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TOPOLOGICAL VECTOR SPACES WITH SOME BAIRE-TYPE PROPERTIES

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Dedicated to the memory of Professor Gottfried Köthe

0. INTRODUCTION

In 1972 Saxon [10] introduced a class of locally convex spaces (called *Baire-like*) containing strictly the class of Baire spaces and which is strictly included in the class of barrelled spaces. A locally convex space (lcs) E is called *Baire-like* if given an increasing sequence of absolutely convex closed subsets of E covering E, there exists one of them which is a neighbourhood of zero in E. By Valdivia [14], Theorem 4, a barrelled space whose completion is Baire is Baire-like. In contrast to Baire spaces, Baire-like spaces enjoy good permanence properties, i.e. products, quotients, countably codimensional subspaces of Baire-like spaces are Baire-like [10]. Much of the importance of Baire-like spaces comes from their connection with the closed graph theorem. In [10], Theorem 2.18, Saxon showed that

(*) *if E is Baire-like, F an (LB)-space with a defining sequence $(F_n)_{n\in\mathbb{N}}$ of Banach spaces and $f: E \to F$ a linear map with closed graph, then $f(E) \subset F_n$ for some $n \in \mathbb{N}$ and f induces a continuous map of E into the Banach space F_n.*

Since barrelled metrizable spaces are Baire-like, it follows that no (LB)-space is metrizable. It is known however that metrizable (LF)-spaces exist, cf. e.g. [7], [12]. Thus (*) may be false when F is an (LF)-space. It turns out that in order to obtain a closed graph theorem which includes (LF)-spaces in the range class, it is enough to assume that the spaces E in the domain class are *suprabarrelled* [16] (or (db)-spaces [9]), i.e. given an increasing sequence of subspaces of E covering E, then one of them is both dense and barrelled. By dropping here the word «increasing» one obtains the definition of an *unordered Baire-like* space (shortly UBL space) in the sense of Todd and Saxon [13]. Clearly we have the following implications: Baire $\Rightarrow$ UBL $\Rightarrow$ suprabarrelled $\Rightarrow$ Baire-like $\Rightarrow$ barrelled. This line of works provided new types of strong barrelledness conditions, a classifiction of (LF)-spaces and several forms of the closed graph theorem. We refer the reader to [3] for detailed informations on this subject.

A natural extension of the Baire-like property to the class of arbitrary topological vector spaces (tvs) was introduced in [5], under name of **-Baire-like*, containing strictly the class of Baire spaces and strictly included in the class of ultrabarrelled spaces. In [5] it is shown that all ultrabarrelled spaces whose completion is Baire (hence all metrizable ultrabarrelled

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spaces) are **-Baire-like*. Among locally convex spaces every $*$- Baire-like space is Baire-like. Another generalization of Baire-likeness and suprabarrelledness was pursued by Pérez Carreras [6].

In the present paper we continue the study on strong (ultra) barrelledness conditions in the (non) convex setting. Section 1 deals with $*$- Baire-like spaces and includes the closed graph theorem and an analogue of the Banach-Steinhaus theorem for such spaces. Moreover, we give a characterization of $(LF)_{tv,i}$-spaces (in the sense of Narayanaswami and Saxon, but considered in the category of arbitrary tvs). The connections between metrizable $(LF)_{tv}$-spaces, $*$- suprabarrelled and $*$- Baire-like spaces are discussed.

All tvs considered in this paper are assumed to be Hausdorff and infinite dimensional over the field $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. For a topological space (E, τ) and $a \in E$, $\mathscr{U}_a(E)$ or $\mathscr{U}_a(\tau)$ will denote the filter of all neighbourhoods of a in (E, τ).

1. RESULTS

Let $E = (E, \tau)$ be a tvs. By a *string* in E we understand (after Adasch [1]) a sequence $(U_j)_{j\in\mathbf{N}}$ of balanced and absorbing subsets U_j of E such that $U_{j+1} + U_{j+1} \subset U_j$ for all $j \in \mathbf{N}$. A string $(U_j)_{j\in\mathbf{N}}$ is called

(a) *closed,* if every U_j is τ-closed;

(b) *topological,* if every U_j is a τ-neighbourhood of zero.

A tvs E is called *ultrabarrelled* if every closed string in E is topological [1]. The following conditions are equivalent:

(1) (E, τ) is ultrabarrelled.

(2) Every linear map from (E, τ) into an F-space with closed graph is continuous.

(3) Every Hausdorff vector topology ϑ on E which is τ-polar, i.e. $\mathscr{U}_0(\vartheta)$ has a basis of τ-closed sets, is coarser than τ (cf. [1], p. 32, p. 44).

Every metrizable and complete tvs will be called an F-space. A double sequence $(K_j^n)_{n,j\in\mathbf{N}}$ of balanced and closed subsets of E such that

(c) $K_j^n \subset K_j^{n+1}$, $K_{j+1}^n + K_{j+1}^n \subset K_j^n$, $n, j \in \mathbf{N}$;

(d) $\bigcup_{n=1}^{\infty} K_j^n$ is absorbing in E for all $j \in \mathbf{N}$,

will be called a *γ-sequence*. A γ- sequence $(K_j^n)_{n,j\in\mathbf{N}}$ is called *topological,* if for every $j \in \mathbf{N}$ there exists $n \in \mathbf{N}$ such that $K_j^n \in \mathscr{U}_0(E)$.

We shall need repeatedly the following fact about γ- sequences.

Lemma 1.0. *Let E be a dense ultrabarrelled subspace of a tvs F. If $(K_j^n)_{n,j\in\mathbf{N}}$ is a γ- sequence in E, then $(\overline{K_j^n})_{n,j\in\mathbf{N}}$ is a γ- sequence in F.*

Proof. Property (c) being clear, it is enough to prove that

$$F = \overline{E} = \overline{\bigcup_{n=1}^{\infty} nK^n_{j+1}} \subset \bigcup_{n=1}^{\infty} \overline{nK^n_j} \quad \text{for all} \quad j \in \mathbb{N}.$$

If $x \notin \bigcup_{n=1}^{\infty} \overline{nK^n_i}$ for some $i \in \mathbb{N}$, then for every $n \in \mathbb{N}$ there exists a topological string $(U^n_j)_{j\in\mathbb{N}}$ in F such that $x \notin \overline{nK^n_i + U^n_i}$. Set

$$V_k = \bigcap_{n=1}^{\infty} \left(\left(\overline{nK^n_{i+k} + U^n_{1+k}} \right) \cap E \right), \qquad k \in \mathbb{N}.$$

Then $(V_k)_{k\in\mathbb{N}}$ is a closed string in E; hence topological. But $x \notin (\bigcup_{n=1}^{\infty} nK^n_{i+1}) + \overline{V_1}$; otherwise for some $m \in \mathbb{N}$, $x \in mK^m_{i+1} + \overline{V_1} \subset mK^m_{i+1} + \overline{mK^m_{i+1} + U^m_2} \subset \overline{mK^m_i + U^m_1}$, a contradiction. Hence $x \notin \overline{\bigcup_{n=1}^{\infty} nK^n_{i+1}}$. ■

A *tvs* E is called *$*$-Baire-like* [5] if every γ-sequence in E is topological. Clearly: Baire $\Rightarrow$ $*$-Baire-like $\Rightarrow$ ultrabarrelled; none of the reverse implications are true [5]. Every locally convex *tvs* which is $*$-Baire-like is Baire-like, but Baire-like spaces which are not $*$-Baire-like do exist [5]. In [5] it was proved that products, quotients and completions of $*$-Baire-like spaces are $*$-Baire-like. Also, by [5], every countable-codimensional subspace F of a $*$-Baire-like space E is $*$-Baire-like if and only if F is ultrabarrelled. Every metrizable and ultrabarrelled *tvs* is $*$-Baire-like. This remark in [5] follows also from the following proposition which is clear from Lemma 1.0.

