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Control affine systems on solvable three-dimensional Lie groups, II

Rory Biggs
Department of Mathematics (Pure and Applied), Rhodes University, 6140 Grahamstown, South Africa
rorybiggs@gmail.com

Claudiu C. Remsing
Department of Mathematics (Pure and Applied), Rhodes University, 6140 Grahamstown, South Africa
c.c.remsing@ru.ac.za

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Abstract. We seek to classify the full-rank left-invariant control affine systems evolving on solvable three-dimensional Lie groups. In this paper we consider only the cases corresponding to the solvable Lie algebras of types III, VI, and VII in the Bianchi-Behr classification.

Keywords: Left-invariant control system, (detached) feedback equivalence, affine subspace, solvable Lie algebra

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1 Introduction

Left-invariant control affine systems constitute an important class of systems, extensively used in many control applications. In this paper we classify, under local detached feedback equivalence, the full-rank left-invariant control affine systems evolving on certain solvable three-dimensional Lie groups. Specifically, we consider only those Lie groups with Lie algebras of types VI_h (including III), VI_0, VII_h, and VII_0 in the Bianchi-Behr classification.

We reduce the problem of classifying such systems to that of classifying affine subspaces of the associated Lie algebras. Thus, for each of the four types of Lie algebra, we need to classify their affine subspaces. A tabulation of the results is included as an appendix.

2 Invariant control systems and equivalence

A left-invariant control affine system Σ is a control system of the form

$$\dot{g} = g\,\Xi\,(\mathbf{1}, u) = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell.$$

Here G is a (real, finite-dimensional) Lie group with Lie algebra $\mathfrak{g}$. Also, the parametrization map $\Xi(\mathbf{1},\cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i.e., $B_1, \ldots, B_\ell$ are linearly independent). The "product" $g\,\Xi\,(\mathbf{1}, u)$ is to be understood as $T_{\mathbf{1}} L_g \cdot \Xi\,(\mathbf{1}, u)$, where $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ is the left translation by g. Note that the dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are invariant under left translations, i.e., $\Xi\,(g, u) = g\,\Xi\,(\mathbf{1}, u)$. We shall denote such a system by $\Sigma = (\mathsf{G}, \Xi)$ (cf. [3]).

The admissible controls are piecewise continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$. A *trajectory* for an admissible control $u(\cdot)$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. We say that a system Σ is *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. For more details about (invariant) control systems see, e.g., [2], [12], [13], [18], [17].

The image set $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot)$, called the *trace* of Σ, is an affine subspace of $\mathfrak{g}$. Specifically,

$$\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle.$$

A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Furthermore, Σ is said to have *full rank* if its trace generates the whole Lie algebra (i.e., the smallest Lie algebra containing Γ is $\mathfrak{g}$). Henceforth, we assume that all systems under consideration have full rank. (The full-rank condition is necessary for a system Σ to be controllable.)

An important equivalence relation for invariant control systems is that of detached feedback equivalence. Two systems are detached feedback equivalent if there exists a "detached" feedback transformation which transforms the first system to the second (see [4], [11]). Two detached feedback equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrized differently by admissible controls. More precisely, let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control affine systems. Σ and Σ' are called *locally detached feedback equivalent* (shortly DF_{loc}-equivalent) at points $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if there exist open neighbourhoods N and N' of a and a', respectively, and a diffeomorphism $\Phi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^{\ell'}$, $(g, u) \mapsto (\phi(g), \varphi(u))$ such that $\phi(a) = a'$ and $T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), \varphi(u))$ for $g \in N$

and $u \in \mathbb{R}^\ell$ (i.e., the diagram

$$\begin{tikzcd} N \times \mathbb{R}^\ell \arrow[r, "\phi\times\varphi"] \arrow[d, "\Xi"'] & N' \times \mathbb{R}^{\ell'} \arrow[d, "\Xi'"] \\ TN \arrow[r, "T\phi"'] & TN' \end{tikzcd}$$

commutes).

Any DF_{loc}-equivalence between two control systems can be reduced to an equivalence between neighbourhoods of the identity. More precisely, Σ and Σ' are DF_{loc}-equivalent at $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if and only if they are DF_{loc}-equivalent at $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1} \in \mathsf{G}'$ ([4]). Henceforth, we will assume that any DF_{loc}-equivalence is between neighbourhoods of identity. We recall an algebraic characterization of this equivalence.

