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THE INITIAL-VALUE PROBLEM FOR SINGULAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor Gottfried Köthe

Let A and B be $n\times n$ matrices, let A be a singular matrix and let $c_1,\ldots,c_n$ be arbitrary numbers. Consider the system of differential equations

$$AY'(t) + BY(t) = 0 \qquad \ldots \qquad t>0, \tag{1}$$

where Y is an unknown n-vector of differentiable functions such that

$$Y_1(0) = c_1,\ldots,Y_n(0) = c_n. \tag{2}$$

G. Doetsch [4] was the first to satisfactorily discuss such problems: the exension of the one-sided Laplace transformation used in [4] is in the present paper replaced by an endomorphism $y \to \mathscr{L}y$ of an algebra of generalized functions; our basic equation $\mathscr{L}Dy = s\mathscr{L}y - y(0)\,\delta$ (where δ is the Dirac distribution) leads to the system of algebraic equations studied in [1]-[6] and [8]; our basic equation is akin to the «generalized Mikusinski formula» obtained in [8] but does not require the very deep properties established in [8]. Initial-value problems for singular systems occur in applications; an example is discussed in §4 and §5.

1. OPERATORS

Test-functions are infinitely differentiable complex-valued functions defined on $(-\infty,\infty)$, they and their derivatives vanish at the origin.

Thus, a test-function $\phi(\ )$ is such that $0 = \phi(0) = \phi^{(n)}(0)$ for every integer $n \geq 0$. The equations $0 = \phi(0)$ and $\phi(t) = \exp(-1/|t|)$ for $-\infty < t < \infty$ define a test-function.

Operators assign test-functions to test-functions. Let p be an operator: if $\phi(\ )$ is a test-function, we denote by $p\colon \phi(\ )$ the test-function that the operator p assigns to the test-function $\phi(\ )$. Let p_1 and p_2 be operators. The operator p_1p_2 is such that

$$p_1p_2\colon\phi(\ ) = p_1\colon[p_2\colon\phi](\ ) \tag{1.1}$$

for every test-function $\phi(\ )$. The equation $p_1 = p_2$ holds only when $p_1\colon\phi(\ ) = p_2\colon\phi(\ )$ for every test-function $\phi(\ )$.

If n is an integer ≥ 0, the operator s^n assigns to each test-function $\phi(\ )$ its derivative $\phi^{(n)}(\ )$. In particular,

$$s\colon\phi(\ ) = \phi'(\ ) \qquad \text{and} \qquad s^0\colon\phi(\ ) = \phi(\ ).$$

Let p, p_1 and p_2 be operators. From (1.1) it follows that

(1.3) $$ps\colon \phi(\ ) = p\colon \phi'(\ ) \qquad \text{and} \qquad sp\colon \phi(\ ) = [p\colon \phi]'(\ )$$

for every test-function $\phi(\ )$. The operator $p_1 \pm p_2$ assigns to each test-function $\phi(\ )$ the test-function $p_1 : \phi(\ ) \pm p_2 : \phi(\ )$:

(1.4) $$[p_1 \pm p_2]\colon \phi(t) = p_1 : \phi(t) \pm p_2 : \phi(t) \ldots - \infty < t < \infty.$$

Let α and α_1 be numbers (possibly complex). The operators αs^0 assigns to each test-function $\phi(\ )$ the function $\alpha\phi(\ )$; consequently,

(1.5) $$[\alpha s^0]p\colon \phi(t) = \alpha[p\colon \phi(t)] \ldots - \infty < t < \infty$$

for every test-function $\phi(\ )$. Addition is associative and commutative. The following equations are easily verified

(1.6) $$[\alpha + \alpha_1]s^0 = \alpha s^0 + \alpha_1 s^0 \qquad \text{and} \qquad \alpha\alpha_1 s^0 = [\alpha s^0]\alpha_1 s^0;$$

also,

(1.7) $$1 s^0 = s_0,$$

(1.8) $$ps^0 = p = s^0 p = [1 s^0]p,$$

(1.9) $$p - p_1 = p + [-1 s^0]p,$$

(1.10) $$0 s^0 + p = p \qquad \text{and} \qquad p - p = 0 s^0,$$

(1.11) $$[p_1 + p_2]p = p_1 p + p_2 p \qquad \text{and} \qquad p_1[p_2 p] = [p_1 p_2]p,$$

2. FUNCTION-LIKE OPERATORS

Let $G(\ )$ be a function whose derivative $G'(\ )$ is *continuous* (i.e., continuous on the interval $(-\infty, \infty)$). Let $Y(\ )$ be piecewise-continuous (only a finite number of jumps in any finite sub-interval of $(-\infty, \infty)$). We write

(2.1) $$Y * G(t) \stackrel{\text{def}}{=} \int_0^t Y(\omega)G(t-\omega)\,d\omega \ldots - \infty < t < \infty,$$

where

$$\int_0^t = -\int_t^0 \ldots t < 0;$$

also, we denote by $I(\ )$ the unit constant:

$$I(x) = 1 \quad \ldots \quad -\infty < x < \infty;$$

accordingly,

$$(2.2) \qquad Y * I(t) = \int_0^t Y(\omega)\,d\omega \quad \ldots \quad -\infty < t < \infty.$$

If

$$(2.3) \qquad F = Y * I,$$

then $F(\ )$ is a continuous function, $0 = F(0)$ and

$$Y * G(t) = \int_0^t F'(\omega)G(t-\omega)\,d\omega = F(\omega)G(t-\omega)\Big|_0^t + \int_0^t F(\omega)G'(t-\omega)\,d\omega$$

when $-\infty < t < \infty$; consequently,

$$(2.4) \qquad Y * G(\ ) = G(0)F(\ ) + F * G'(\ )$$

2.5 Remark If $\phi(\ )$ is a test-function, then $\phi(0) = 0$ and (2.2) gives

$$(2.6) \qquad \phi' * I(t) = \int_0^t \phi'(\omega)\,d\omega = \phi(t) - 0 \quad \ldots \quad -\infty < t < \infty.$$

2.7. Remark. The space of continuous functions forms a commutative ring (see [9]); in particular,

$$G * H(\ ) = H * G(\ ) \quad \text{and} \quad G * [H * H_1](\ ) = [G * H] * H_1(\ )$$

for continuous functions $G(\ ), H(\ )$, and $H_1(\ )$.

2.8. Remark. If $G(\ )$ is continuous, then $[G * I]'(\ ) = G(\ )$. To verify that $[G * I]'(0) = G(0)$, note that

$$G * I(\varepsilon) - G * I(0) = \int_0^\varepsilon G(\omega)\,d\omega = G(\omega_\varepsilon)\varepsilon,$$

so that $\lim[G * I(\varepsilon) - G * I(0)]/\varepsilon = \lim G(\omega_\varepsilon) = G(0)$ as $\varepsilon \to 0$: these equations come from the Mean Value Theorem and $0 < |\omega_\varepsilon| < \varepsilon$.