Proposition 1.1. *An ultrabarrelled tvs E which is dense in a $*$-Baire-like space F is $*$-Baire-like.* ■

Our first theorem is connected with the closed graph theorem for $*$-Baire-like spaces. First we recall the following two notions: E is said to be boundedly summing [1], if for every bounded subset B of E there exists a scalar sequence $(\lambda_j)_{j\in\mathbb{N}}$, $\lambda_j > 0$, such that

$$\sum_{n=1}^{\infty *} \lambda_n B := \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} \lambda_k B$$

is bounded. All metrizable *tvs* are boundedly summing; locally pseudo-convex and almost convex spaces are boundedly summing, [1], p. 76. A sequence $(A_j)_{j\in\mathbb{N}}$ of balanced subsets of E such that $A_{j+1} + A_{j+1} \subset A_j$ for all $j \in \mathbb{N}$ is said to be completing if given any sequence $x_j \in A_j$, $j \in \mathbb{N}$, then the series $\sum_{j=1}^{\infty} x_j$ converges in E. This implies that the filter basis $(A_j)_{j\in\mathbb{N}}$ is finer than $\mathscr{U}_0(E)$.

We shall need also the following variant of Theorem 9.1.44 of [3].

Lemma 1.2. *Let $(E,\tau),(F,\vartheta)$ be tvs and $f:(E,\tau)\to(F,\vartheta)$ a linear map with closed graph. If there exists a completing sequence $(A_n)_{n\in\mathbb{N}}$ in F such that for every $n\in\mathbb{N}$ the closure of $f^{-1}(A_n)$ is a τ-neighbourhood of zero, then f is continuous.*

Proof. We start with the special case that (E,τ) is metrizable. Let $(U_n)_{n\in\mathbb{N}}$ be a basis of τ-neighbourhoods of zero in E such that $U_{n+1}\subset U_n$, $n\in\mathbb{N}$. Let $K_n=f^{-1}(A_n)$, $n\in\mathbb{N}$. We can find an increasing sequence (m_n) in $\mathbb{N}$ such that $U_{m_n}\subset K_n+U_{m_{n+1}}$, $n\in\mathbb{N}$. It is enough to show that $f(U_{m_n})\subset\overline{A}_n$, $n\in\mathbb{N}$. Fix $n\in\mathbb{N}$ and $x_1\in U_{m_n}$. We find two sequences $(x_j)_{j\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ such that $x_j=y_j+x_{j+1}$, $j\in\mathbb{N}$, and $x_j\in U_{m_{n+j-1}}$, $f(y_j)\in A_{n+j-1}$. Therefore $x_1=\sum_{j=1}^{\infty}y_j$. By assumption there exist $y\in F$ such that $y=\sum_{j=1}^{\infty}f(y_j)$. Since $\sum_{j=1}^{m}f(y_j)\in\sum_{j=1}^{m}A_{n+j}\subset A_n$, $m\in\mathbb{N}$, then $y\in\overline{A}_n$. The graph of f being closed, we have $f(x_1)=y$, which completes the proof. Now we turn to the case of an arbitrary tvs (E,τ). First we show that $P:=\bigcap_{n=1}^{\infty}\overline{f^{-1}(A_n)}$ is equal to the closed subspace $f^{-1}(0)$. In fact, $f^{-1}(0)\subset P$ is trivial, and on the other hand $P\subset$ $\subset\bigcap_{V\in\mathscr{U}_0(F)}\overline{f^{-1}(V)}$ since the filter basis $(A_n)_{n\in\mathbb{N}}$ is finer than $\mathscr{U}_0(F)$. Hence

$$P\subset\cap\{U+f^{-1}(V):U\in\mathscr{U}_0(E),\ V\in\mathscr{U}_0(F)\},\quad\text{so}$$
$$f(P)\subset\cap\{f(U)+V:U\in\mathscr{U}_0(E),\ V\in\mathscr{U}_0(F)\}=\{0\}$$

since graph f is closed. So $P\subset f^{-1}(0)$. Let $q:E\to E/P$ be the quotient map. There is a linear map $g:E/P\to F$ such that $g\circ q=f$. We endow E/P with the topology α whose basis of the neighbourhoods of zero is given by $\overline{q(f^{-1}(A_n))}$, $n\in\mathbb{N}$, which is metrizable. α is coarser than the quotient topology τ/P. Observe that $g:(E/P,\alpha)\to(F,\vartheta)$ has closed graph in $(E/P,\alpha)\times(F,\vartheta)$. In fact, since graph f is closed, there exists a Hausdorff vector topology $\beta\le\vartheta$ on F such that $f:(E,\tau)\to(F,\beta)$ is continuous. Let $V\in\mathscr{U}_0(\beta)$ be closed. There exists $k\in\mathbb{N}$ such that $A_k\subset V$. It follows that $f^{-1}(A_k)\subset f^{-1}(V)$. Therefore $\overline{q(f^{-1}(A_k))}\subset g^{-1}(V)$, and $g:(E/P,\alpha)\to(F,\beta)$ is continuous. So $g:(E/P,\alpha)\to(F,\vartheta)$ has closed graph. On the other hand $\overline{g^{-1}(A_n)}^{\alpha}$ is an α-neighbourhood of zero for all $n\in\mathbb{N}$. In fact, $\overline{q(f^{-1}(A_n))}\subset\overline{g^{-1}(A_n)}^{\alpha}$. Hence the assumptions of Lemma 1.2, metrizable case, are satisfied for g. Therefore $g:(E/P,\alpha)\to(F,\vartheta)$ is continuous. Since $g:(E,\tau)\to(E/P,\alpha)$ is also continuous, we obtain that f is continuous.