Proposition 1 ([4]). *Σ and Σ' are DF_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

For the purpose of classification, we may assume that Σ and Σ' have the same Lie algebra $\mathfrak{g}$. We will say that two affine subspaces Γ and Γ' are $\mathfrak{L}$-*equivalent* if there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$. Then Σ and Σ' are DF_{loc}-equivalent if and only if there traces Γ and Γ' are $\mathfrak{L}$-equivalent. This reduces the problem of classifying under DF_{loc}-equivalence to that of classifying under $\mathfrak{L}$-equivalence. Suppose $\{\Gamma_i : i \in I\}$ is an exhaustive collection of (non-equivalent) class representatives (i.e., any affine subspace is $\mathfrak{L}$-equivalent to exactly one Γ_i). For each $i \in I$, we can easily find a system $\Sigma_i = (\mathsf{G}, \Xi_i)$ with trace Γ_i. Then any system Σ is DF_{loc}-equivalent to exactly one Σ_i.

3 Affine subspaces of three-dimensional Lie algebras

The classification of three-dimensional Lie algebras is well known. The classification over $\mathbb{C}$ was done by S. Lie (1893), whereas the standard enumeration of the real cases is that of L. Bianchi (1918). In more recent times, a different (method of) classification was introduced by C. Behr (1968) and others (see [15], [14], [16] and the references therein); this is customarily referred to as the *Bianchi-Behr classification* (or even the "Bianchi-Schücking-Behr classification"). Any solvable three-dimensional Lie algebra is isomorphic to one of nine types (in fact, there are seven algebras and two parametrized infinite families of algebras). In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutator operation is given by

$$\begin{aligned}
[E_2, E_3] &= n_1 E_1 - a E_2 \\
[E_3, E_1] &= a E_1 + n_2 E_2 \\
[E_1, E_2] &= 0.
\end{aligned}$$

The (Bianchi-Behr) structure parameters a, n_1, n_2 for each type are given in table 1. For the two infinite families, VI_h and VII_h, each value of the parameter

Type	Notation	a	n_1	n_2	Representatives
I	$3\mathfrak{g}_1$	0	0	0	$\mathbb{R}^3$
II	$\mathfrak{g}_{3.1}$	0	1	0	$\mathfrak{h}_3$
$III = VI_{-1}$	$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	1	1	-1	$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$
IV	$\mathfrak{g}_{3.2}$	1	1	0	
V	$\mathfrak{g}_{3.3}$	1	0	0	
VI_0	$\mathfrak{g}_{3.4}^0$	0	1	-1	$\mathfrak{se}(1,1)$
$VI_h,\ {\scriptstyle h<0 \atop h\neq -1}$	$\mathfrak{g}_{3.4}^h$	$\sqrt{-h}$	1	-1	
VII_0	$\mathfrak{g}_{3.5}^0$	0	1	1	$\mathfrak{se}(2)$
$VII_h,\ {\scriptstyle h>0}$	$\mathfrak{g}_{3.5}^h$	$\sqrt{h}$	1	1	

Table 1. Bianchi-Behr classification (solvable)

h yields a distinct (i.e., non-isomorphic) Lie algebra.

In this paper we will only consider types III, VI_0, VI_h, VII_0, and VII_h. The other solvable Lie algebras (i.e., those of types II, IV, and V) are treated in [7]. (For the Abelian Lie algebra $3\mathfrak{g}_1$ the classification is trivial.) Furthermore, type $III = VI_{-1}$ will be considered as part of VI_h.

An affine subspace Γ of a Lie algebra $\mathfrak{g}$ is written as

$$\Gamma = A + \Gamma^0 = A + \langle B_1, B_2, \ldots, B_\ell \rangle$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$. Let Γ_1 and Γ_2 be two affine subspaces of $\mathfrak{g}$. Γ_1 and Γ_2 are $\mathfrak{L}$-equivalent if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. $\mathfrak{L}$-equivalence is a genuine equivalence relation. (Note that $\Gamma_1 = A_1 + \Gamma_1^0$ and $\Gamma_2 = A_2 + \Gamma_2^0$ are $\mathfrak{L}$-equivalent if and only if there exists an automorphism ψ such that $\psi \cdot \Gamma_1^0 = \Gamma_2^0$ and $\psi \cdot A_1 \in \Gamma_2$.) An affine subspace Γ is said to have *full rank* if it generates the whole Lie algebra. The full-rank property is invariant under $\mathfrak{L}$-equivalence. Henceforth, we assume that all affine subspaces under consideration have full rank.

In this paper we classify, under $\mathfrak{L}$-equivalence, the (full-rank) affine subspaces of $\mathfrak{g}_{3.4}^{h}$ (including $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$), $\mathfrak{g}_{3.4}^{0}$, $\mathfrak{g}_{3.5}^{h}$, and $\mathfrak{g}_{3.5}^{0}$. Clearly, if Γ_1 and Γ_2 are $\mathfrak{L}$-equivalent, then they are necessarily of the same dimension. Furthermore, $0 \in \Gamma_1$ if and only if $0 \in \Gamma_2$. We shall find it convenient to refer to an ℓ-dimensional affine subspace Γ as an $(\ell, 0)$-affine subspace when $0 \in \Gamma$ (i.e., Γ is a vector subspace) and as an $(\ell, 1)$-affine subspace, otherwise. Alternatively, Γ is said to be homogeneous if $0 \in \Gamma$, and inhomogeneous otherwise.