2.9 Theorem. *Let $Y(\ )$ be piecewise-continuous: also, let $F = Y * I$. If $\phi(\ )$ is a test-function, then $F * \phi(\ )$ is a test-functions*

$$Y * \phi(\ ) = F * \phi'(\ ) = [F * \phi]'(\ ) \tag{2.10}$$

and

$$Y * \phi'(\ ) = [Y * \phi]'(\ ). \tag{2.11}$$

Proof. Set $G = \phi$ in (2.4) to obtain $Y * \phi(\ ) = F * \phi'(\ )$; this establishes the left side of (2.10). Let n be any integer ≥ 0 and set $\phi_n(\ ) = \phi^{(n)}(\ )$; in particular, $\phi_0(\ ) = \phi(\ )$. Note that $\phi_n(\ )$ is a test-function. Since both $F(\ )$ and $\phi_n'(\ )$ are continuous, it follows from 2.7 that

$$[F * \phi_n'] * I(\ ) = F * [\phi_n' * I](\ ) = F * \phi_n(\ ); \tag{3}$$

the right-hand equation comes from (2.6); since $F * \phi_n'(\ )$ is continuous, we can set $G = F * \phi_n'$ in 2.8 to obtain

$$G(\ ) = [G * I]'(\ ) = [F * \phi_n]'(\ );$$

the right-hand equation comes from $G = F * \phi_n'$ and (3); thus,

$$F * \phi_n'(\ ) = [F * \phi_n]'(\ ). \tag{4}$$

Choosing $n = 0$ in (4) gives the right-hand equation in (2.10), namely, $F * \phi'(\ ) = [F * \phi_n]'(\ )$.

To verifty that the equation

$$F * \phi^{(m)}(\ ) = [F * \phi]^{(m)}(\ ) \tag{5}$$

holds for every integer $m \geq 1$, we proceed by induction. Set $n = m$ in (4) to obtain

$$F * \phi^{(m+1)}(\ ) = [F * \phi^{(m)}]'(\ ) = [F * \phi]^{(m+1)}(\ );$$

the right-hand equation comes from the induction hypothesis (5). Thus, (5) holds for every integer $m \geq 1$; since $\phi^{(m)}(\ )$ is continuous, the function $F * \phi^{(m)}(\ )$ is continuous and vanishes at the origin; accordingly, (5) states the m^{th} derivative of $F{*}\phi(\ )$ has the properties; consequently, that function is a test-function.

It remains to verify (2.11). From (2.10) we get

$$[Y * \phi]'(\ ) = [F * \phi]^{(2)}(\ ) = F * \phi^{(2)}(\ ); \tag{6}$$

the right-hand equation comes from (5); replacing ϕ by ϕ' in (2.10), we get

$$Y * \phi'(\ ) = F * \phi^{(2)}(\ ) = [Y * \phi]'(\ );$$

the right-hand equation comes from (6).

2.12 Definition. *Let* $Y(\ )$ *be piecewise-continuous. We shall denote by* $\{Y\}$ *the operator that assigns to each test-function* $\phi(\ )$ *the test-function* $Y * \phi(\ )$ *:*

$$(2.13) \qquad \{Y\}\colon \phi(t) = Y * \phi(t) = \int_0^t Y(\omega)\phi(t-\omega)\,d\omega \ldots -\infty < t < \infty.$$

2.14. Again, let $\phi(\ )$ be a test-function. If $H(\ )$ is continous, it follows from (1.3) that

$$(2.15) \qquad \{H\}s\colon \phi(\ ) = \{H\}\colon \phi'(\ ) = H * \phi'(\ );$$

the right-hand equation can be obtained by replacing Y by H in (2.13). As in 2.9, suppose that $F(\ ) = Y * I(\ )$ and let $Y(\ )$ be piecewise-continuous. From (2.15) and (2.10) we get

$$\{F\}s\colon \phi(\ ) = F * \phi'(\ ) = Y * \phi(\ ) = \{Y\}\colon \phi(\ );$$

the right-hand equation comes from (2.13); since $\phi(\ )$ is arbitrary, we conclude that $\{F\}s = \{Y\}$, namely,

$$(2.16) \qquad \{Y * I\}s = \{Y\}.$$

From (2.15) we get

$$\{I\}s\colon \phi(\ ) = I * \phi'(\ ) = \phi' * I(\ ) = \phi(\ ) = s^0 \colon \phi(\ ) :$$

the right-hand equations come from 2.7, (2.6), and (1.2); consequently,

$$(2.17) \qquad \{I\}s = s^0.$$

2.18 Theorem. *If* $Y(\ )$ *is piecewise-continuous, then* $\{Y\}s = s\{Y\}$.

Proof. Let $\phi(\ )$ be an arbitrary test-function. From (1.3) we have

$$\{Y\}s\colon \phi(\ ) = \{Y\}\colon \phi'(\ ) = Y * \phi'(\ ) = [Y * \phi]'(\ ) = [\{Y\}\colon \phi]'(\ ) = s\{Y\}\colon \phi(\ );$$

the right-hand equations come from (2.11), (2.13) and (1.3). Consequently,

$$\{Y\}s\colon \phi(\ ) = s\{Y\}\colon \phi(\ )$$

for every test-function $\phi(\ )$. The conclusion $\{Y\}s = s\{Y\}$ is at hand.

2.19 Theorem. *Let $H_1(\ )$ and $H_2(\ )$ be continuous; also, let a be a number. If $H(\ ) = aH_1(\ ) + H_2(\ )$, then*

$$\{H\} = [\alpha s^0]\{H_1\} + \{H_2\}. \tag{6}$$

Proof. Let $\phi(\ )$ be an arbitrary test-function. Set $\alpha_1 = \alpha$ and $\alpha_2 = 1$. From (2.1) we have

$$H * \phi(t) = \int_0^t \sum_{k=1}^{2} \alpha_k H_k(\omega)\phi(t-\omega)\,d\omega = \sum_{k=1}^{2} \alpha_k[H_k * \phi](t)$$

when $-\infty < t < \infty$; from (2.12) it results that

$$\{H\}: \phi(t) = \alpha[\{H_1\}: \phi(t)] + \{H_2\}: \phi(t) = [\alpha s^0]\{H_1\}: \phi(t) + \{H_2\}: \phi(t)$$

the right-hand equation comes from (1.5). Since $\phi(\ )$ is arbitrary, the conclusion (6) comes from (1.4).

2.20 Theorem. *If $H(\ )$ is a continuous function whose derivative $H'(\ )$ is piecewise-continuous, then*

$$\{H\}s = H(0)s^0 + \{H'\}. \tag{7}$$

Proof. From the Fundamental Theorem of Calculus we have

$$H(t) = H(0) + \int_0^t H'(\omega)\,d\omega = H(0)I(t) + H' * I(t)$$

the right-hand equation comes from (2.2); since this holds when $-\infty < t < \infty$, it results form 2.19 that

$$\{H\} = [H(0)s^0]\{I\} + \{H' * I\}$$

to which we may apply the right-factorization (1.11) to get

$$\{H\}s = [H(0)s^0]\{I\}s + \{H' * I\}s = [H(0)s^0]s^0 + \{H'\}:$$

the right-hand equation comes from (2.17) and (2.16): the conclusion (7) is obtained by setting $p = H(0)s^0$ in (1.8).