Theorem 1.3. *Let (E,τ) be a $*$-Baire-like space and let (Y,ϑ) be the inductive limit of an increasing sequence $(Y_n,\vartheta_n)_{n\in\mathbb{N}}$ of boundedly summing tvs (Y_n,ϑ_n) such that $\vartheta_{n+1}|Y_n\le$ $\le\vartheta_n$ for all $n\in\mathbb{N}$. Assume that every (Y_n,ϑ_n) has a fundamental sequence of bounded* *balanced sets which are complete. If $f : (E, \tau) \to (Y, \vartheta)$ is a linear map with closed graph, then there exists $m \in \mathbb{N}$ such that $f(E) \subset Y_m$ and $f : (E, \tau) \to (Y_m, \vartheta_m)$ is continuous.*

Proof. For every $n \in \mathbb{N}$ let $(A^n_m)_{m\in\mathbb{N}}$ be a fundamental sequence of balanced ϑ_n-bounded subsets of A_n which are ϑ_n-complete. We may assume that $A^n_m + A^n_m \subset A^n_{m+1}$, $n, m \in \mathbb{N}$. Since (E, τ) *is *-Baire-like, there exists* $n \in \mathbb{N}$ such that $f^{-1}(Y_p)$ is τ-dense for all $p \geq n$. Without loss of generality we may assume that $n = 1$. Let $\tau_n = \tau | f^{-1}(Y_n)$, $n \in \mathbb{N}$. First we prove that there are $n, m \in \mathbb{N}$ such that $\overline{f^{-1}(A^n_m)}^{\tau_n}$ is a τ_n-neighbourhood of zero in $f^{-1}(Y_n)$. Suppose this is not the case. Hence none of the sets $\overline{f^{-1}(A^n_m)}^{\tau}$ is a τ-neighbourhood of zero. We construct two sequences $(S_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ of balanced subsets of E and Y, respectively, and a sequence $(p(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $S_n + S_n \subset S_{n+1}$, $B_n + B_n \subset B_{n+1}$, $n \in \mathbb{N}$, and such that:

(a) $B_n = \sum_{k=1}^{n} A^k_n + A^n_{p(n)} + \sum_{j=1}^{\infty *} \lambda^n_j A^n_{p(n)}$, where $(\lambda^n_j)_{j\in\mathbb{N}}$ is such that $0 < \lambda^n_{j+1} < \lambda^n_j$ and $\sum_{j=1}^{\infty *} \lambda^n_j A^n_{p(n)}$ is ϑ_n-bounded.

(b) $B_n + B_n \subset A^{n+1}_{p(n+1)}$, $n \in \mathbb{N}$.

(c) $S_n = \overline{f^{-1}(B_n)}^{\tau}$ and $S_n \notin \mathscr{U}_0(\tau)$, $n \in \mathbb{N}$.

We construct both sequences by induction. Since Y_1 is boundedly summing, there exists a scalar sequence $(\lambda^1_j)_{j\in\mathbb{N}}$, $0 < \lambda^1_{j+1} < \lambda^1_j$, $j \in \mathbb{N}$, such that $\sum_{j=1}^{\infty *} \lambda^1_j A^1_1$ is ϑ_1-bounded. Set $p(1) = 1$ and $B_1 = A^1_1 + A^1_1 + \sum_{j=1}^{\infty *} \lambda^1_j A^1_1$ and $S_1 = \overline{f^{-1}(B_1)}^{\tau}$. Then $B_1 + B_1 \subset A^2_{p(2)}$ for some $p(2) \in \mathbb{N}$. Hence S_1 is not a τ-neighbourhood of zero. Suppose, we have already found sets $B_1, B_2, \ldots, B_n$; $S_1, S_2, \ldots, S_n$, with the above conditions. Choose $p(n+1) \in \mathbb{N}$ such that $B_n + B_n \subset A^{n+1}_{p(n+1)}$. There exists a sequence $(\lambda^{n+1}_j)_{j\in\mathbb{N}}$, $0 < \lambda^{n+1}_j < \lambda^{n+1}_j$, $j \in \mathbb{N}$, such that $\sum_{j=1}^{\infty *} \lambda^{n+1}_j A^{n+1}_{p(n+1)}$ is ϑ_{n+1}-bounded. Set

$$B_{n+1} = \sum_{k=1}^{n+1} A^k_{n+1} + A^{n+1}_{p(n+1)} + \sum_{j=1}^{\infty *} \lambda^{n+1}_j A^{n+1}_{p(n+1)}, S_{n+1} = \overline{f^{-1}(B_{n+1})}^{\tau}.$$

Then $B_n + B_n \subset B_{n+1}$, $S_n + S_n \subset S_{n+1}$. Since $B_{n+1} + B_{n+1} \subset A^{n+1}_{p(n+2)}$ for some $p(n+2) \in \mathbb{N}$, S_{n+1} is not a τ-neighbourhood of zero. This completes the inductive step. By (a), the sets

$$T^n_j = \sum_{k=1}^{\infty *} \lambda^n_{2^{j-1}k} A^n_{p(n)}, \ n, j \in \mathbb{N},$$

satisfy

$$T^n_j \subset B_n, T^n_{j+1} + T^n_{j+1} \subset T^n_j, \ n, j \in \mathbb{N},$$

and every T_j^n is balanced and ϑ_n-bounded.

The sets

$$K_j^n = T_j^1 + T_j^2 + \ldots + T_j^n, \; n, j \in \mathbf{N}$$

are balanced in Y. Clearly, $K_j^n \subset K_j^{n+1}$, $K_{j+1}^n + K_{j+1}^n \subset K_j^n$, $n, j \in \mathbf{N}$,

$$(*) \qquad K_j^n \subset B_1 + B_2 + \ldots + B_n \subset B_{n+1}, \; n, j \in \mathbf{N}.$$

Moreover, for every $j \in \mathbf{N}$ the set $\bigcup_{n=1}^{\infty} K_j^n$ is absorbing in Y. In fact, if $x \in Y$, then $x \in B_m$ for some $m \in \mathbf{N}$. Hence $x \in A_{p(m+1)}^{m+1}$ by (b). Fix $j \in \mathbf{N}$. Then $\lambda_{2^j-1}^{m+1} x \in \lambda_{2^j-1}^{m+1} A_{p(m+1)}^{m+1} \subset \sum_{k=1}^{\infty *} \lambda_{2^{j-1}k}^{m+1} A_{p(m+1)}^{m+1} = T_j^{m+1} \subset K_j^{m+1}$. This implies that $\overline{(f^{-1}(K_j^n))}_{n,j \in \mathrm{N}}$ is a γ-sequence in E which, because of (*) and (c), is not topological, a contradiction, since (E, τ) is $*$-Baire-like. We have proved that there are numbers $n, m \in \mathbf{N}$ such that $\overline{f^{-1}(A_m^n)}^{\tau_n}$ is a τ_n-neighbourhood of zero in $f^{-1}(Y_n)$. Using this we find on Y_n a complete vector topology σ_n and a completing sequence $(W_p)_{p \in \mathbf{N}}$ in (Y_n, σ_n) such that $\overline{f^{-1}(W_p)}^{\tau_n}$ is a τ_n-neighbourhood of zero for all $p \in \mathbf{N}$. In fact, since (Y_n, ϑ_n) is boundedly summing, there exists a scalar sequence $(\lambda_\iota)_{\iota \in \mathbf{N}}$, $0 < \lambda_{\iota+1} < \lambda_\iota$, such that $\sum_{\iota=1}^{\infty *} \lambda_\iota A_m^n$ is ϑ_n-bounded. The ϑ_n-bounded sets

$$W_p = \sum_{j=1}^{\infty *} \lambda_{2^{p-1}j} A_m^n, \quad p \in \mathbf{N},$$

satisfy

$$(**) \qquad W_{p+1} + W_{p+1} \subset W_p \subset W_1 = \sum_{j=1}^{\infty *} \lambda_j A_m^n.$$