Remark 1. No $(1,0)$-affine subspace has full rank. A $(1,1)$-affine subspace has full rank if and only if A, B_1, and $[A, B_1]$ are linearly independent. A $(2,0)$-affine subspace has full rank if and only if B_1, B_2, and $[B_1, B_2]$ are linearly independent. Any $(2,1)$-affine subspace or $(3,0)$-affine subspace has full rank.

Clearly, there is only one affine subspace whose dimension coincides with that of the Lie algebra $\mathfrak{g}$, namely the space itself. From the standpoint of classification this case is trivial and hence will not be covered explicitly.

Let us fix a three-dimensional Lie algebra $\mathfrak{g}$ (together with an ordered basis). In order to classify the affine subspaces of $\mathfrak{g}$, we require the (group of) automorphisms of $\mathfrak{g}$. These are well known (see, e.g., [9], [10], [16]); a summary is given in table 3. For each type of Lie algebra, we construct class representatives (by considering the action of automorphisms on a typical affine subspace). By using some classifying conditions, we explicitly construct $\mathfrak{L}$-equivalence relations relating an arbitrary affine subspace to a fixed representative. Finally, we verify that none of the representatives are equivalent.

The following simple result is useful.

Proposition 2. *Let Γ be a $(2,0)$-affine subspace of a Lie algebra $\mathfrak{g}$. Suppose $\{\Gamma_i \,:\, i \in I\}$ is an exhaustive collection of $\mathfrak{L}$-equivalence class representatives for $(1,1)$-affine subspaces of $\mathfrak{g}$. Then Γ is $\mathfrak{L}$-equivalent to at least one element of $\{\langle \Gamma_i \rangle \,:\, i \in I\}$.*

Proof. We have $\Gamma = \langle A, B \rangle$ for some $A, B \in \mathfrak{g}$. Thus $\Gamma' = A + \langle B \rangle$ is a $(1,1)$-affine subspace and so there exists an automorphism ψ such that $\psi \cdot \Gamma' = \Gamma_i$ for some $i \in I$. Hence $\psi \cdot \Gamma = \langle \Gamma_i \rangle$. QED

4 Type VI

The family $\mathfrak{g}_{3.4}^{h}$ of Lie algebras (including cases $h = 0$ and $h = -1$) has commutator relations

$$[E_2, E_3] = E_1 - aE_2, \qquad [E_3, E_1] = aE_1 - E_2, \qquad [E_1, E_2] = 0$$

in terms of an (appropriate) ordered basis (E_1, E_2, E_3). Here a is a non-negative parameter ($a = \sqrt{-h}$). However, we shall choose a different basis with respect to which the automorphisms take a simpler form (see table 3). Specifically, we will consider the basis

$$F_1 = E_3 \qquad F_2 = E_1 + E_2 \qquad F_3 = -E_1 + E_2$$

which has commutator relations

$$[F_2, F_3] = 0, \qquad [F_3, F_1] = -(a+1)F_3, \qquad [F_1, F_2] = (a-1)F_2.$$

4.1 Type VI_h

Consider the Lie algebra $\mathfrak{g}_{3.4}^h$ with $h < 0$. Recall that type III is included in this discussion. With respect to the ordered basis (F_1, F_2, F_3), the group of automorphisms takes the form

$$\mathsf{Aut}\big(\mathfrak{g}_{3.4}^h\big) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & u & 0 \\ y & 0 & v \end{bmatrix} \, : \, x, y, u, v \in \mathbb{R},\, uv \neq 0 \right\}.$$

We start the classification with the inhomogeneous one-dimensional case.

Proposition 3. *Any* $(1,1)$*-affine subspace of* $\mathfrak{g}_{3.4}^h$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = F_2 + F_3 + \langle F_1 \rangle \qquad \Gamma_{2,\alpha} = \alpha F_1 + \langle F_2 + F_3 \rangle\,.$$

Here $\alpha \neq 0$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Proof. Let $\Gamma = A + \Gamma^0$ be a $(1,1)$-affine subspace of $\mathfrak{g}_{3.4}^h$. First assume that $F_1^*(\Gamma^0) \neq \{0\}$. (Here F_1^* denotes the corresponding element of the dual basis.) Then $\Gamma = \sum_{i=1}^3 a_i F_i + \left\langle \sum_{i=1}^3 b_i F_i \right\rangle$ with $b_1 \neq 0$. Hence $\Gamma = a_2' F_2 + a_3' F_3 + \langle F_1 + b_2' F_2 + b_3' F_3 \rangle$. The condition that Γ has full rank is then equivalent to $a_2' a_3' \neq 0$. Thus

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ b_2' & a_2' & 0 \\ b_3' & 0 & a_3' \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_1 = \Gamma$.