2.21 Remark. Let $H_1(\ )$ and $H_2(\ )$ be continuous. The equations

$$\{H_1 \pm H_2\} = \{H_1\} \pm \{H_2\} \qquad \text{and} \qquad \{0\} = 0\, s^0$$

can be obtained by setting $a = 1$ and $a = -1$, respectively, in 2.19 and taking into account (1.7)-(1.10); for the right-hand equation, note that $I - I(\ )$ is the constant $0(\ )$ and $\{I - I\} = \{I\} - \{I\} = 0\, s^0$, in view of (1.10).

If $H_1(\ ) = H_2(\ )$, then $\{0\} = \{H_1 - H_2\} = \{H_1\} - \{H_2\}$ and it follows from (1.11)-(1.12) that $\{H_1\} = \{H_2\}$.

2.22 Lemma. *If $G(\ )$ and $H(\ )$ are continuous, then*

$$\{G * H\} = \{G\}\{H\} = \{H\}\{G\}. \tag{8}$$

Proof. Let $\phi(\ )$ be an arbitrary test-function. From (2.13) and 2.7 we have

$$\{G * H\}: \phi(\ ) = [G * H] * \phi(\ ) = G * [H * \phi](\ ) = \{G\}: [\{H\}: \phi](\ );$$

similarly,

$$\{H * G\}: \phi(\ ) = \{H\}: [\{G\}: \phi](\ ) = \{H\}\{G\}: \phi(\ );$$

the right-hand equations come from (1.1); since $H * G(\ ) = G * H(\ )$ (see 2.7), the conclusion (8) is at hand.

2.23 Theorem. *Let $G_1(\ )$ and $G_2(\ )$ be continuous. If $\{G_1\} = \{G_2\}$, then $G_1(\ ) = G_2(\ )$.*

Proof. This is an immediate consequence of Titchmarsh's Convolution Theorem: instead using it, we shall use the Bounded Convergence Theorem.

Let k be any integer ≥ 0. The equations

$$0 = \phi_k(0) \qquad \text{and} \qquad \phi_k(\omega) = \exp\left(\frac{-1}{k|\omega|}\right) \ldots \omega \neq 0$$

define a test-function $\phi_k(\ )$ such that

$$\lim_{k \to \infty} \phi_k(\omega) = 1 \qquad \ldots \qquad \omega \neq 0. \tag{9}$$

Replacing Y by G_n in (2.13), we get

$$\{G_n\}: \phi_k(\ ) = G_n * \phi_k(\ ) = \phi_k * G_n(\ ); \tag{10}$$

the right-hand equation comes from 2.7. Our hypothesis implies that $G_1 : \phi_k(\ ) = G_2 : \phi_k(\ )$; consequently, it results from (10) that $\phi_k * G_1(\ ) = \phi_k * G_2(\ )$ whence, in view of (2.1)

$$\int_0^\tau \phi_k(\omega) G_1(\tau-\omega)\,d\omega = \int_0^\tau \phi_k(\omega) G_2(\tau-\omega)\,d\omega \ldots -\infty < \tau \neq 0 < \infty,$$

so that

$$(11) \qquad 0 = \int_0^\tau \phi_k(\omega)[G_1(\tau-\omega) - G_2(\tau-\omega)]\,d\omega = \int_0^\tau f_k(\omega)\,d\omega,$$

where

$$(12) \qquad f_k(\omega) = \phi_k(\omega)[G_1(\tau-\omega) - G_2(\tau-\omega)].$$

Since $0 \leq \phi_k(x) \leq 1$ for $-\infty < x < \infty$ we have

$$(13) \qquad |f_k(\omega)| \leq \max_{0 \leq |x| \leq |\tau|} |G_1(x) - G_2(x)|;$$

also, the function $f_k(\ )$ is continuous and

$$(14) \qquad f_\infty(\omega) \stackrel{\text{def}}{=} \lim_{k\to\infty} f_k(\omega) = G_1(\tau-\omega) - G_2(\tau-\omega) \ldots \omega \neq 0$$

the right-hand equation comes from (12) and (9). Since $f_\infty(\ )$ is integrable, taking into account (13)-(14), we may apply Arzelà's theorem to obtain from (11) that

$$0 = \lim_{k\to\infty} \int_0^\tau f_k(\omega)\,d\omega = \int_0^\tau f_\infty(\omega)\,d\omega = \int_0^\tau [G_1(\tau-\omega) - G_2(\tau-\omega)]\,d\omega;$$

the right-hand equation comes from (14); therefore,

$$\int_0^\tau G_1(\tau-\omega)\,d\omega = \int_0^\tau G_2(\tau-\omega)\,d\omega \qquad -\infty < \tau \neq 0 < \infty;$$

consequently, $I * G_1(\ ) = I * G_2(\ )$, whence $G_1 * I(\ ) = G_2 * I(\ )$, hence $[G_1 * I]'(\ ) = [G_2 * I]'(\ )$ and we may use 2.8 to conclude that $G_1(\ ) = G_2(\ )$.

2.24. The unit step-function

We shall denote by $U(\ )$ the function

$$U(t) = \begin{cases} 0 & \ldots \quad t < 0 \\ 1 & \ldots \quad t \geq 0; \end{cases}$$

and by δ the operator such that

$$(15) \qquad \delta : \phi(t) = U(t)\phi(t) \qquad \ldots \qquad -\infty < t < \infty$$

for every test-function $\phi(\ )$.

2.25 Theorem. *Let $X(\ )$ be piecewise-continuous. If $UX(\ )$ is the function such that*

$$UX(t) = U(t)X(t) = \begin{cases} 0 & \dots \quad t < 0 \\ X(t) & \dots \quad t \geq 0, \end{cases}$$

then $\delta\{X\} = \{UX\}$.

Proof. Let $\phi(\ )$ be an arbitrary test-function; also, suppose that $-\infty < t < \infty$. From (1.1) it results that

$$\delta\{X\}: \phi(t) = \delta: [\{X\}: \phi](t) = U(t)[\{X\}: \phi(t)];$$

the right-hand equation is obtained by replacing ϕ in (15) by $\{X\}: \phi$; from (2.13) we therefore have

$$\delta\{X\}: \phi(t) = U(t) \int_0^t X(\omega)\phi(t-\omega)\, d\omega = \int_0^t UX(\omega)\phi(t-\omega)\, d\omega;$$

we can therefore replace Y by UX in (2.13) to obtain $\delta\{X\}: \phi(t) = \{UX\}: \phi(t)$ for $-\infty < t < \infty$. Since $\phi(\ )$ is arbitrary, the conclusion $\delta\{X\} = \{UX\}$ is at hand.

