 4. *For $\boldsymbol{v}_0 = DD_\gamma(s_0, \theta_0)$, we have the following*:
(1) $\boldsymbol{\gamma}$ *has at least* 2*-point contact with* $ES(\boldsymbol{v}_0, 0)$ *at* s_0 *if and only if* $\theta_0 = \pi/2$ *or* $3\pi/2$.
(2) $\boldsymbol{\gamma}$ *has* 3*-point contact with* $ES(\boldsymbol{v}_0, 0)$ *at* s_0 *if and only if* $\theta_0 = \pi/2$ *or* $3\pi/2$ *and* $\tau_h(s_0) \neq 0$.
(3) $\boldsymbol{\gamma}$ *has* 4*-point contact with* $ES(\boldsymbol{v}_0, 0)$ *at* s_0 *if and only if* $\theta_0 = \pi/2$ *or* $3\pi/2$, $\tau_h(s_0) = 0$ *and* $\tau_h'(s_0) \neq 0$.

By Theorem 1, we have the following geometric characterization of the singularities of DD_γ as follows:

Theorem 6. *Let* $\boldsymbol{\gamma} : I \longrightarrow H_+^3(-1)$ *be a unit speed hyperbolic space curve with* $\kappa_h \neq 0$. *For* $\boldsymbol{v}_0 = DD_\gamma(s_0, \theta_0)$, *we have the following*:
(1) *The de Sitter dual surface* DD_γ *is singular at a point* (s_0, θ_0) *if and only if* $\boldsymbol{\gamma}$ *has at least* 2*-point contact with* $ES(\boldsymbol{v}_0, 0)$ *at* s_0.
(2) *The de Sitter dual surface* DD_γ *of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the cuspidal edge* $C \times \mathbb{R}$ *at* (s_0, θ_0) *if* $\boldsymbol{\gamma}$ *has* 3*-point contact with* $ES(\boldsymbol{v}_0, 0)$ *at* s_0.
(3) *The de Sitter dual surface* DD_γ *of* $\boldsymbol{\gamma}$ *is locally diffeomorphic to the swallow tail* SW *at* (s_0, θ_0) *if* $\boldsymbol{\gamma}$ *has* 4*-point contact with* $ES(\boldsymbol{v}_0, 0)$ *at* s_0.

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