Next assume $F_1^*(\Gamma^0) = \{0\}$ and $F_1^*(A) = \alpha \neq 0$. Then $\Gamma = \alpha F_1 + a_2 F_2 + a_3 F_3 + \langle b_2 F_2 + b_3 F_3 \rangle$. A simple calculation shows that $b_2 b_3 \neq 0$ and so

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_2}{\alpha} & b_2 & 0 \\ \frac{a_3}{\alpha} & 0 & b_3 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_{2,\alpha} = \Gamma$.

Lastly, we verify that none of these representatives are equivalent. $\Gamma_{2,\alpha}$ cannot be equivalent to Γ_1 as F_2 and F_3 are eigenvectors of every automorphism, $\Gamma^0_{2,\alpha} = \langle F_2 + F_3 \rangle$, and $\Gamma^0_1 = \langle F_1 \rangle$. On the other hand, as $F_1^*(\psi \cdot \alpha F_1) = \alpha$ for any automorphism ψ, it follows that $\Gamma_{2,\alpha}$ and $\Gamma_{2,\alpha'}$ are $\mathfrak{L}$-equivalent only if $\alpha = \alpha'$. $\boxed{QED}$

We obtain the result for the homogeneous two-dimensional case by use of propositions 2 and 3.

Proposition 4. *Any* $(2,0)$*-affine subspace of* $\mathfrak{g}^h_{3.4}$ *is* $\mathfrak{L}$*-equivalent to* $\langle F_1, F_2 + F_3 \rangle$.

We now move on to the inhomogeneous two-dimensional case.

Proposition 5. *Any* $(2,1)$*-affine subspace of* $\mathfrak{g}^h_{3.4}$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = F_3 + \langle F_1, F_2 \rangle \qquad \Gamma_2 = F_2 + \langle F_3, F_1 \rangle$$
$$\Gamma_3 = F_2 + \langle F_1, F_2 + F_3 \rangle \qquad \Gamma_{4,\alpha} = \alpha F_1 + \langle F_2, F_3 \rangle .$$

Here $\alpha \neq 0$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Proof. Let $\Gamma = A + \Gamma^0$ be a $(2,1)$-affine subspace of $\mathfrak{g}^h_{3.4}$. If $F_1^*(\Gamma^0) \neq \{0\}$, then not both F_2 and F_3 can be in Γ^0.

Assume that $F_1^*(\Gamma^0) \neq \{0\}$ and $F_2 \in \Gamma^0$. Then $\Gamma = a_1F_1 + a_3F_3 + \langle b_1F_1 + b_3F_3, F_2 \rangle$ with $b_1 \neq 0$. Hence $\Gamma = (a_3 - \frac{a_1b_3}{b_1})F_3 + \left\langle F_1 + \frac{b_3}{b_1}F_3, F_2 \right\rangle$. Thus $(a_3 - \frac{a_1b_3}{b_1}) \neq 0$ and

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{b_3}{b_1} & 0 & a_3 - \frac{a_1b_3}{b_1} \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_1 = \psi \cdot (F_3 + \langle F_1, F_2 \rangle) = \Gamma$.

Next assume that $F_1^*(\Gamma^0) \neq \{0\}$ and $F_3 \in \Gamma^0$. By a similar argument, we get $\Gamma = a_2F_2 + \langle F_3, F_1 + b_2F_2 \rangle$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ b_2 & a_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_2 = \psi \cdot (F_2 + \langle F_3, F_1 \rangle) = \Gamma$.

Now, assume that $F_1^*(\Gamma^0) \neq \{0\}$, $F_2 \notin \Gamma^0$, and $F_3 \notin \Gamma^0$. Then $\Gamma = \sum_{i=1}^3 a_i F_i + \left\langle \sum_{i=1}^3 b_i F_i, \sum_{i=1}^3 c_i F_i \right\rangle$ with $b_1 \neq 0$. Hence $\Gamma = a_2' F_2 + a_3' F_3 + \langle F_1 + b_2' F_2 + b_3' F_3, c_2' F_2 + c_3' F_3 \rangle$ with $c_2' \neq 0$ and $c_3' \neq 0$. Therefore $\Gamma = a_2'' F_2 + \langle F_1 + b_2' F_2 + b_3' F_3, F_2 + \frac{c_3'}{c_2'} F_3 \rangle$ with $a_2'' \neq 0$. Thus

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ b_2' & a_2'' & 0 \\ b_3' & 0 & \frac{a_2'' c_3'}{c_2'} \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_3 = \psi \cdot (F_2 + \langle F_1, F_2 + F_3 \rangle) = \Gamma$.