Since

$$\lambda_{2^{p-1}} A_m^n \subset W_p,$$

then $\overline{f^{-1}(W_p)}^{\tau_n}$ is a τ_n-neighbourhood of zero in $f^{-1}(Y_n)$, $p \in \mathbf{N}$. Let σ_n be the finest vector topology on Y_n agreeing with ϑ_n on all sets A_k^n, $k \in \mathbf{N}$. Then $\vartheta_n \leq \sigma_n$ and (Y_n, σ_n) is complete, [1], 16 (13). Moreover, by 16(3) of [1], every ϑ_n-bounded set is σ_n-bounded. Therefore and because of (**) the sequence $(W_p)_{p \in \mathbf{N}}$ is completing in (Y_n, σ_n). We may apply Lemma 1.2 to deduce that

$$f | f^{-1}(Y_n) : f^{-1}(Y_n) \to (Y_n, \sigma_n)$$

is continuous. Since $f^{-1}(Y_n)$ is dense in E and (Y_n, σ_n) is complete, there exists a continuous linear extension g of $f|f^{-1}(Y_n)$ to the whole space E. It is easy to see that $f = g$. This completes the proof. ∎

We shall say that a *tvs* (E,τ) is an *$(LF)_{tv}$-space* (resp. *$(LB)_{tv}$-space*) if (E,τ) is the inductive limit of a strictly increasing sequence $(E_n, \tau_n)_{n\in\mathbf{N}}$ of F-spaces (resp. locally bounded F-spaces) such that $\tau_{n+1}|E_n \leq \tau_n$ for all $n \in \mathbf{N}$. We call $(E_n, \tau_n)_{n\in\mathbf{N}}$ a *defining sequence* for (E,τ).

Corollary 1.4. *Let E be a $*$-Baire-like space and Y an $(LB)_{tv}$-space. Then every linear map $f : E \to Y$ with closed graph is continuous.* ∎

Remark 1.5. If Y is not an $(LB)_{tv}$-space, then the conclusion can fail, even under the hypothesis that Y be a metrizable $(LF)_{tv}$-space and E is metrizable and ultrabarrelled. Indeed, it is enough to show that every metrizable $(LF)_{tv}$-space (E,τ) admits a strictly weaker metrizable and ultrabarrelled topology. Since (E,τ) is ultrabarrelled [1], 6 (4), but non-complete, (E,τ) is not an infra-s-space (in the sense of Adasch, [1], p. 44; cf. also 10 (10)). Hence E admits a strictly weaker Hausdorff vector topology ϑ such that the associated ultrabarrelled topology ϑ^t is strictly weaker than $\tau^t = \tau$. Let φ be the vector topology on E which has $\{\overline{U}^{\vartheta^t} : U \in \mathscr{U}_0(\tau)\}$ as a basis of $\mathscr{U}_0(\varphi)$. Then $\vartheta^t \leq \varphi$, φ is ϑ^t-polar and φ is metrizable. Hence $\vartheta^t = \varphi$. Recall that metrizable (even non-locally convex) $(LF)_{tv}$-spaces do exist, [7], [12].

Corollary 1.6. *Let (E,τ) be the inductive limit of a strictly increasing sequence of complete boundedly summing ultrabarrelled tvs (E_n, τ_n) such that every (E_n, τ_n) has a fundamental sequence of bounded sets. Then (E,τ) is not metrizable. In particular, no $(LB)_{tv}$-space is metrizable.* ∎

This extends Corollary 3 of [7].

For ultrabarrelled *tvs*, which are the inductive limits of an increasing sequence of metrizable *tvs*, we have the following characterization of $*$-Baire-likeness.

Theorem 1.7. *Let (E,τ) be an ultrabarrelled tvs which is the inductive limit of an increasing sequence $(E_n, \tau_n)_{n\in\mathbf{N}}$ of tvs.*

A. *If every (E_n, τ_n) is metrizable, then the following properties are equivalent:*

A1. *(E,τ) is $*$-Baire-like.*

A2. *(E,τ) is metrizable.*

B. *Let $Bd(\tau_n)$ be the set of all τ_n-bounded sets $n \in \mathbf{N}$. Suppose that $\mathscr{U}_0(\tau_n) \cap \cap Bd(\tau_{n+1}) \neq \emptyset$ for all $n \in \mathbf{N}$. Then the following properties are equivalent:*

B1. (E,τ) *is* $*$*-Baire-like.*

B2. *There is* $n \in \mathbb{N}$ *and* $U \in \mathscr{U}_0(\tau_n) \cap Bd(\tau_{n+1})$ *such that the* τ*-closure* $\overline{U}$ *of* U *is a* τ*-neighbourhood of zero.*

B3. (E,τ) *is locally bounded.*

B4. *The sequential closure of any subset of* (E,τ) *is sequentially closed.*

B5. *For any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* E *there exists a scalar sequence* $(\varrho_n)_{n\in\mathbb{N}}$, $\varrho_n > 0$, *such that 0 belongs to the* τ*-closure of* $\{\varrho_n x_n : n \in \mathbb{N}\}$.

The hypothesis of B. is clearly satisfied when each (E_n, τ_n) is locally bounded or when the inclusion map of (E_n,τ_n) into (E_{n+1},τ_{n+1}) is compact (or precompact) for each $n \in \mathbb{N}$.

To prove B. we shall need the following two lemmas.

Lemma 1.8 (H. *Pfister).* *Let* (E,τ) *be a tvs and* $(x_n)_{n\in\mathbb{N}}$ *a sequence in* E. *Assume the following condition: B3'. The sequential closure of any countable subset of* (E,τ) *is sequentially closed. Then there exists a scalar sequence* $(\varrho_n)_{n\in\mathbb{N}}$, $\varrho_n > 0$, *such that a subsequence of* $(\varrho_n x_n)_{n\in\mathbb{N}}$ *converges to 0; in particular, B4 holds.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E. We shall construct the sequence $(\varrho_n)_{n\in\mathbb{N}}$ so that 0 is even in the sequential closure of $\{\varrho_n x_n : n \in \mathbb{N}\}$. Choose $a \in E \setminus (\{nk^{-1}x_n : n,k \in \mathbb{N}\} \cup \{0\})$ and put $H = \{n^{-1}a - k^{-1}x_n : n,k \in N\}$. Then $n^{-1}a$ belongs to the sequential closure $\widehat{H}$ of H, and 0 is in the sequential closure of $\widehat{H}$. Hence $0 \in \widehat{H}$ by B3', i.e. there are sequences (n_l) and (k_l) in $\mathbb{N}$ such that

$$(*)\qquad n_l^{-1}a - k_l^{-1}x_{n_l} \to 0 \quad \text{for} \quad l \to \infty.$$

