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A NOTE ON Σ -COMPLETE LOCALLY CONVEX SPACES

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Dedicated to the memory of Professor Gottfried Köthe

Let E be a separated locally convex topological vector space with topology τ.

A sequence, $\{x_n\}_{n=1}^{\infty}$, of elements of E is said to have *finite variation* if

$$\sum_{j=1}^{\infty} p(x_j - x_{j-1}) < \infty,$$

for every τ-continuous seminorm, p, on E. Here and in the sequel, we put $x_0 = 0$.

A set, $A \subset E$, is said to be Σ *-closed* if the limit of every convergent sequence of elements of A having finite variation belongs to A.

A set, $A \subset E$, is said to be Σ *-complete* if every sequence of its elements having finite variation is convergent and its limit belongs to A.

In this note, we are concerned mainly with the Σ -complete locally convex spaces. Such spaces, or classes of such spaces, were studied already a long time ago, although this terminology was not necessarily used. For example, the celebrated theorem of C. Bessaga and A. Pełczynski about weakly unconditionally summable sequences, see [2], p. 98, can be reformulated by saying that a Banach space is Σ -complete in its weak topology if and only if it does not contain an isomorphic copy of the space c_0. Banach spaces or general locally convex spaces which are Σ -complete in their weak topology play a significant role in the theory of vector valued measures and vector integration. In fact, G.E.F. Thomas seems to be the first to use, in [3], the term «weakly Σ -complete space».

We may note in this context that there exist Banach spaces which are Σ -complete but not sequentially complete in the weak topology. The space J of R.C. James is an example of such a space. For, if e_n, $n = 1, 2, \ldots$, are elements of its natural base, see [2], p. 25, then the sequence of vectors $\{x_n\}_{n=1}^{\infty}$, where

$$x_n = \sum_{j=1}^{n} e_j,$$

for $n = 1, 2, \ldots$, is weakly Cauchy but not weakly convergent, because $\{e_n : n = 1, 2, \ldots\}$ is a shrinking base. And yet, J does not contain an isomorphic copy of c_0 and, hence, by the

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Bessaga-Pełczynski theorem, it is Σ-complete. So, the class of Σ-complete locally convex spaces is non-trivial and thus, it may be worth-while to study it in its own right.

We are going to show that in some well-known theorems on sequentially complete locally convex spaces, the assumption of sequential completeness can be relaxed to Σ-completeness. So we prove that every Σ-complete bornological space is an inductive limit of Banach spaces. We also prove the Banach-Mackey theorem under the assumption of Σ-completeness instead of sequential completeness, see [1].

A sequence, $\{x_n\}_{n=1}^{\infty}$, of elements of the space E is said to be *summable* if there exists an element, s, of E such that

$$s = \lim_{n\to\infty} \sum_{j=1}^{n} x_j.$$

If it exists, such an element, s, is unique and is called the sum of the sequence $\{x_n\}_{n=1}^{\infty}$.

Proposition 1. *i) Let* $x_n \in E, n = 1,2,\ldots, x_0 = 0$. *The sequence* $\{x_n\}_{n=1}^{\infty}$ *has finite variation if and only if, for every absolutely convex neighbourhood,* U, *of zero, there exists a sequence,* $\{\lambda_j\}_{j=1}^{\infty} \in l^1$, *such that* $(x_j - x_{j-1}) \in \lambda_j U$, *for every* $y = 1,2,\ldots$

ii) Every sequence having finite variation is Cauchy. If the space E *is metrizable, then every Cauchy sequence has a subsequence which has finite variation.*

iii) If a subset of E *is sequentially closed, or sequentially complete, then it is* Σ*-closed, or* Σ*-complete, respectively. If the space* E *is metrizable, then a subset of* E *is sequentially closed, or sequentially complete, if and only if it is* Σ*-closed, or* Σ*-complete, respectively.*

iv) Every Σ*-complete subset of* E *is* Σ*-closed.*

v) If the space E *is* Σ*-complete, then every* Σ*-closed subset of* E *is* Σ*-complete.*

vi) A vector subspace, F, *of* E *is* Σ*-complete, if and only if, every sequence,* $\{x_n\}_{n=1}^{\infty}$, *of its elements such that*

$$\sum_{j=1}^{\infty} p(x_j) < \infty,$$

for every τ*-continuous seminorm,* p, *on* E, *is summable and its sum belongs to* F.

Proof. We prove only that, if the space E is metrizable, then every Cauchy sequence in E has a subsequence with finite variation. So, let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in E and let $\{p_k : k = 1,2,\ldots\}$ be a family of seminorms determining the topology τ.

Choose inductively strictly increasing sequences $\{m_l(n)\}_{n=1}^{\infty}$ of natural numbers, $l = 0,1,2,\ldots,$ so that

i) $m_0(n) = n$, for every $n = 1,2,\ldots$; and

ii) for every $l = 1,2,\ldots, \{m_l(n)\}_{n=1}^{\infty}$ is a subsequence of $\{m_{l-1}(n)\}_{n=1}^{\infty}$ such that

$$p_k(x_{m_l(n)} - x_{m_l(n-1)}) < \frac{1}{2^{n+1}},$$

for every $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, l$.