Lastly, assume that $F_1^*(\Gamma^0) = \{0\}$ and $F_1^*(A) = \alpha \neq 0$. Then $\Gamma_0 = \langle F_2, F_3 \rangle$ and so $\Gamma = \alpha F_1 + \langle F_2, F_3 \rangle = \Gamma_{4,\alpha}$.

We verify that none of these representatives are equivalent. F_2 and F_3 are eigenvectors of every automorphism. Hence, none of $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{4,\alpha}$ can be equivalent to any of $\Gamma_1, \Gamma_2, \Gamma_3$, other than matching a representative with itself. On the other hand, as $F_1^*(\psi \cdot \alpha F_1) = \alpha$ for any automorphism ψ, it follows that $\Gamma_{4,\alpha}$ and $\Gamma_{4,\alpha'}$ are $\mathfrak{L}$-equivalent only if $\alpha = \alpha'$. $\boxed{QED}$

4.2 Type VI_0

Consider the Lie algebra $\mathfrak{g}_{3.4}^0$. With respect to the ordered basis (F_1, F_2, F_2), the group of automorphisms takes the form

$$\mathsf{Aut}(\mathfrak{g}_{3.4}^0) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & u & 0 \\ y & 0 & v \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ x & 0 & u \\ y & v & 0 \end{bmatrix} \; : \; x, y, u, v \in \mathbb{R},\ uv \neq 0 \right\}.$$

We start with the inhomogeneous one-dimensional case. The proof of the following result is very similar to that of proposition 3 and will be omitted. However, the automorphism

$$\psi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

now allows us to change the sign of α (and thus α can always be taken to be positive).

Proposition 6. *Any $(1,1)$-affine subspace of $\mathfrak{g}_{3.4}^0$ is $\mathfrak{L}$-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = F_2 + F_3 + \langle F_1 \rangle \qquad\qquad \Gamma_{2,\alpha} = \alpha F_1 + \langle F_2 + F_3 \rangle .$$

Here $\alpha > 0$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

The result for the homogeneous two-dimensional case follows from propositions 2 and 6.

Proposition 7. *Any* $(2,0)$*-affine subspace of* $\mathfrak{g}_{3.4}^0$ *is* $\mathfrak{L}$*-equivalent to* $\langle F_1, F_2 + F_3\rangle$*.*

Lastly, we consider the inhomogeneous two-dimensional case. Again, the proof of the following result is very similar to that of proposition 5 and will be omitted. However, the two affine subspaces $F_3 + \langle F_1, F_2\rangle$ and $F_2 + \langle F_1, F_3\rangle$ are now $\mathfrak{L}$-equivalent, the required automorphism relating the two being

$$\psi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Also, the parameter α can be made positive.

Proposition 8. *Any* $(2,1)$*-affine subspace of* $\mathfrak{g}_{3.4}^0$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = F_3 + \langle F_1, F_2\rangle \qquad \Gamma_2 = F_2 + \langle F_1, F_2 + F_3\rangle$$
$$\Gamma_{3,\alpha} = \alpha F_1 + \langle F_2, F_3\rangle .$$

Here $\alpha > 0$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

4.3 Summary

We express the results for type VI in terms of the ordered basis (E_1, E_2, E_3). In some cases, additional automorphisms are used to simplify expressions.

Theorem 1. *Any affine subspace of* $\mathfrak{g}_{3.4}^0$ *(type* VI_0*) is* $\mathfrak{L}$*-equivalent to exactly one of* $E_2+\langle E_3\rangle$, $\alpha E_3+\langle E_2\rangle$, $\langle E_2, E_3\rangle$, $E_1+\langle E_2, E_3\rangle$, $E_1+\langle E_1 + E_2, E_3\rangle$*, and* $\alpha E_3 + \langle E_1, E_2\rangle$*, where* $\alpha > 0$*. Any affine subspace of* $\mathfrak{g}_{3.4}^h$ *(type* VI_h*, including* $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 = \mathfrak{g}_{3.4}^{-1}$*) is* $\mathfrak{L}$*-equivalent to exactly one of the above formal list for* $\mathfrak{g}_{3.4}^0$ *or* $E_1 + \langle E_1 - E_2, E_3\rangle$*, but with* $\alpha \neq 0$*. In both cases* α *parametrizes families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

5 Type VII

The family of Lie algebras $\mathfrak{g}_{3.5}^h$ (including cases $h = 0$ and $h = -1$), has commutator relations

$$[E_2, E_3] = E_1 - aE_2, \qquad [E_3, E_1] = aE_1 + E_2, \qquad [E_1, E_2] = 0$$

in terms of an (appropriate) ordered basis (E_1, E_2, E_3). Here a is a non-negative parameter $(a = \sqrt{h})$.