Then (n_l) tends to infinity. For otherwise (n_l) would have a constant subsequence and this would violate (*). Without loss of generality we may assume that $n_l < n_{l+1}$, $l \in \mathbb{N}$. So (*) implies that $(k_l^{-1}x_{n_l})$ is a nullsequence. Defining now $\varrho_{n_l} = k_l^{-1}$ for $l \in \mathbb{N}$ and $\varrho_n = 1$ for $n \in \mathbb{N} \setminus \{n_l : l \in \mathbb{N}\}$ (recall $n_l < n_{l+1}$), the sequence $(\varrho_n)_{n\in\mathbb{N}}$ is as claimed. ∎

Remark 1.9. (1) It is easy to see that, conversely, $0 \in \widehat{H}$ if there is a scalar sequence $(\varrho_n)_{n\in\mathbb{N}}$, $\varrho_n > 0$, such that 0 is in the sequential closure of $\{\varrho_n x_n : n \in \mathbb{N}\}$. (2) Since only locally convergent sequences appear in the proof, the hypothesis B3' could be «localized».

Lemma 1.10. *Let* $(K_j^n)_{n,j\in\mathbb{N}}$ *be a* γ*-sequence in an ultrabarrelled space* (E,τ). *If* (E,τ) *has property B5, then for every* $j \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *such that* K_j^n *is absorbing in* E.

Proof. Assume there exists $i \in \mathbb{N}$ such that none of the sets $K_i^n, n \in \mathbb{N}$, is absorbing in E. Hence for every $n \in \mathbb{N}$ there exists $x_n \in E$ which is not absorbed by K_i^n. Choose a sequence $(\varrho_n)_{n\in\mathbb{N}}$ according to B5. Then $\varrho_n x_n \notin K_i^n$, $n \in \mathbb{N}$. Let $(U_n)_{n\in\mathbb{N}}$ be a topological string in (E,τ) such that

$$\varrho_n x_n \notin \overline{K_i^n + U_n}, n \in \mathbb{N}.$$

Set

$$W_j = \bigcap_{m=1}^{\infty} \overline{\left(K_{i+j-1}^m + U_{m+j-1}\right)}.$$

Then $(W_j)_{j\in\mathbb{N}}$ is a closed string in (E,τ); hence topological. But $\varrho_n x_n \notin W_1$, $n \in \mathbb{N}$, a contradiction to B4. ∎

Proof of Theorem 1.7. A1 $\Rightarrow$ A2: For every $n \in \mathbb{N}$ let $(U_j^n)_{j\in\mathbb{N}}$ be a basis of balanced τ_n-neighbourhoods of zero in E_n such that $U_{j+1}^n + U_{j+1}^n \subset U_j^n$ for all $j \in \mathbb{N}$. Let $F = \{\overline{\sum_{(l,j)\in\Delta} U_j^l} : \Delta \subset \mathbb{N} \times \mathbb{N}, \Delta \text{ finite}\}$, where the closure is taken in τ. Clearly card $F = \aleph_0$. We prove that every closed τ-neighbourhood U of zero contains an element from F which is a τ-neighbourhood of zero. Choose in (E,τ) a topological string $(U^n)_{n\in\mathbb{N}}$ such that $U^1 + U^1 \subset U$. Then for every $n \in \mathbb{N}$ there exists $j_n \in \mathbb{N}$ such that $U^n \cap E_n \supset U_{j_n}^n$. Hence

$$U \supset \sum_{l=1}^{\infty *} U^l \cap E_l \supset \sum_{l=1}^{\infty *} U_{j_l}^l,$$

and

$$U = \overline{U} \supset \overline{U_{j_1}^1 + U_{j_2}^2 + \ldots U_{j_n}^n}, n \in \mathbb{N}.$$

Set

$$K_j^n = \overline{U_{j_1+j}^1 + U_{j_2+j}^2 + \ldots + U_{j_n+j}^n}, n, j \in \mathbb{N}.$$

Clearly $K_j^n \subset K_j^{n+1}$, $K_{j+1}^n + K_{j+1}^n \subset K_j^n$, $n \in \mathbb{N}$. Moreover $\bigcup_{n=1}^{\infty} K_j^n$ is absorbing in E for all $j \in \mathbb{N}$. Since (E,τ) is $*$-Baire-like, then for $j = 1$ there is $m \in \mathbb{N}$ such that $K_1^m \in \mathscr{U}_0(\tau)$. Clearly $K_1^m \subset U$. This completes the proof. A2 $\Rightarrow$ A1: This follows from Proposition 1.1.

Now we prove part B. The implications B2 $\Rightarrow$ B3 $\Rightarrow$ B4 are obvious. B4 $\Rightarrow$ B5: This follows from Lemma 1.8. B5 $\Rightarrow$ B2: Choose a sequence $(U_n)_{n\in\mathbb{N}}$ such that

$$U_n \in \mathscr{U}_0(\tau_n) \cap Bd(\tau_{n+1}) \quad \text{and} \quad U_n + U_n \subset U_{n+1}, \ n \in \mathbb{N}.$$

B2 is proved when we show that $\overline{U}_n \in \mathscr{U}_0(\tau)$ for some $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ choose a sequence $(U_j^n)_{j\in\mathbb{N}}$ of balanced sets such that $U_j^n \in \mathscr{U}_0(\tau_n)$ and $U_{j+1}^n + U_{j+1}^n \subset U_j^n \subset U_n$, $j \in \mathbb{N}$. Clearly the sets

$$K_j^n = \overline{U_j^1 + U_j^2 + \ldots + U_j^n}, n, j \in \mathbb{N},$$

form a γ-sequence, and $K_j^n \subset \overline{U}_{n+1}$ for $n, j \in \mathbb{N}$. Moreover, one has $K_1^n \subset \subset \overline{U_1 + U_2 + \ldots + U_n}$, where $U_1, U_2, \ldots, U_n$ are τ_{n+1}-bounded. Therefore, for all $n, j \in \in \mathbb{N}$, there is $\alpha > 0$ such that $U_1 + \ldots + U_n \subset \alpha U_j^{n+1}$, whence $K_1^n \subset \alpha K_j^{n+1}$. Now using Lemma 1.10 one obtains that there is $m \in \mathbb{N}$ such that $(K_j^m)_{j\in\mathbb{N}}$ is a τ-closed string in E; hence $(K_j^m)_{j\in\mathbb{N}}$ is topological. This implies that $\overline{U}_{m+1} \in \mathscr{U}_0(\tau)$. B1 $\Rightarrow$ B2: Replace in the last proof the role of Lemma 1.10 by the assumption B1. B3 $\Rightarrow$ B1: Apply Proposition 1.1. ∎

In Theorem 1.7, the equivalence B1 $\Longleftrightarrow$ B3 remains true under the weaker assumption $\mathscr{U}_0(\tau_n) \cap Bd(\tau) \neq \emptyset$ for all $n \in \mathbb{N}$ instead of $\mathscr{U}_0(\tau_{n+1}) \neq \emptyset$: Obviously B3 $\Rightarrow$ B1 holds, and B1 $\Rightarrow$ B3 follows by an obvious change in the proof of B5 $\Rightarrow$ B2.