2.27. The equations

$$\delta = \{U\}s \qquad \text{and} \qquad \delta\delta = \delta$$

can be verified as follows. From (1.8) and (2.17) it follows that

$$\delta = \delta s^0 = \delta[\{I\}s] = [\delta\{I\}]s = [\{UI\}]s = \{U\}s;$$

the right-hand equations are from 2.25. Next,

$$\delta\delta = \delta[\{U\}s] = [\delta\{U\}]s = \{UU\}s = \{U\}s = \delta.$$

3. DISTRIBUTION-LIKE OPERATORS

An operator will be called *distribution-like* if it has the form $\{H_1\}s^m$, where $H_1(\ )$ is a continuous function and m is an integer ≥ 0. If $Y(\ )$ is piecewise-continuous, it follows from (2.16) that $\{Y\} = \{H_1\}s$, where $H_1(\ )$ is the continuous function $Y * I(\ )$; consequently, the operator $\{Y\}$ is distribution-like.

3.1 Let a be a number. To verify that as^0 is distribution-like, let $H(\ )$ be the constant $a(\ )$; since $H(0) = a$ and since $H'(\ ) = 0(\ )$, it follows from 2.20 and 2.21 that

$$\{H\}s = H(0)s^0 + \{0\} = as^0 + 0s^0 = as^0;$$

consequently, $as^0 = \{H\}s$ and as^0 is distribution-like; setting $a = 1$ and using (1.7), we find that s^0 is distribution-like. Since $\delta = \{U\}s$, that operator is also distribution-like. Since $s = s^0 s = \{I\}ss = \{I\}s^2$ in view of (2.17), the operator s is also distribution-like.

3.2 Let $\mathscr{C}$ be the ring of all operators of the form $\{H_1\}$, where $H_1(\ )$ is continuous. If g, h, and h_1 belong to $\mathscr{C}$, then

$$g = \{G\}, \quad h = \{H\}, \quad \text{and} \quad h_1 = \{H_1\}$$

for continuous functions $G(\ ), H(\ )$, and $H_1(\ )$, from 2.22 and 2.21 we have

$$(1) \qquad gh = \{G * H\} = hg \quad \text{and} \quad g + h = \{G * H\};$$

consequently, the operators gh and $g + h$ belong to $\mathscr{C}$.

3.3. Remark. An operator c is distribution-like only if there is an integer $k \geq 0$ such that $c = h_1 s^k$ for some operator h_1 belonging to $\mathscr{C}$.

As we shall see, distribution-like operators form a commutative algebra. Repeated use will be made of the equations

$$(2) \qquad p_1[p_2 p] = [p_1 p_2]p, \qquad ps^0 = p = s^0 p$$

and

$$(3.4) \qquad [p_1 + p_2]p = p_1 p + p_2 p,$$

which hold for any three operators p, p_1, and p_2.

3.5. Set $i = \{I\}$. From (2.17) it follows that

$$(3.5) \qquad s^0 = \{I\}s = s\{I\} = si = is;$$

the middle equation comes form 2.18. Since the operator i belongs to the ring $\mathscr{C}$, it follows from 3.2 that the operators $ii = i^2, \ldots, i^n i = i^{n+1}, \ldots$ all belong to the ring $\mathscr{C}$. To verify that the equation

$$(3.6) \qquad i^k s^k = s^0$$

holds for every integer $k \geq 1$, we proceed by induction:

$$i^{k+1}s^{k+1} = i^k i[ss^k] = i^k[is]s^k = i^k s^0 s^k = i^k s^k = s^0$$

the right-hand equations come from (3.5) and the induction hypothesis (3.6).

3.7 Remark. Let $y = \{Y\}$ for some continuous function $Y(\ )$. From (2.18) it follows that $sy = ys$. To verify that the equation

$$s^k y = ys^k \tag{3.8}$$

holds for every integer $k \geq 1$, we proceed by induction:

$$s^{k+1}y = ss^k y = s[s^k y] = s[ys^k] = [sy]s^k = [ys]s^k = y[ss^k] = ys^{k+1},$$

the right-hand equations come from the induction hypothesis (3.8) and from the equation $sy = ys$.

3.9 Theorem. *If a and b are distribution-like, then ab and $a + b$ are distribution-like; also, $ab = ba$.*

Proof. Since a and b are distribution-like it follows from 3.3 the existence of integers m and $n \geq 0$ such that

$$a = gs^m \qquad \text{and} \qquad b = hs^n \tag{3}$$

for some operators g and h belonging to the commutative ring $\mathscr{C}$. Consequently,

$$ab = gs^m hs^n = g[s^m h]s^n = g[hs^m]s^n = g[hs^m s^n] = ghs^{m+n}; \tag{4}$$

the right-hand equations come from (3.8) and (2); since it follows from 3.2 that gh belongs to the ring $\mathscr{C}$, it follows from (3)-(4) and 3.3 that ab is distribution-like. By exchanging the positions of a and b in (4), we get

$$ba = hgs^{n+m} = ghs^{m+n} = ab;$$

the right-hand equations come from (1) and (4).

It only remains to consider the operator $a + b$. From (3) we have

$$a + b = gs^0 s^m + hs^0 s^n = g[i^n s^n]s^m + h[i^m s^m]s^n$$
$$= gi^n[s^n s^m] + hi^m[s^m s^n] = [gi^n + hi^m]s^{m+n},$$

the right-hand equations come from (3.6), from (2), and from (3.4). Set $h_1 = gi^n + hi^m$, since

$$a + b = h_1 s^k \qquad \ldots \quad \text{if } k = m + n, \tag{5}$$

it will result from 3.3 that $a + b$ is distribution-like once it has been established that $gi^n + hi^m$ belongs to $\mathscr{C}$; to that effect, note that (as pointed out above) both i^n and i^m belong to $\mathscr{C}$; from 3.2 we may therefore infer that gi^n and hi^m also belong to the ring $\mathscr{C}$; we may now replace g by gi^n and h by hi^m in 3.2 to conclude that $gi^n + hi^m$ belongs to $\mathscr{C}$. Since $h_1 = gi^n + hi^m$ in (5), we conclude that $a + b$ is distribution-like.

3.10. Suppose that a, b and c are distribution-like. From 3.9 we get

$$ba + bc = ab + cb = [a + c]b = b[a + c];$$

the right-hand equations come from (3.4) and by replacing a by $a + c$ in 3.9. *Conclusion: distribution-like operators form a commutative ring.*

Let α be a number; we write

$$(3.11) \qquad \alpha a \overset{\text{def}}{=} [\alpha s^0]a \qquad \text{and} \qquad b \pm \alpha \overset{\text{def}}{=} b \pm \alpha s^0;$$

since it follows from 3.1 that αs^0 is distribution-like, the operators in (3.11) are distribution-like. If $H(\ )$ is the constant $0(\ )$, it follows from 2.21 and (1.7)-(1.11) that

$$\{H\} = \{0\} = 0s^0 = 0a = b - b + -1b$$

and $a + \{0\} = a = 1a$. All the laws of algebra hold; for example,

$$\alpha[ab] = [\alpha a]b = \alpha ab \qquad \text{and} \qquad \alpha b + \beta b = [\alpha + \beta]b$$

for any numbers α and β.

3.12. Invertibility

An operator v is called *invertible* if it is distribution-like and if $vb = s^0$ for some distribution-like operator b. Let v be invertible; we denote by s^0/v the unique distribution-like operator b such that $vb = s^0$; if a is an operator, then

$$\frac{a}{v} \overset{\text{def}}{=} a\frac{s^0}{v}.$$

Let a and b be distribution-like. If v and w are invertible, then $a/v = b/w$ if (and only if) $aw = bv$. All the familiar rules for fractions apply.