Now, let $m(n) = m_n(n)$, for every $n = 1, 2, \ldots$. Then

$$p_k(x_{m(n)} - x_{m(n-1)}) < \frac{1}{2^n},$$

for every $k = 1, 2, \ldots$ and $n = k + 1, k + 2, \ldots$. Therefore, $\{x_{m(n)}\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ which has finite variation.

The Σ *-closure*, $\Sigma(A)$, of a set $A \subset E$ is defined to be the intersection of all Σ -closed subsets of E that contain A. A set, A, is said to be Σ -dense in a set, B, if $B \subset \Sigma(A)$.

Theorem 1. *For every separated locally convex space, E, there is a Σ -complete space $\hat{E}$ such that E is isomorphic to a Σ -dense subspace of $\hat{E}$. The space $\hat{E}$ is unique up to isomorphims.*

Proof. For $\hat{E}$ take the Σ -closure of E in the completion of E.

The space $\hat{E}$ of Theorem 1 is called the Σ *-completion* of E. As customary, the space E is identified with its image in $\hat{E}$. So, one writes simply $E \subset \hat{E}$.

A locally convex topology, ρ, on E is said to be Σ -linked with the topology τ if it is finer than τ, that is, $\tau \subset \rho$, and if there exists a base of neighbourhoods of zero for ρ consisting of absolutely convex sets which are Σ -closed with respect to τ.

Lemma 1. *Let ρ be a locally convex topology on E which is Σ -linked with the topology τ. Then every sequence, $\{x_n\}_{n=1}^{\infty}$, of elements of the space E having finite variation with respect to ρ which converges in the topology τ to an element, x, of E also converges to x in the topology ρ.*

Proof. Let U be an absolutely convex ρ-neighbourhood of 0 which is Σ -closed with respect to τ. Let $\{\lambda_j\}_{j=1}^{\infty} \in l^1$ be a sequence such that $x_j - x_{j-1} \in \lambda_j U$, for every $j = 1, 2, \ldots$. Let $N > 1$ be an integer such that

$$\sum_{j=N}^{\infty} |\lambda_j| < 1.$$

Then

$$(x_{n+k} - x_{n-1}) \in \left(\sum_{j=n}^{n+k} \lambda_j\right) U \subset U,$$

for every $n = N, N + 1, N + 2, \ldots$ and every $k = 1, 2, \ldots$.

Now, for every $n = N, N+1, N+2, \ldots$, the sequence $\{x_{n+k} - x_{n-1}\}_{k=1}^{\infty}$ has finite variation with respect to both topologies, ρ and τ, and converges to $x - x_{n-1}$ in the topology τ. Moreover, $(x - x_{n-1}) \in U$, for every $n = N, N+1, \ldots$, because the set U is Σ-closed with respect to τ. Since U is an arbitrary member of a base of neighbourhoods of 0 for the topology ρ, the sequence $\{x_n\}_{n=1}^{\infty}$ converges in ρ to x.

Theorem 2. *Let ρ be a locally convex topology on E which is Σ-linked with the topology τ. Then every subset of the space E which is Σ-complete with respect to τ is also Σ-complete with respect to ρ.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of a Σ-complete, in τ, subset, A, of the space E which has finite variation with respect to ρ. Because the set A is Σ-complete in τ, the sequence $\{x_n\}_{n=1}^{\infty}$ converges in τ to an element, x, of A. By Lemma 1, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x in the topology ρ.

Theorem 3. *Let A be an absolutely convex, bounded and Σ-complete subset of the space E. Let F be the vector space spanned by the set A. Let q be the gauge of the set A in the space F. Then q is a norm which makes of F a Banach space.*

Proof. Because it is absolutely convex, the set A is absorbing in F. Moreover, q is a norm on F because A is a bounded set. Hence, the topology, ρ, induced by the norm q is finer than the relative topology, $\tau|F$, induced by τ on the space F. The collection of sets $\{\varepsilon A : \varepsilon > 0\}$ is a base of neighbourhoods of zero for the topology ρ. It consists of absolutely convex sets which are Σ-complete with respect to $\tau|F$. By Theorem 2, for $\varepsilon > 0$, the set εA is Σ-complete with respect to the topology ρ, that is, with respect to the norm q, and, hence, it is complete. Because every q-Cauchy sequence in F is bounded, it is contained in a set εA, for some $\varepsilon > 0$, and, therefore, it is q-convergent.

We are grateful to the referee for turning our attention to an interesting consequence of Theorem 3. Namely, it implies that a Σ-complete locally convex space is locally complete, that is Mackey sequentially complete.

This notion is of course derived from the notion of a locally convergent sequence and the related notion of a locally Cauchy sequence. Let us recall that a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space E is said to be locally convergent to x if there exists a closed, bounded and absolutely convex set $A \subset E$ such that $x_n \in A, n = 1, 2, \ldots, x \in A$ and, if q is the gauge of the set A in the space spanned by A, then $q(x_n - x) \to 0$ as $n \to \infty$; see [1], §28, 3.

So, we can obtain many consequences of Theorem 3 by merely listing the properties of locally complete spaces. We would like to mention just two such corollaries of Theorem 3, viz., that a Σ-complete bornological space is ultrabornological and that the conclusion of the

Banach-Mackey theorem holds for all Σ -complete and not only sequentially complete locally convex spaces.

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