Corollary 1.11. *Let (E, τ) be the inductive limit of an increasing sequence of tvs (E_n, τ_n) such that for every $n \in \mathbb{N}$ the inclusion map of (E_n, τ_n) into (E_{n+1}, τ_{n+1}) is compact. Then E contains a subset whose sequential closure is not sequentially closed.*

Proof. By [1], 18 (iv), p. 108, (E, τ) is ultrabarrelled. Since (E, τ) is not metrizable (cf. [1], 18 (i) and 16 (10) and recall our convention to consider only infinite dimensional *tvs*) it is enough to apply Theorem 1.7 part B.

Remark 1.12. Note that there exist $*$-Baire-like (even Baire) spaces for which condition B5 from Theorem 7.1 is not satisfied: Consider the space $E = \mathbf{R}^{\mathbf{R}}$ endowed with the product topology τ. Then (E, τ) is a Baire space. There exists a sequence $(x_k)_{k\in\mathbb{N}}$ in E, $x_k = = (x_{k,\alpha})_{\alpha\in\mathbf{R}}$, such that $\{(x_{k,\alpha})_{k\in\mathbb{N}} : \alpha \in \mathbf{R}\} = \mathbf{R}^{\mathbf{N}}$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{R}, \alpha_n > 0$, $n \in \mathbb{N}$. Then there exists $\gamma \in \mathbf{R}$ such that $(x_{k,\gamma})_{k\in\mathbb{N}} = (\alpha_k^{-1})_{k\in\mathbb{N}}$. Then $0 \notin \overline{\{\alpha_k x_k : k \in \mathbb{N}\}}^{\tau}$.

Following Pérez Carreras [6] we call a *tvs* E a *$*$-suprabarrelled* space if given any increasing sequence of subspaces of E covering E, one of them is both dense and ultrabarrelled. Further we shall say that E is *$*$-quasi-Baire* if E is ultrabarrelled and if E is covered by an increasing sequence of subspaces of E, then one of them is dense.

Clearly

$$
\begin{array}{ccccc}
*-Baire-like & \Rightarrow & *-quasi-Baire & \Rightarrow & ultrabarrelled \\
 & & & & \uparrow \\
 & & & & *-suprabarrelled.
\end{array}
$$

Using Proposition 1.1 one obtains that within metrizable *tvs* $*$-suprabarrelled $\Rightarrow$ $*$-Baire-like $\Longleftrightarrow$ $*$-quasi-Baire $\Longleftrightarrow$ ultrabarrelled. Using Lemma 1.0 one obtains easily the following analog of Proposition 1.1. If E is an ultrabarrelled dense subspace of a $*$-quasi-Baire space then E is $*$-quasi-Baire. The analog for $*$-suprabarrelled spaces fails, since there exist metrizable $(LF)_{tv}$-spaces and these spaces are not $*$-suprabarrelled (see below).

It is known that all $(LF)_{tv}$-spaces are ultrabarrelled. On the other hand, Adasch's closed graph theorem [1], 8 (6), applies to show that no $(LF)_{tv}$-space is $*$-suprabarrelled. Our Corollary 1.6 and Theorem 1.7 show that no $(LB)_{tv}$-space is $*$-Baire-like. A very similar argument to that which was used in the proof of Theorem 4 and Corollary 6 of [12] enables one to show that all F-spaces with an unconditional basis contain proper dense subspaces which are $(LF)_{tv}$-spaces. Following Narayanaswami and Saxon [7] we partition all $(LF)_{tv}$-spaces into three mutually disjoint non-empty classes as follows:

An $(LF)_{tv}$-space (E,τ) is an $(LF)_{tv,i}$-space if it satisfies the condition (i) below, $i = 1,2,3$.

(1) (E,τ) *has a defining sequence none of whose members is dense in* (E,τ).

(2) (E,τ) *is non-metrizable and has a defining sequence each of whose members is dense in* (E,τ).

(3) (E,τ) *is metrizable.*

Examples of $(LF)_{tv,i}$-spaces can be found in [3], [7]. Using Theorem 1.7 we obtain the following characterization of $(LF)_{tv,i}$-spaces in terms of Baire-type properties defined above.

Proposition 1.13. *Let (E,τ) be an $(LF)_{tv}$-space. Then:*

(a) (E,τ) is an $(LF)_{tv,3}$-space iff (E,τ) is $*$-Baire-like.

(b) (E,τ) is an $(LF)_{tv,2}$-space iff (E,τ) is $*$-quasi-Baire but not $*$-Baire-like.

(c) (E,τ) is an $(LF)_{tv,1}$-space iff (E,τ) is not $*$-quasi-Baire.

Proof. (a) Follows from Theorem 1.7. (b) If (E,τ) is $*$-quasi-Baire but not $*$-Baire-like, then by Theorem 1.7, part A, (E,τ) is an $(LF)_{tv,2}$-space. Now assume that (E,τ) is an $(LF)_{tv,2}$-space. Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of τ-closed subspaces of E covering E. By assumption, (E,τ) has a defining sequence $(E_n,\tau_n)_{n\in\mathbb{N}}$ of τ-dense F-spaces. If $G_n = E_n \cap F_n$, $n\in\mathbb{N}$, then $(G_n,\tau_n|G_n)_{n\in\mathbb{N}}$ is an increasing sequence of F-spaces covering E. Let (E,ϑ) be the inductive limit of $(G_n,\tau_n|G_n)_{n\in\mathbb{N}}$. Then $\tau\le\vartheta$. By Adasch's closed graph theorem [1], 10 (11), and a remark after it, $\tau=\vartheta$ and for every $n\in\mathbb{N}$ there exists $m\in\mathbb{N}$ such that $E_n\subset G_m$. Consequently G_m is τ-dense and so is F_m. Therefore (E,τ) is $*$-quasi-Baire. On the other hand, by Theorem 1.7, (E,τ) is not $*$-Baire-like. (c) Assume (E,τ) is not $*$-quasi-Baire and not an $(LF)_{tv,1}$-space, i.e. given a defining sequence $(E_n,\tau_n)_{n\in\mathbb{N}}$ of F-spaces, some E_n is τ-dense. Then there exists a strictly increasing sequence $(F_n)_{n\in\mathbb{N}}$ of τ-closed subspaces of E covering E. If $G_n = E_n\cap F_n$, $n\in\mathbb{N}$, then (E,τ) is the inductive limit of the sequence $(G_n,\tau_n|G_n)_{n\in\mathbb{N}}$, cf. the proof of (b). Taking $n\in\mathbb{N}$ such that E_n is τ-dense, then $E_n\subset G_m$ for some $m\in\mathbb{N}$, cf. the proof of case (b). Hence F_m is τ-dense, a contradiction. The reverse implication in (c) is obvious. ∎

As we have observed every metrizable $(LF)_{tv}$-space is $*$-Baire-like but need not be $*$-supra- barrelled. Now we discuss the occurence of proper dense non-*-suprabarrelled subspaces of F-spaces, extending Theorem 1 of [18] and [11]. First, if an F-space (E,τ) contains a proper dense ultrabarrelled subspace G, which is an $(LF)_{tv}$-space for a topology finer than $\tau|G$, then $(G,\tau|G)$ is not $*$-suprabarrelled. This follows from the closed graph theorem of [1], 8 (6). In the converse direction we have the following more interesting proposition.