3.13. Let α be a number. If $H(t) = e^{\alpha t}$ for $-\infty < t < \infty$, it follows from 2.20 that

$$\{H\}s = H(0)s^0 + \{H'\} = 1s^0 + [\alpha s^0]\{H\}:$$

the right-hand equation comes from 2.19; consequently, $s\{H\} - \alpha s^0\{H\} = s^0$, whence $[s - \alpha s^0]\{H\} = s^0$; the distribution-like operator $s - \alpha$ is invertible and $\{H\} = s^0/[s - \alpha]$.

3.14. In particular, the operator $s-0$ is invertible. Let $Y(\ )$ be piecewise-continuous. Since it follows from (2.16) that $s\{Y*I\} = \{Y\}$ and since $\{Y*I\}$ is distribution-like, we conclude that

$$\{Y * I\} = \{Y\}/s. \tag{3.15}$$

3.16. Suppose that $G(\ )$ is continuous. If $G'(\ )$ is piecewise-continuous, it follows from 2.20 that

$$\{G'\} = s\{G\} - G(0)\,s^0. \tag{3.17}$$

3.18. Suppose that $F(t) = \sin t$ for $-\infty < t < \infty$. From (3.17) it follows that $\{F'\} = s\{F\}$; replacing G by F' in (3.17), we get

$$\{F''\} = s\{F'\} - F'(0)\,s^0 = s^2\{F\} - s^0,$$

thus,

$$s^0 = s^2\{F\} - \{F''\} = s^2\{F\} + \{F\} = [s^2 + 1]\{F\};$$

therefore, $s^2 + 1$ is invertible and

$$\{\sin \tau\} \overset{\text{def}}{=} \{F\} = \frac{s^0}{s^2+1}. \tag{3.19}$$

3.20. Let $\alpha_0, \dots, \alpha_n$ and $\beta_0, \dots, \beta_n$ be numbers, $n \geq 1$ and $\alpha_n \neq 0$; set

$$v = \alpha_n s^n + \dots + \alpha_0 \qquad \text{and} \qquad b = \beta_n s^n + \dots + \beta_0.$$

The operator v is invertible; if $F(\ )$ is piecewise-continuous, then

$$\frac{b}{v}\{F\} = \left\{\frac{\beta_n}{\alpha_n}F + F*G\right\},$$

where $F*G(\ )$ is continuous and has a piecewise-continuous derivative. If y is an operator such that

$$[\alpha_n s^n + \dots + \alpha_0]y = \beta_{n-1}\,s^{n-1} + \dots + \beta_0,$$

then $y = \{P_n\}$ for some infinitely differentiable function $P_n(\ )$ such that $P_n(0) = \beta_{n-1}/\alpha_n$.

4. A SINGULAR SYSTEM OF DIFFERENTIAL EQUATIONS

With the help of 3.20, 2.19 and elementary results such as the ones in 3.15-3.19, non-singular systems of differential equations on the whole real line $(-\infty, \infty)$ can be solved by proceeding as when using one-sided Laplace transforms.

4.1 Definition. *If* $Y(\ )$ *is piecewise-continuous, we write*

(1) $$D\{Y\} \stackrel{\text{def}}{=} s\{Y\} - Y(0-)s^0.$$

4.2 Theorem. *Let* $Y(\ )$ *be a piecewise-continuous function which is continuous except possibly at the origin. If* $Y'(\ )$ *is piecewise-continuous, then*

(2) $$D\{Y\} = \{Y'\} + [Y(0+) - Y(0-)]\delta.$$

Proof. Set

(3) $$G(t) = Y(t) - [Y(0+) - Y(0-)]U(t) \qquad \ldots -\infty < t \neq 0 < \infty$$

and $G(0) = Y(0-)$. Since $G(0+) = Y(0-) = G(0)$ and $G(0-) = Y(0-)$, the function $G(\ )$ is continuous and $G'(t) = Y(t)$ for $t \neq 0$; consequently, $G'(\ )$ is piecewise-continuous and it follows from (3.17) that

$$\begin{aligned} \{G'\} &= \{G\}s - G(0)s^0 \\ &= [\{Y\} - [Y(0+) - Y(0-)]\{U\}]s - G(0)s^0 \ldots \text{from (3) and (2.19)} \\ &= \{Y\}s - [Y(0+) - Y(0-)]\delta - Y(0-)s^0; \end{aligned}$$

this last equation comes from 2.27 and $G(0) = Y(0-)$. Therefore,

(4) $$s\{Y\} - Y(0-)s^0 = \{G'\} + [Y(0+) - Y(0-)]\delta.$$

Since $G'(t) = Y'(t)$ when $t \neq 0$, we have $G' * I(\ ) = Y' * I(\ )$, so that $\{G' * I\}s = \{Y' * I\}s$ and (3.14) gives $\{G'\} = \{Y'\}$; consequently, the conclusion (2) now comes from (4) and (1).

4.3. The operator δ corresponds to the Dirac distribution: see 5.15; from (1) and 2.27 it follows that $\delta = D\{U\}$. The equation (2) holds for the distributional derivative of the distributions generated by the functions $Y(\ )$ and $Y'(\ )$.

4.4. Notation

If a is an operator, then

$$\mathscr{L}a \overset{\text{def}}{=} \delta a. \tag{5}$$

From (1) it follows that

$$\mathscr{L}D\{Y\} = s\delta\{Y\} - Y(0-)\delta = s\{UY\} - Y(0-)\delta. \tag{6}$$

the right-hand equation comes from 2.25. If $Y(\;)$ is continuous and if $Y'(\;)$ is piecewise-continuous, it follows from 4.2 and since $Y(0-) = Y(0) = Y(0+)$ that

$$\mathscr{L}D\{Y\} = \mathscr{L}\{Y'\} = s\mathscr{L}\{Y\} - Y(0)\delta :$$

the right-hand equation comes from (6), from $\{UY\} = \delta\{Y\}$ (see 2.25), and from (5). Note the resemblance with the identity involvings Laplace transforms - often abusively applied to singular systems of differential equations.

If $X(\;)$ is piecewise-continuous, it follows from (5) and 2.25 that

$$\mathscr{L}\{X\} = \delta\{X\} = \{UX\}. \tag{7}$$

4.5. Suppose that $H_1(\;)$ and $H_2(\;)$ are piecewise-continuous on the interval $[0,\infty)$ .If

$$H_1(t) = H_2(t) \quad \ldots \quad t > 0, \tag{8}$$

then $UH_1(\;) = UH_2(\;)$; therefore, $\{UH_1\} = \{UH_2\}$ and it follows from (7) that

$$\mathscr{L}\{H_1\} = \mathscr{L}\{H_2\}. \tag{9}$$

4.6. A singular system

Let $G_1(\;)$ and $G_2(\;)$ be continuous on the interval $[0,\infty)$. Consider the system

$$a_k^1 Y_1'(t) + a_k^2 Y_2'(t) + b_k^1 Y_1(t) + b_k^2 Y_2(t) = G_k(t) \ldots t > 0, \tag{10}$$

where $k = 1,2$. The coefficients a_k^n and b_k^n are numbers, $0 \neq a_1^1 a_2^2 = a_2^1 a_1^2$: the system is singular, it may govern the currents in an electric circuit obtained by switching off (at time $t = 0$) an earlier circuit which has determined the values $Y_1(0-)$ and $Y_2(0-)$.