5.1 Type VII_h

Consider the Lie algebra $\mathfrak{g}_{3.5}^h$ with $h > 0$. With respect to the basis (E_1, E_2, E_3), the group of automorphisms takes the form

$$\mathsf{Aut}\big(\mathfrak{g}_{3.5}^h\big) = \left\{ \begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix} \,:\, x, y, u, v \in \mathbb{R},\, x^2 + y^2 \neq 0 \right\}.$$

We start with the inhomogeneous one-dimensional case.

Proposition 9. *Any $(1,1)$-affine subspace of $\mathfrak{g}_{3.5}^h$ is $\mathfrak{L}$-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = E_2 + \langle E_3 \rangle \qquad\qquad \Gamma_{2,\alpha} = \alpha E_3 + \langle E_2 \rangle \,.$$

Here $\alpha \neq 0$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Proof. Let $\Gamma = A + \Gamma^0$ be a $(1,1)$-affine subspace of $\mathfrak{g}_{3.5}^h$. First, assume that $E_3^*(\Gamma^0) \neq \{0\}$. (Here E_3^* denotes the corresponding element of the dual basis.) Then $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i \right\rangle$ with $b_3 \neq 0$. Hence $\Gamma = a_1' E_1 + a_2' E_2 + \langle b_1' E_1 + b_2' E_2 + E_3 \rangle$. Thus

$$\psi = \begin{bmatrix} a_2' & a_1' & b_1' \\ -a_1' & a_2' & b_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_1 = \psi \cdot (E_2 + \langle E_3 \rangle) = \Gamma$.

Now assume $E_3^*(\Gamma^0) = \{0\}$ and $E_3^*(A) = \alpha \neq 0$. Then $\Gamma = a_1 E_1 + a_2 E_2 + \alpha E_3 + \langle b_1 E_1 + b_2 E_2 \rangle$. Thus

$$\psi = \begin{bmatrix} b_2 & b_1 & \frac{a_1}{\alpha} \\ -b_1 & b_2 & \frac{a_2}{\alpha} \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma_{2,\alpha} = \psi \cdot (\alpha E_3 + \langle E_2 \rangle) = \Gamma$.

We verify that none of the class representatives are equivalent. As $\langle E_1, E_2 \rangle$ is invariant under automorphisms, $\Gamma_{2,\alpha}^0 = \langle E_2 \rangle$, and $\Gamma_1^0 = \langle E_3 \rangle$, it follows that $\Gamma_{2,\alpha}$ cannot be $\mathfrak{L}$-equivalent to Γ_2. On the other hand, as $E_3^*(\psi \cdot \alpha E_3) = \alpha$ for any automorphism ψ, it follows that $\Gamma_{2,\alpha}$ and $\Gamma_{2,\alpha'}$ are $\mathfrak{L}$-equivalent only if $\alpha = \alpha'$. $\boxed{QED}$

The result for the homogeneous two-dimensional case follows by propositions 2 and 9.

Proposition 10. *Any* $(2,0)$*-affine subspace of* $\mathfrak{g}_{3.5}^h$ *is* $\mathfrak{L}$*-equivalent to* $\langle E_2, E_3\rangle$*.*

Lastly, we consider the inhomogeneous two-dimensional case.

Proposition 11. *Any* $(2,1)$*-affine subspace of* $\mathfrak{g}_{3.5}^h$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following subspaces*

$$\Gamma_1 = E_1 + \langle E_2, E_3\rangle \qquad\qquad \Gamma_{2,\alpha} = \alpha E_3 + \langle E_1, E_2\rangle\,.$$

Here $\alpha \neq 0$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Proof. Let $\Gamma = A + \Gamma^0$ be a $(2,1)$-affine subspace. First, assume $E_3^*(\Gamma^0) \neq \{0\}$. Then $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\rangle$ with $c_3 \neq 0$. Hence $\Gamma = a_1' E_1 + a_2' E_2 + \langle b_1' E_1 + b_2' E_2, c_1' E_1 + c_2' E_2 + E_3\rangle$. Now either $b_1' \neq 0$ or $b_2' \neq 0$, and so

$$\begin{bmatrix} b_2' & -b_1' \\ b_1' & b_2' \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_2' \\ a_1' \end{bmatrix}$$

has a unique solution. By a simple argument, it follows that $v_2 \neq 0$ (as Γ is inhomogeneous). Therefore

$$\psi = \begin{bmatrix} v_2 b_2' & v_2 b_1' & c_1' \\ -v_2 b_1' & v_2 b_2' & c_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi\cdot\Gamma_1 = \psi\cdot(E_1 + \langle E_2, E_3\rangle) = \Gamma$. Indeed, $\psi\cdot\Gamma_1^0 = \Gamma^0$ and

$$\begin{aligned} \psi \cdot E_1 &= v_2 b_2' E_1 - v_2 b_1' E_2 \\ &= (a_1' - v_1 b_1')E_1 + (a_2' - v_1 b_2')E_2 \\ &= a_1' E_1 + a_2' E_2 - v_1(b_1' E_1 + b_2' E_2) \in \Gamma. \end{aligned}$$