Proposition 1.14. *Let (E,τ) be an F-space and F a dense subspace which is ultrabarrelled (equivalently $*$-Baire-like) but not $*$-suprabarrelled. Then (E,τ) contains a proper dense ultrabarrelled subspace G such that $F \subset G$ and G is an $(LF)_{tv}$-space for a topology finer than $\tau|G$.*

Proof. By assumption there exists an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of subspaces of F covering F such that no F_n is both dense and ultrabarrelled. Using Proposition 1.1 we may assume that all F_n are dense in F. Let $(V_j)_{j\in\mathbb{N}}$ be a basis of balanced τ-closed neighbourhoods of zero in (E,τ) such that $V_{i+1}+V_{i+1}\subset V_j$, $j\in\mathbb{N}$. By assumption on F for every $n\in\mathbb{N}$ there exists in F_n a closed string $(W_j^n)_{j\in\mathbb{N}}$ such that $W_j^n \notin \mathscr{U}_0(F_n)$, $j\in\mathbb{N}$. Set

$$V_j^n = \overline{W_j^n}\cap V_j, \quad Q_j^{n,p} = V_j^p \cap V_j^{p+1}\cap\ldots\cap V_j^n,$$

$j\in\mathbb{N}$, $n\geq p$, $n,p\in\mathbb{N}$, where the closure is taken in τ. Let

$$G_j^{n,p} = \{\lambda Q_j^{n,p} : \lambda\in\mathbf{K}\},\ G_p = \bigcap_{j=1}^{\infty}\bigcap_{n\geq p} G_j^{n,p}.$$

Then G_p is a subspace of E, $F_p\subset G_p$, $G_p\subset G_{p+1}$, $p\in\mathbb{N}$. Let $\tau_{n,p}$ be the metrizable vector topology on G_p defined by the string $(G_p\cap Q_j^{n,p})_{j\in\mathbb{N}}$. Then $\tau|G_p \leq \tau_{n,p}$. Set $\tau_p=\sup\{\tau_{n,p} : n\geq p\}$. Then (G_p,τ_p) is an F-space. In fact, since τ_p is $\tau|G_p$-polar, it is enough to show that every Cauchy sequence $(x_k)_{k\in\mathbb{N}}$ in (G_p,τ_p) converges in $\tau|G_p$. Fix $j\in\mathbb{N}$, $n\geq p$. There exists $\lambda\in\mathbf{K}$ such that $x_k\in\lambda Q_j^{n,p}$ for all $k\in\mathbb{N}$. Since $(x_k)_{k\in\mathbb{N}}$ is τ-Cauchy, $x_k\to x$ in τ for some $x\in E$. Hence $x\in\lambda Q_j^{n,p}$, which implies that $x\in G_j^{n,p}$. Therefore $x\in G_p$. Moreover, $\tau_{p+1}|G_p\leq\tau_p$, $\tau|G_p\leq\tau_p$, $p\in\mathbb{N}$. Let (G,ϑ) be the inductive limit of the sequence $(G_p,\tau_p)_{p\in\mathbb{N}}$, where $G=\bigcup_{p=1}^{\infty}G_p$. Then $\tau|G\leq\vartheta$. Suppose that $G=E$. Then using Adasch's closed graph theorem [1], 10 (11) and a remark after it, one obtains that $\tau=\vartheta$, $G_l=E$ and $\tau_l=\tau$ for some $l\in\mathbb{N}$. Therefore $(\overline{W_j^l}\cap V_j)_{j\in\mathbb{N}}$ is a topological string in (E,τ). Hence $W_j^l=\overline{W_j^l}\cap F_l\in\mathscr{U}_0(F_l)$, a contradiction. ∎

Corollary 1.15. *An F-space (E,τ) contains a dense non-ultrabarrelled subspace iff (E,τ) contains a dense subspace which is not $*$-suprabarrelled.*

Proof. If (E,τ) contains a dense non-ultrabarrelled subspace F, then F is not $*$-supra- barrelled. Now suppose that (E,τ) contains a dense subspace F which is not $*$-suprabarrelled. If F is ultrabarrelled (otherwise there is nothing to show), then by Proposition 1.14 there exists in E a dense ultrabarrelled subspace $G \supset F$ such that G is an $(LF)_{tv}$-space for a topology $\vartheta \geq \tau|G$. Let $(G_n)_{n\in\mathbb{N}}$ be a defining sequence of F-spaces for (G,ϑ). Since $(G,\tau|G)$ is metrizable and ultrabarrelled, it is $*$-Baire-like. So there is $m \in \mathbb{N}$ such that G_m is τ-dense. Then $(G_m, \tau|G_m)$ is not ultrabarrelled by the closed graph theorem [1], 8 (6). ∎

Now we come to results related to the Banach-Steinhaus theorem which involve $*$-Baire-like (Baire-like) spaces. In [4],§ 3, ex. 1.1, Bourbaki proved that every separately equicontinuous set $\mathscr{F}$ of bilinear maps $f : E \times T \to F$ is equicontinuous, provided E is metrizable and barrelled, T is a metrizable locally convex space and F is a locally convex space. In [17] Valdivia extended this result to Baire-like spces E. The following proposition extends both results.

Proposition 1.16. *Let E be a $*$-Baire-like space, F a tvs, and T a topological space whose points have countable bases of neighbourhoods. Let $\mathcal{F}$ be a set of maps $f : E \times T \to F$ with the following properties:*

(I_1) For each $t \in T$, $\{f(\cdot,t) : f \in \mathcal{F}\}$ is an equicontinuous set of linear maps from E into F.

(I_2) For each $x \in E$, $\{f(x,\cdot) : f \in \mathcal{F}\}$ is equicontinuous.

Then the set $\mathcal{F}$ is equicontinuous. The same conclusion holds when E is Baire-like and F is a locally convex space.