Since (8) implies (9), the equations (10) imply the equations

$$\mathscr{L}\{a_k^1 Y_1' + a_k^2 Y_2' + b_k^1 Y_1 + b_k^2 Y_2\} = \mathscr{L}\{G_k\} \ldots k = 1,2 \tag{11}$$

To find the *physically acceptable* particular solution of the equations (10), we replace (11) by

$$(12)\qquad a_k^1\mathscr{L}D\{Y_1\}+a_k^2\mathscr{L}D\{Y_2\}+b_k^1\mathscr{L}\{Y_1\}+b_k^2\mathscr{L}\{Y_2\}=\mathscr{L}\{G_k\}\ldots k=1,2;$$

since it follows from (7) and (6) that

$$\mathscr{L}\{Y_k\}=\{UY_k\}\quad\text{and}\quad \mathscr{L}D\{Y_k\}=s\{UY_k\}-Y_k(0-)\delta,$$

the system (12) becomes the system of algebraic equations

$$[a_k^1 s+b_k^1]\{UY_1\}+[a_k^2 s+b_k^2]\{UY_2\}=\{UG_k\}+[a_k^1Y_1(0-)+a_k^2Y_2(0-)]\delta,$$

where $k=1,2$. We suppose that $\beta_1 a_2^2\neq\beta_2 a_2^1$, where

$$\beta_k \overset{\text{def}}{=} a_1^1 b_2^k - a_2^1 b_1^k.$$

Solving for $\{UY_1\}$ the above system of algebraic equations, we get

$$(13)\qquad \{UY_1\}=\{F\}+\{F_0\}\delta=\{F+UF_0\},$$

where

$$\{F\}=\frac{[a_2^2 s+b_2^2-\beta_2]\{a_1^2G_2-a_2^2G_1\}}{[\beta_1a_2^2-\beta_2a_2^1]s+\beta_1b_2^2-\beta_2b_2^1}$$

and

$$\{F_0\}=\frac{-[a_2^1Y_1(0-)+a_2^2Y_2(0-)]\beta_2}{[\beta_1a_2^2-\beta_2a_2^1]s+\beta_1b_2^2-\beta_2b_2^1}.$$

From 3.20 it results that

$$F(0+)=\frac{a_2^2}{\beta_1a_2^2-\beta_2a^1}[a_1^2G_2(0+)-a^2G_1(0+)]$$

and

$$F_0(0+)=\frac{-\beta_2[a_2^1Y_1(0-)+a_2^2Y_2(0-)]}{\beta_1a_2^2-\beta_2a_2^1}$$

From (13) we have $UY_1(\ )=F(\ )+UF_0(\ )$, so that $Y_1(t)=F(t)+F_0(t)$ for $t>0$. Similarly, $UY_2(\ )=G(\ )+UG_0(\ )$ for continuous functions $G(\ )$ and $G_0(\ )$. It turns out that

$$(14)\qquad 0=a_2^1[Y_1(0+)-Y_1(0-)]+a_2^2[Y_2(0+)-Y_2(0-)]$$

and

$$Y_1(0+) - Y_1(0-) \over a_2^2 \qquad (15)$$

$$\frac{Y_1(0+) - Y_1(0-)}{a_2^2} = \frac{a_1^2 G_2(0+) - a_2^2 G_1(0+) - \beta_1 Y_1(0-) - \beta_2 Y_2(0-)}{\beta_1 a_2^2 - \beta_2 a_2^1} \tag{15}$$

4.7. Continuous transition

As can be seen from (15), the equation $Y_1(0+) = Y_1(0-)$ holds only when

$$a_1^2 G_2(0+) - a_2^2 G_1(0+) = \beta_1 Y_1(0-) + \beta_2 Y_2(0-);$$

in the words of G. Doetsch [4, p. 73], this equation ensures «a continuous transition from the past into the future».

4.8. The conservation property

Written in the form

$$a_2^1 Y_1(0+) + a_2^2 Y_2(0+) = a_2^1 Y_1(0-) + a_2^2 Y_2(0-),$$

the equation (14) yields various physical conservation principles. For example when the system (10) governs the currents in a perfectly coupled transformer, that equation states the principle of «conservation of flux».

4.9. If $\beta_1 a_2^2 = \beta_2 a_2^1$, there are no functions $Y_k(\ )$ satisfying the equation (12); the response of the circuit is impulsive – see 5.12.

5. INITIAL VALUES AT THE ORIGIN

5.1. Let $(\mathscr{S})$ be the linear space generated by the family of operators of the form $s^k\{X\}$, where k is an integer ≥ 1 and $X(\ )$ is a piecewise-continuous function such that $0 = X(0-)$ and $X'(\omega) = 0$ when $X(\omega_-^+) = X(\omega)$ and $-\infty < \omega < \infty$.

5.2. Since $\delta s = \{U\}$, the operator δ belongs to $(\mathscr{S})$. The space $(\mathscr{S})$ consists of singular distributions; indeed, if $Y(\ )$ is piecewise-continuous and such that $\{Y\}$ belongs to $(\mathscr{S})$ then $\{Y\} = \{0\}$ (see 6.2). If $p \in (\mathscr{S})$ then it follows from 2.25 that $\delta p \in (\mathscr{S})$; moreover, $s^k p \in (\mathscr{S})$ for every integer $k \geq 0$.

5.3 Definition. *An operator y will be called* differentiable *if there is a piecewise-continuous function $Y(\ )$ such that $y - \{Y\}$ belongs to $(\mathscr{S})$; if so, then*

$$y(t) \overset{\text{def}}{=} Y(t-) \quad \dots \quad -\infty < t < \infty \tag{1}$$

and

(2) $$Dy \overset{\text{def}}{=} sy - y(0)s^0.$$

5.4. *The definition (1) does not depend on the choice of the function* $Y(\ )$ *: if there are piecewise-continuous functions* $Y_k(\ )$ *such that* $y - \{Y_k\}$ *belongs to* $(\mathscr{S})$ *for* $k = 1,2$, *then* $Y_1(t-)$ *when* $-\infty < t < \infty$. *The proof is given in 6.3.*

5.5. If $y = \{Y\}$ for some piecewise-continuous function $Y(\ )$, then $y - \{Y\} = \{0\} \in (\mathscr{S})$; therefore, Definition (1) becomes $y(t) = Y(t-)$ (when $-\infty < t < \infty$) and

$$Dy = D\{Y\} = s\{Y\} - Y(0-)s^0$$

which agrees with 4.1. If $y = p \in (\mathscr{S})$, then $y - \{Y\}$ belongs to $(\mathscr{S})$ with $Y(\ ) = 0(\ )$; therefore, $y(\ ) = 0(\ )$ and $Dy = sp$; in particular, since $\delta \in (\mathscr{S})$, we have $D\delta = s\delta$.