Now assume $E_3^*(\Gamma^0) = \{0\}$ and $E_3^*(A) = \alpha \neq 0$. Then $\Gamma^0 = \langle E_1, E_2\rangle$ and so $\Gamma = \alpha E_3 + \langle E_1, E_2\rangle = \Gamma_{2,\alpha}$.

Finally, we verify that none of the class representatives are equivalent. As $\langle E_1, E_2\rangle$ is invariant under automorphisms, it follows that $\Gamma_{2,\alpha}$ cannot be equivalent to Γ_1. Then again, as $E_3^*(\psi\cdot\alpha E_3) = \alpha$ for any automorphism ψ, it follows that $\Gamma_{2,\alpha}$ is $\mathfrak{L}$-equivalent to $\Gamma_{2,\alpha'}$ only if $\alpha = \alpha'$. $\boxed{QED}$

5.2 Type VII_0

Consider the Lie algebra $\mathfrak{g}_{3.5}^0$. With respect to the ordered basis (E_1, E_2, E_3), the group of automorphisms takes the form

$$\mathsf{Aut}\big(\mathfrak{g}_{3.5}^0\big) = \left\{ \begin{bmatrix} x & y & u \\ -\varsigma y & \varsigma x & v \\ 0 & 0 & \varsigma \end{bmatrix} \, : \, x, y, u, v \in \mathbb{R}, \, x^2 + y^2 \neq 0, \, \varsigma = \pm 1 \right\}.$$

The proof of the following result is very similar to that of propositions 9 and 11, and will be omitted. (Again, the homogeneous two-dimensional case follows by proposition 2.) However, the automorphism $\psi = \mathrm{diag}(1, -1, -1)$ now allows us to change the sign of α (and thus α can always be taken to be positive).

Proposition 12. *Any affine subspace of $\mathfrak{g}_{3.5}^0$ is $\mathfrak{L}$-equivalent to exactly one of $E_2 + \langle E_3 \rangle$, $\alpha E_3 + \langle E_2 \rangle$, $\langle E_2, E_3 \rangle$, $E_1 + \langle E_2, E_3 \rangle$, and $\alpha E_3 + \langle E_1, E_2 \rangle$. Here $\alpha > 0$ parametrizes families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

5.3 Summary

Theorem 2. *Any affine subspace of $\mathfrak{g}_{3.5}^0$ or $\mathfrak{g}_{3.5}^h$ (type VII_0 or VII_h, respectively) is $\mathfrak{L}$-equivalent (with respect to the different ordered bases) to exactly one of $E_2 + \langle E_3 \rangle$, $\alpha E_3 + \langle E_2 \rangle$, $\langle E_2, E_3 \rangle$, $E_1 + \langle E_2, E_3 \rangle$, and $\alpha E_3 + \langle E_1, E_2 \rangle$, where $\alpha > 0$ for $\mathfrak{g}_{3.5}^0$ and $\alpha \neq 0$ for $\mathfrak{g}_{3.5}^h$. In both cases α parametrizes families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

6 Final remarks

A description of controllable single-input right-invariant systems on simply connected solvable Lie groups, up to dimension six, was obtained by Sachkov [19]. These results are of a different nature to those obtained here. (Besides, no equivalence relations were considered in [19].)

Agrachev and Barilari [1] recently classified the invariant sub-Riemannian structures on three-dimensional Lie groups. We, however, are concerned with the equivalence of the underlying invariant distributions. Invariant sub-Riemannian structures can be related to certain classes of invariant optimal control problems; two such problems are cost-equivalent only when the underlying invariant control systems are detached feedback equivalent ([8]).

The present paper forms part of a series in which the full-rank left-invariant control affine systems, evolving on three-dimensional Lie groups, are classified.

A summary of this classification can be found in [5]. The other solvable cases are treated in [7], whereas the semisimple cases are treated in [6].