Proof. Because of $f(x,t) - f(a,c) = f(x-a,t) + (f(a,t) - f(a,c))$ for $x, a \in E, t, c \in T$ and (I_2) it suffices to show the equicontinuity at points $(0,c) \in E \times T$. Let $(W_n)_{n\in\mathbb{N}}$ be a decreasing base of $\mathscr{U}_c(T)$ and let $V \in \mathscr{U}_0(F)$. We show that there are $U \in \mathscr{U}_0(E)$ and $m \in \mathbb{N}$ such that $f(U \times W_m) \subset V$ for all $f \in \mathcal{F}$. Let $(V_j)_{j\in\mathbb{N}}$ be a closed topological string in F such that $V_1 + V_1 \subset V$. The sets $K_j^n = \{x \in E : f(x,t) \in V_j, f \in \mathcal{F}, t \in W_n\}$ with $n, j \in \mathbb{N}$ are closed (by (I_1)) and balanced and satisfy $K_j^n \subset K_j^{n+1}$, $K_{j+1}^n + K_{j+1}^n \subset K_j^n$, $n, j \in \mathbb{N}$. We show that $E = \bigcup_{n=1}^{\infty} nK_j^n$, $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$ and choose $x \in E$. Because of (I_1) there exists $l \in \mathbb{N}$ such that $f(x,t) - f(x,c) \in V_{j+1}$, $t \in W_l$, $f \in \mathcal{F}$. Moreover, by (I_1) there exists $r > 1$ such that $f(x,c) \in rV_{j+1}$, $f \in \mathcal{F}$. Hence $f(x,t) \in$ $\in f(x,c) + V_{j+1} \subset rV_{j+1} + V_{j+1} \subset rV_j$, $t \in W_l$, $f \in \mathcal{F}$. Therefore $x \in sK_j^s$ for some

$s \in \mathbf{N}$. We have proved that $(K_j^n)_{j,n\in\mathbf{N}}$ is a γ-sequence in E. Hence there is $n \in \mathbf{N}$ such that $U = K_1^n \in \mathscr{U}_0(E)$. The second part of Proposition 1.16 is obtained similarly. ∎

Proposition 1.17. *Let E be a barrelled space. Then E is Baire-like iff, for every topological space T whose points have countable bases of neighbourhoods and every locally convex space F, any set $\mathscr{F}$ of maps $f : E\times T \to F$ with the properties (I_1) and (I_2) of Proposition 1.16 is equicontinuous.*

Proof. If E is Baire-like, the conclusion holds by Proposition 1.16. Now assume that E is not Baire-like. Since E is barrelled, there exists an increasing sequence $(K_n)_{n\in\mathbf{N}}$ of closed absolutely convex subsets of E covering E such that none of the sets K_n is absorbing in E. Set $T = E'$ equipped with the topology ϑ of the uniform convergence on the sets $nK_n, n \in \mathbf{N}$, where E' denotes the topological dual of E. Then (T, ϑ) is a metrizable topological vector group (in the sense of Raikov [8]) and $\sigma(E', E) \leq \vartheta$. Let $f : E \times T \to \mathbf{K}$ be the evaluation map $(x,t) \mapsto t(x)$, $x \in E, t \in T$, and put $\mathscr{F} = \{f\}$. Then the conditions (I_1) and (I_2) of Proposition 1.16 are satisfied. However f is discontinuous at $(0,0)$. For if f were continuous at $(0,0)$, there would be $V \in \mathscr{U}_0(E)$ and $m \in \mathbf{N}$ such that $|t(x)| \leq 1$, $x \in V$ and $t \in m^{-1}K_m^0$. This means that $V \subset (m^{-1}K_m^0)^0 = mK_m$, and K_m would be a neighbourhood of zero in E, a contradiction. ∎

Remark 1.18. From Proposition 1.17 and its proof we have the following: Let (E, τ) be a barrelled space, E' its topological dual and f the evaluation map $(x,t) \mapsto t(x), x \in E, t \in E'$. Then E is Baire-like iff for every metrizable vector group topology ϑ on E' the map $f : (E,\tau) \times (E,\vartheta) \to \mathbf{K}$ is continuous at zero. With the same technique one proves: let (E,τ) be a quasi-barelled space with a fundamental sequence of bounded sets. Then (E,τ) is normed iff the evaluation map $(x,t) \mapsto t(x)$, $x \in E$, $t \in E'$ is continuous at zero as a map from $(E,\tau) \times (E', \beta(E',E))$ into $\mathbf{K}$.

From Proposition 1.16 we derive an analogue of the Banach-Steinhaus theorem:

Proposition 1.19. *Let E, F and T be spaces as in Proposition 1.16. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence of maps $f_n : E \times T \to F$ with the following properties:*

(1) For each $n \in \mathbf{N}$ and $t \in T$, $f_n(\cdot, t)$ is a linear and continuous map from E into F.
(2) $(f_n)_{n\in\mathbf{N}}$ converges pointwise to a map $g : E \times T \to F$.
(3) For each $x \in E, \{f_n(x,\cdot) : n \in \mathbf{N}\}$ is equicontinuous.
Then $g : E \times T \to F$ is continuous.

Proof. Since every pointwise bounded set of continuous linear maps from an ultrabarrelled space into a tvs is equicontinuous, [1], 7 (3), the set $\mathscr{F} = \{f_n : n \in \mathbf{N}\}$ satisfies the conditions of Proposition 1.16. Hence the sequence of maps $f_n : E \times T \to F$ is equicontinuous. Therefore for $a \in E$, $c \in T$, and closed $V \in \mathscr{U}_0(E)$ there exist $U \in \mathscr{U}_0(E)$,

$W \in \mathscr{U}_c(T)$ such that $f_n(U \times W) - f_n(a,c) \subset V, n \in \mathbb{N}$. Hence $g(U \times W) - g(a,c) \subset V$, which completes the proof. ∎

It is known that the product of two metrizable Baire *tvs* may be not Baire, cf. e.g. [15]. Hence the property of being a Baire *tvs* is not a three-space property, i.e. there exists a *tvs* E containing a closed subspace F such that E/F and F are Baire spaces but E is not a Baire space. We conclude this section by showing that $*$-Baire-likeness is a three-space property. A similar result for Baire-likeness was obtained in [2].

Proposition 1.20. *Let F be a closed subspace of a tvs E such that E/F and F are $*$-Baire-like. Then E is $*$-Baire-like.*

Proof. Let $(K_j^n)_{n \in \mathbb{N}}$ be a γ-sequence in E. Fix $i \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $K_{i+1}^m \cap F \in \mathscr{U}_0(F)$. Choose $U \in \mathscr{U}_0(E)$ such that $(U - U - U) \cap F \subset K_{i+1}^m$. Let $(U_j)_{j \in \mathbb{N}}$ be a topological string in E such that $U_1 + U_1 \subset U$, and let $q : E \to E/F$ be the quotient map. Then $\overline{(q(U_j \cap K_j^n))}_{n,j \in \mathbb{N}}$ is a γ-sequence in E/F. Since E/F is $*$-Baire-like, there exists $n \in \mathbb{N}$ such that $\overline{q(U_{i+1} \cap K_{i+1}^n)} \in \mathscr{U}_0(E/F)$. There exists $V \in \mathscr{U}_0(E)$ such that $V \subset U$ and $V \subset U_{i+1} \cap K_{i+1}^n + W \cap U + F$ for all $W \in \mathscr{U}_0(E)$. Hence $V \subset U_{i+1} \cap K_{i+1}^n + W \cap U + F \cap (U - U - U)$. Therefore $V \subset \overline{K_{i+1}^n + K_{i+1}^m} \subset K_i^p$ for $p = \max(n, m)$. We proved that $(K_j^n)_{n,j \in \mathbb{N}}$ is topological; hence E is $*$-Baire-like. ∎

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