5.6. Let z be differentiable. Consequently, there is a piecewise-continuous function $Z(\ )$ such that $z = \{Z\} + p$ for some operator $p \in (\mathscr{S})$ and

$$\delta z = \delta\{Z\} + \delta p = \{UZ\} + \delta p;$$

the right-hand equation comes from 2.26 and 2.25; since it follows from 5.2 that $\delta p \in (\mathscr{S})$, we have $\delta z - \{UZ\} \in (\mathscr{S})$; replacing y by δz in 5.3, we find that δz is differentiable and

(5.7) $$\delta z(t) = UZ(t-) = U(t-)Z(t-) = U(t-)z(t) \ldots -\infty < t < \infty;$$

the middle equation comes from 2.25; to obtain the right-hand equation, note that $z - \{Z\} \in (\mathscr{S})$, whence $z(t) = Z(t-)$. Consequently,

(5.8) $$\delta z(t) = \begin{cases} 0 & \ldots \quad t \le 0 \\ z(t) & \ldots \quad t > 0. \end{cases}$$

From 5.3 it results that

$$\delta Dz = s\delta z - z(0)\delta = \delta[sz - z(0)\delta];$$

the right-hand equation comes from recalling that $\delta\delta = \delta$ (see 2.27); in view of 4.4 we therefore have

(5.9) $$\mathscr{L}Dz = s\mathscr{L}z - z(0)\delta = \mathscr{L}[sz - z(0)\delta].$$

5.10. It is now possible to re-state the singular system in 4.6. There are differentiable operators y_1 and y_2 such that

$$(3)\qquad \mathscr{L}[a_k^1 Dy_1 + a_k^2 Dy_2 + b_k^1 y_1 + b_k^2 y_2] = \mathscr{L}\{G_k\} \qquad \ldots \quad k = 1,2$$

and numbers $Y_1(0-)$ and $Y_2(0-)$ (originating from the past of the circuit) such that $y_1(0) = Y_1(0-)$ and $y_2(0) = Y_2(0-)$; since it follows from (5.9) that $\mathscr{L}Dy_k = s\mathscr{L}y_k - Y_k(0-)\delta$, the system (3) becomes

$$(4)\qquad [a_k^1 s + b_k^1]\mathscr{L}y_1 + [a_k^2 s + b_k^2]\mathscr{L}y_2 + b_k^1\mathscr{L}y_1 + b_k^2\mathscr{L}y_2 = \mathscr{L}\{G_k\}\ldots k = 1,2;$$

5.11. If z_1 and z_2 are differentiable, we shall write $z_1 = z_2$ *on* $(0,\infty)$ to indicate that $\delta z_1 = \delta z_2$. Therefore, in view of 4.4,

$$(5)\qquad \mathscr{L}z_1 = \mathscr{L}z_2 \qquad \text{if (and only if)} \qquad z_1 = z_2 \quad on\,(0,\infty)$$

5.12 Solving for $\mathscr{L}y_1$ the algebraic system (4) in case $\beta_1 a_2^2 \neq \beta_2 a_2^1$, we find that $\mathscr{L}y_1 = \{UY_1\}$, where $UY_1(\ )$ is the function $F(\ ) + F_0(\ )$ obtained in 4.6. Now suppose that $\beta_1 a_2^2 = \beta_2 a_2^1$ and $G_1(\ ) = 0(\ ) = G_2(\ )$: in this case,

$$(6)\qquad \mathscr{L}y_1 = \alpha\delta = \delta\alpha\delta = \mathscr{L}\alpha\delta,$$

where $\alpha = [a_2^1 Y_1(0-) + a_2^2 Y_2(0-)]\beta_2/[\beta_2 b_2^1 - \beta_1 b_2^2]$. In view of (5) the equation (6) can be written

$$y_1 = \alpha\delta \quad on\ (0,\infty)$$

this is the *physically acceptable* particular solution of the singular system (3), namely, of the system

$$(7)\qquad a_k^1 Dy_1 + a_k^2 Dy_2 + b_k^1 y_1 + b_k^2 y_2 = \{0\}\ on\ (0,\infty) \qquad \ldots k = 1,2;$$

in simple circuits, agreement with physical reality is readily observed – for example, when the circuit involves a capacitor short-circuited at time $t = 0$ (causing an impulsive surge of current), or the impulsive surge of voltage caused by opening a switch on a RL-loop circuit at time $t = 0$.

Since (5.9) combines with (5) to give

$$Dz = sz - z(0)\delta \quad on\,(0,\infty),$$

this equation combines with (7) to give the system of algebraic equations we have been dealing with all along, without the symbol $\mathscr{L}$.

5.13. Let $H_1(\ )$ and $H_2(\ )$ be piecewise-continuous on $[0,\infty)$. If

(8) $$H_1(t) = H_2(t) \quad \ldots \quad t > 0$$

it follows from 4.5 and (5) that

$$\{H_1\} = \{H_2\} \quad on\,(0,\infty).$$

Conversely, if (8), it results from 4.4 that $\{UH_1\} = \{UH\}_2$ and it results from 6.3 that

$$H_1(t-) = H_2(t-) \quad \ldots \quad t > 0.$$

5.14. Let $F(\ )$ be continuous on $[0,\infty)$. If

$$z = \{F\} \quad on\,(0,\infty),$$

then $z(t) = F(t)$ for $t > 0$; also, if $F'(\ )$ is piecewise-continuous on $[0,\infty)$, the

$$Dz = \{UF'\} + [z(0+) - z(0)]\delta \quad on\,(0,\infty).$$

5.15 We return briefly to the interval $(-\infty,\infty)$ to illustrate the effect of the operator δ as an impulse input. Suppose that y and Dy are differentiable and such that

(9) $$D^2y + y = \delta;$$

since it follows from (2) that

$$D^2y = s^2y - y(0)s - Dy(0)s^0,$$

the equation (9) implies

(10) $$y = \frac{\delta}{s^2+1} + \{H\} = \delta\{\sin\tau\} + \{H\} = \delta\{F\} + \{H\}.$$

where $H(\ )$ is the infinitely differentiable function such that $\{H\} = [y(0)s + Dy(0)]/[s^2 + 1]$ and $F(t) = \sin t$ for $-\infty < t < \infty$; the right-hand equation comes from (3.19). From 2.26 and 2.19 it results that $y = \{UF + H\}$; from 5.5 we conclude that

(11) $$y(t) = U(t-)F(t-) + h(t-) = \begin{cases} H(t) & \ldots \quad t \le 0 \\ H(t) + \sin t & \ldots \quad t \ge 0. \end{cases}$$

5.16. Zero state at $t = 0$. If (9) and $0 = y(0) = Dy(0)$, then $H(\ ) = 0(\ )$ and $y = \{UF\}$. Conversely, suppose that $y = \{UF\}$; setting $Y = UF$ in 4.2, we obtain $Dy = D\{UF\} = \{UF'\}$; replacing y by Dy and Y by UF' in (1), we get

$$Dy(t) = U(t-)F'(t-) = \begin{cases} 0 & \dots \quad t \leq 0 \\ \cos t & \dots \quad t > 0; \end{cases}$$

consequently, $Dy(0) = 0 \neq Dy(0+) = 1$; the equations $y(0) = 0 = y(0+)$ are immediate from (11). In order to verify (9), another application of 4.2 with $Y = UF'$ gives

$$D^2 y + y = D\{UF'\} + y = \{UF''\} + F'(0+)\delta + y = -y + y + \delta.$$

6. APPENDIX

The aim of this section is to establish the assertion in 5.4 and a remark in 5.2.