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Type	(ℓ, ε)	Conditions		Equiv. repr.
VI_0	$(1,1)$	$F_1^*(\Gamma^0) \neq \{0\}$		$F_2 + F_3 + \langle F_1 \rangle$
		$F_1^*(\Gamma^0) = \{0\},\ F_1^*(A) = \pm\alpha,\ \alpha > 0$		$\alpha F_1 + \langle F_2 + F_3 \rangle$
	$(2,0)$			$\langle F_1, F_2 + F_3 \rangle$
	$(2,1)$	$F_1^*(\Gamma^0) \neq \{0\}$	$F_2 \in \Gamma^0 \ \vee \ F_3 \in \Gamma^0$	$F_3 + \langle F_1, F_2 \rangle$
			$F_2 \notin \Gamma^0 \ \wedge \ F_3 \notin \Gamma^0$	$F_2 + \langle F_1, F_2 + F_3 \rangle$
		$F_1^*(\Gamma^0) = \{0\},\ F_1^*(A) = \pm\alpha,\ \alpha > 0$		$\alpha F_1 + \langle F_2, F_3 \rangle$
VI_h	$(1,1)$	$F_1^*(\Gamma^0) \neq \{0\}$		$F_2 + F_3 + \langle F_1 \rangle$
		$F_1^*(\Gamma^0) = \{0\},\ F_1^*(A) = \alpha \neq 0$		$\alpha F_1 + \langle F_2 + F_3 \rangle$
	$(2,0)$			$\langle F_1, F_2 + F_3 \rangle$
	$(2,1)$	$F_1^*(\Gamma^0) \neq \{0\}$	$F_2 \in \Gamma^0$	$F_3 + \langle F_1, F_2 \rangle$
			$F_3 \in \Gamma^0$	$F_2 + \langle F_3, F_1 \rangle$
			$F_2 \notin \Gamma^0 \ \wedge \ F_3 \notin \Gamma^0$	$F_2 + \langle F_1, F_2 + F_3 \rangle$
		$F_1^*(\Gamma^0) = \{0\},\ F_1^*(A) = \alpha \neq 0$		$\alpha F_1 + \langle F_2, F_3 \rangle$
VII_0	$(1,1)$	$E_3^*(\Gamma^0) \neq \{0\}$		$E_2 + \langle E_3 \rangle$
		$E_3^*(\Gamma^0) = \{0\},\ E_3^*(A) = \pm\alpha,\ \alpha > 0$		$\alpha E_3 + \langle E_2 \rangle$
	$(2,0)$			$\langle E_2, E_3 \rangle$
	$(2,1)$	$E_3^*(\Gamma^0) \neq \{0\}$		$E_1 + \langle E_2, E_3 \rangle$
		$E_3^*(\Gamma^0) = \{0\},\ E_3^*(A) = \pm\alpha,\ \alpha > 0$		$\alpha E_3 + \langle E_1, E_2 \rangle$
VII_h	$(1,1)$	$E_3^*(\Gamma^0) \neq \{0\}$		$E_2 + \langle E_3 \rangle$
		$E_3^*(\Gamma^0) = \{0\},\ E_3^*(A) = \alpha \neq 0$		$\alpha E_3 + \langle E_2 \rangle$
	$(2,0)$			$\langle E_2, E_3 \rangle$
	$(2,1)$	$E_3^*(\Gamma^0) \neq \{0\}$		$E_1 + \langle E_2, E_3 \rangle$
		$E_3^*(\Gamma^0) = \{0\},\ E_3^*(A) = \alpha \neq 0$		$\alpha E_3 + \langle E_1, E_2 \rangle$

Table 2. Affine subspaces of Lie algebras

Type	Commutators	Automorphisms
VI_0	$[F_2, F_3] = 0$ $[F_3, F_1] = -F_3$ $[F_1, F_2] = -F_2$	$\begin{bmatrix} 1 & 0 & 0 \\ x & u & 0 \\ y & 0 & v \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ x & 0 & u \\ y & v & 0 \end{bmatrix};\ uv \neq 0$
VI_h	$[F_2, F_3] = 0$ $[F_3, F_1] = -(a+1)F_3$ $[F_1, F_2] = (a-1)F_2$	$\begin{bmatrix} 1 & 0 & 0 \\ x & u & 0 \\ y & 0 & v \end{bmatrix};\ uv \neq 0$
VI_0	$[E_2, E_3] = E_1$ $[E_3, E_1] = -E_2$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x & y & u \\ -y & -x & v \\ 0 & 0 & -1 \end{bmatrix};\ x^2 \neq y^2$
VI_h	$[E_2, E_3] = E_1 - aE_2$ $[E_3, E_1] = aE_1 - E_2$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix};\ x^2 \neq y^2$
VII_0	$[E_2, E_3] = E_1$ $[E_3, E_1] = E_2$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x & y & u \\ y & -x & v \\ 0 & 0 & -1 \end{bmatrix};\ x^2 \neq -y^2$
VII_h	$[E_2, E_3] = E_1 - aE_2$ $[E_3, E_1] = aE_1 + E_2$ $[E_1, E_2] = 0$	$\begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix};\ x^2 \neq -y^2$

Table 3. Lie algebra automorphisms