6.1 Lemma. *Let $G(\ )$ be continuous. If $\{G\} = \{X\}$ for some piecewise-continuous function $X(\ )$, then*

$$(7) \qquad X(\omega-) = G(\omega) \quad \dots \quad -\infty < \omega < \infty;$$

moreover, if $0 = X(0-)$ and if

$$(8) \qquad 0 = X'(x) \quad \text{when} \quad X(x) = X(x^+_-) \dots -\infty < x < \infty,$$

then $\{G\} = \{0\} = \{X\}$.

Proof. By hypothesis, $\{G\}/s = \{X\}/s$; form (3.14) we find that $\{G * I\} = \{X * I\}$; since both $G * I(\ )$ and $X * I(\ )$ are continuous, it results from 2.23 that $G * I(\ ) = X * I(\ )$; therefore, it follows from 2.8 that

$$G(t) = [G * I]'(t) = \frac{d}{dt}\int_0^t X(\omega)d\omega) \dots -\infty < t < \infty;$$

the right-hand equation comes from (2.2); consequently,

$$(9) \qquad G(x) = X(x) \quad \text{when } X(x\pm) = X(x).$$

If $-\infty < \omega < \infty$ there is a point $t_1 < \omega$ such that the function $X(\ )$ is continuous on the open interval (t_1, ω); from (9) it follows that $X(t) = G(t)$ when $t_1 < t < \omega$, whence

$$(10) \qquad X(\omega-) = G(\omega-) = G(\omega) \quad \ldots \quad -\infty < \omega < \infty;$$

the right-hand equation comes from the continuity of $G(\ )$.

Having thus verified (7), suppose that (8) obtains:

$$(11) \qquad 0 = X'(x) = G'(x) \quad \ldots \text{ when } \quad X(x\pm) = X(x);$$

the right-hand equation comes from (9); consequently,

$$(12) \qquad G' * I(t) = 0 \quad \ldots \quad -\infty < t < \infty,$$

and from (3.15) we therefore have

$$(13) \qquad \{G'\} = \{G' * I\}s = \{0\}s = \{0\}.$$

Since $X(\ )$ is piecewise-continuous, it follows from (11) that $G'(\ )$ is piecewise-continuous; since $G(\ )$ is continuous, it results form 2.20 that

$$(14) \qquad s\{G\} = G(0)s^0 + \{G'\} = X(0-)s^0 + \{0\} = X(0-)s^0;$$

the right-hand equations come from (10) and (13); since the operator s is invertible, the equation (14) gives $\{G\} = X(0-)s^0/s$; our hypothesis $X(0-) = 0$ (see (8)) now yields our conclusion $\{G\} = \{0\}$.

6.2 Theorem. *Let $X_0(\ )$ be piecewise-continuous. If $\{X_0\}$ belongs to $(\mathscr{S}\ )$, then* $\{X_0\} = \{0\}$.

Proof. Since, by hypothesis, $\{X_0\}$ belongs to $(\mathscr{S}\ )$,

$$(15) \qquad \{X_0\} = \sum_{k=1}^{n} s^k\{X_k\},$$

where $X_k(\ )$ is piecewise-continuous, $0 = X_k(0-)$, and

$$(16) \qquad 0 = X_k'(x) \quad \text{when} \quad X_k(x) = X_k(x\pm) \quad \ldots -\infty < x < \infty.$$

To show that

$$(17) \qquad \{0\} = \{X_1\} = \ldots = \{X_{n-1}\} = \{X_n\},$$

we proceed by contradiction.

Suppose that $\{X_m\} \neq 0$ for some integer $m \geq 1$; let n be the least integer ≥ 1 such that $\{X_n\} \neq 0$; from (15) it results that

$$(18) \qquad \frac{\{X_0\}}{s^n} = \frac{s}{s^n}\{X_1\} + \ldots + \frac{s^{n-1}}{s^n}\{X_{n-1}\} + \{X_n\}.$$

Suppose that $0 \leq k \leq n-1$: in view of 3.20, the equation

$$\frac{s^k}{s^n}\{X_k\} = \{X_k * G_k\}$$

holds for some infinitely differentiable function $G_k(\ )$; also, the function $X_k * G_k(\ )$ is continuous. Therefore, (18) becomes

$$\{G_0 * X_0\} = \{G_1 * X_1\} + \ldots + \{G_{n-1} * X_{n-1}\} + \{X_n\};$$

consequently, $\{G\} = \{X_n\}$, where

$$G(\ ) = G_0 * X_0(\ ) - G_1 * X_1(\ ) - \ldots - G_{n-1} * X_{n-1}(\ );$$

since $G(\ )$ is continuous, it results from $\{G\} = \{X_n\}$ and since (16) holds for $k = n$, we may apply 6.1 to conclude that $\{G\} = \{0\} = \{X_n\}$, equations which contradict our assumption $\{X_n\} \neq \{0\}$. This establishes (17), whence (15) gives $\{X_0\} = \{0\}$.

6.3 Theorem. *Let $Y_1(\ )$ and $Y_2(\ )$ be piecewise-continuous. If $y - \{Y_1\}$ and $y - \{Y_2\}$ belong to $(\mathscr{S})$, then $Y_1(t-) = Y_2(t-)$ for $-\infty < t < \infty$.*

Proof. By hypothesis, $y = \{Y_k\} + p_k$ for some operators p_1 and p_2 belonging to $(\mathscr{S})$; therefore, $\{Y_1\} + p_1 = \{Y_2\} + p_2$, hence $\{Y_1\} - \{Y_2\} = p_2 - p_1$; since $p_2 - p_1$ belongs to $(\mathscr{S})$, we infer that

$$(19) \qquad \{Y_1\} - \{Y_2\} \text{ belongs to } (\mathscr{S}).$$

Let $X(\ )$ be the piecewise-continuous function such that

$$(20) \qquad X(t) = Y_1(t) - Y_2(t)$$

at every point t where both $Y_1(\ )$ and $Y_2(\ )$ are continuous; the reasoning in 2.19 shows that $\{X\} = \{Y_1\} - \{Y_2\}$; from (19) we conclude that $\{X\}$ belongs to $(\mathscr{S})$ and it results from 6.2 that $\{X\} = \{0\}$; thus, $\{X\} = \{G\}$, where $G(\ ) = 0(\ )$ (the constant zero); from 6.2 we obtain

$$X(\omega-) = G(\omega) = 0 \quad \ldots \quad -\infty < \omega < \infty,$$

so that (20) yields the conclusion $Y_1(\omega-) - Y_2(\omega-) = 0$.

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